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On the $\mathcal{Z}_N$-Nakano Sequence Space

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Abstract

Some results in this paper may give ideas of how ones should formulate the corresponding results for the more general cases. In this paper, using the Zweier operator and a modular, we defined the $\mathcal{Z}_N$- Nakano sequence space and to show that the $\mathcal{Z}_N$- Nakano sequence space equipped with the Z-Luxemburg norm is rotund and posses property-H (or Kadec-Klee property) when $p = (p_k)$ is bounded with $p_k > 1$ for all $k \in N^0$. In addition, we have showed the $\mathcal{Z}_N$- Nakano sequence space is isomorphic to the Cesàro sequence space $Ces(p)$.

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1 Introduction and Preliminary

A sequence space is linear subspace of w, where $w = \{x = (x_k)_{k\in\mathbb{N}} : \quad x : \mathbb{N} \to \mathbb{R} \quad (or \quad \mathbb{C}), \quad k \to x_k = x(k)\}$. An FK- space whose topology is normable is called a BK- space [2].

Let λ be an subset space of w. For a Banach space λ, we denote by $S(\lambda)$ and $B(\lambda)$ the unit sphere and unit ball of λ, respectively. A point $x_0 \in S(\lambda)$ is called:

$a)$ an extreme point if for every $x, y \in S(\lambda)$ the equality $2x_0 = x+y$ implies $x = y$;

$b)$ an H-point if for any sequence (x_n) in λ such that $\| x \| \to 1$ as $n \to \infty$, the weak convergence of (x_n) to x implies that $\| x_n - x \| \to 0$ as $n \to \infty$.

A Banach space λ is said to be rotund, if every point of $S(\lambda)$ an extreme point. A Banach space λ is said posses H-property provided every point of $S(\lambda)$ is an H-point.

Let λ be an arbitrary vector space over $\mathbb{C}$. In this case:

a) A functional $m : \lambda \to [0, \infty]$ is called modular if the following conditions hold:

M1) $m(x) = 0 \Leftrightarrow x = 0$,

M2) $m(rx) = m(x)$ for $r \in \mathbb{R}$ (or $\mathbb{C}$) with $|r| = 1$, for all $x \in \lambda$,

M3) $m(rx + sy) \leq m(x) + m(y)$ if $r, \quad s \geq 0, \quad r + s = 1$, for all $x, y \in \lambda$.

b) If *M3* is replaced by;

M4) $m(rx + sy) = r^{\mu}m(x) + s^{\mu}m(y)$ if $r, \quad s \geq 0, \quad r^{\mu} + s^{\mu} = 1$, with an $\mu \in [0, 1]$ then the modular m is called an s-convex modular; and if $\mu = 1$, m is called a convex modular.

c) A modular m defines the corresponding modular space, i.e, the space λ_m given by

$$\lambda_m = \{x \in w : m(tx) \to 0 \ \ as \ \ t \to \infty\}.$$

Recall that for given any $\epsilon > 0$, a sequence (x_n) is said to be an $\epsilon-$*separated sequence* if

$$\text{sep}(x_n) = \inf\{\| x_n - x_k \| : n \neq k\} > \epsilon.$$

We say Banach space λ has β-property if for every $\epsilon > 0$ such that, for each element $x \in B(\lambda)$ and each sequence $(x_n) \in B(\lambda)$ with $\text{sep}(x_n) \geq \epsilon$, there exists an index k such that

$$\| \frac{x + x_k}{2} \| \leq 1 - \delta.$$

The Nakano sequence space $\ell(p)$ is defined by

$$\ell(p) = \{x = (x_k) \in w : m(tx) < \infty \quad \text{for some} \ \ t > 0\},$$

where $m(x) = \sum_k |x_k|^{p_k}$ and $p = (p_k)$ is a sequence of positive real numbers with $p_k \geq 1$ for all $k \in \mathbb{N}$. The space $\ell(p)$ is a Banach space with the norm

$$\| x \| = \inf\left\{t > 0 : m(\frac{x}{t}) \leq 1\right\}.$$

If $p = (p_k)$ is bounded, we have

$$\ell(p) = \left\{x = (x_k) \in w : \sum_k |x_k|^{p_k} < \infty\right\}.$$

Also, some geometric properties of $\ell(p)$ were studied in [1] and [3].

For $1 \leq p < \infty$, the Cesàro sequence space is defined by

$$ces_p = \left\{ x = (x_k) \in w : \left(\sum_n (\frac{1}{n} \sum_{k=1}^{n} |x_k|)^p \right)^{\frac{1}{p}} < \infty \right\} \tag{1.1}$$

equipped with the norm

$$\| x \| = \left(\sum_n (\frac{1}{n} \sum_{k=1}^{n} |x_k|)^p \right)^{\frac{1}{p}}.$$

This space was introduced by Shue [12]. Some geometric properties of sequence spaces ces_p and $\ell(p)$ have been thoroughly discussed by many mathematicians in the literature. It is known that ces_p is locally uniform rotund and posses property-H [5]. Cui and Hudzik [3] proved that ces_p has the Banach-Saks of type p if $p > 1$, and it was shown in [4] that ces_p has β-property. For more information, see [6], [7], [8] and [9]. Also, we know that Nakano sequence spaces are special cases of Musielak- Orlicz sequence spaces. Some results in this paper may give ideas of how ones should formulate the corresponding results for the more general cases. In this paper, we extend the study on another sequence space which we describe below.

2 Main Results

The space $ces(p)$ [11] is defined by

$$ces(p) = \{ x \in w : \rho(tx) < \infty \quad \text{for some} \;\; t > 0 \}, \tag{2.1}$$

where $\rho(x) = \sum_n \left(\frac{1}{n+1} \sum_{k=0}^{n} |x_k| \right)^{p_n}$. The space $ces(p)$ is a Banach space with the norm

$$\| x \| = \inf \left\{ t > 0 : \rho(\frac{x}{t}) \leq 1 \right\}$$

and if $p = (p_n)$ is bounded then we have

$$ces(p) = \left\{ x \in w : \sum_n \left(\frac{1}{n+1} \sum_{k=0}^{n} |x_k| \right)^{p_n} < \infty \right\}.$$

Several geometric properties of $ces(p)$ were studied in [11]. Define the sequence $y = (y_n)$, which will be frequently used, as the Z-transform of a sequence $x = (x_k)$, i.e.,

$$(Zx)_n = y_n = \alpha x_k + (1 - \alpha) x_{k-1} \tag{2.2}$$

where, $Z = (z_{nk})$ is defined by

$$z_{nk} = \begin{cases} \alpha & , \quad (k = n) \\ 1-\alpha & , \quad (k = n-1) \\ 0 & , \quad (otherwise) \end{cases} ; \; (\; n, \;\; k \in \mathbb{N}), \;\; \alpha \in \mathbb{K} \backslash \{0\} \, , \tag{2.3}$$

and $\mathbb{K}$ is the field of all complex or real numbers.

Now, we wish to introduce the $\mathcal{Z}$-Nakano sequence space $\mathcal{Z}_N(p)$, as the set of all sequences such that $Z-$ transforms of them are in the space $Ces(p)$, that is

$$\mathcal{Z}_N(p) = \{x = (x_k) \in w : (Zx) \in Ces(p)\} \tag{2.4}$$

or in another word

$$\mathcal{Z}_N(p) = \{x \in w : m^*(tx) < \infty \;\; \text{for some} \;\; t > 0 \;\; \} \, ,$$

where $m^*(x) = \sum_n (\frac{1}{n+1} \sum_{i=0}^n |\alpha x_i + (1-\alpha) x_{i-1}|)^{p_n} < \infty$. We consider the space $\mathcal{Z}_N(p)$ equipped with the so - called Z-Luxemburg norm

$$\| \, x \, \|_{\mathcal{Z}} = \inf \left\{ t > 0 : m^*(\frac{x}{t}) \leq 1 \right\} .$$

If $p = (p_n)$ is bounded, then we have

$$\mathcal{Z}_N(p) = \left\{ x \in w : \sum_n \left(\frac{1}{n+1} \sum_{i=0}^n |\alpha x_i + (1-\alpha) x_{i-1}| \right)^{p_n} < \infty \right\} .$$

The purpose of this note is to define and to investigate the $\mathcal{Z}_N$- Nakano sequence space and show that $\mathcal{Z}$- Nakano sequence space equipped with the Luxemburg norm is rotund and posses H-property when $p = (p_k)$ is bounded with $p_k > 1$ for all $k \in \mathbb{N}$. Also, throughout this paper we assume that $p = (p_i)$ is bounded with $p_i > 1$ for all $i \in \mathbb{N}$ and $K = \sup_i p_i$.

Clearly, in the special case $\alpha = 1$ we have $\mathcal{Z}_N(p) = Ces(p)$, [11]. Now, we may begin with the following theorem which is essential in the text:

Theorem 2.1. *The functional m^* on the $\mathcal{Z}_N$-Nakano sequence space is a convex modular.*

Proof. Let $x, y \in \mathcal{Z}_N(p)$. It is clearly that $m^*(x) = 0 \Leftrightarrow x = 0$ and $m^*(rx) = m^*(x)$ for all scalar r with $|r| = 1$, so we omit it. Again let suppose that $x, y \in \mathcal{Z}_N$-Nakano sequence space and $r \geq 0, \;\; s \geq 0$ with $r + s = 1$. By the convexity of the function $u \to |u|^{p_n}$; $n \in \mathbb{N}$, we have:

$$\begin{aligned}
m^*(rx+sy) &= \sum_n \left(\frac{1}{n+1} \sum_{i=0}^{n} |r(\alpha x_i + (1-\alpha)x_{i-1}) + s(\alpha y_i + (1-\alpha)y_{i-1})| \right)^{p_n} \\
&\leq \sum_n \left(\frac{1}{n+1} \sum_{i=0}^{n} |r(\alpha x_i + (1-\alpha)x_{i-1}) \right. \\
&\quad \left. + \frac{1}{n+1} \sum_{i=0}^{n} |s(\alpha y_i + (1-\alpha)y_{i-1})| \right)^{p_n} \\
&= \sum_n |r|^{p_n} \left(\frac{1}{n+1} \sum_{i=0}^{n} |(\alpha x_i + (1-\alpha)x_{i-1})| \right)^{p_n} \\
&\quad + \sum_n |s|^{p_n} \left(\frac{1}{n+1} \sum_{i=0}^{n} |(\alpha y_i + (1-\alpha)y_{i-1})| \right)^{p_n} \\
&\leq r \sum_n \left(\frac{1}{n+1} \sum_{i=0}^{n} |(\alpha x_i + (1-\alpha)x_{i-1})| \right)^{p_n} \\
&\quad + s \sum_n \left(\frac{1}{n+1} \sum_{i=0}^{n} |(\alpha y_i + (1-\alpha)y_{i-1})| \right)^{p_n} = rm^*(x) + sm^*(y).
\end{aligned}$$

□

Theorem 2.2. *The $\mathcal{Z}_N$-Nakano sequence space is the BK- space with the norm $\| x \|_{\mathcal{Z}_N(p)} = \| Zx \|_{\ell(p)}$.*

Proof. Since (2.2) holds and $\ell(p)$ is the $BK-$space [10] with respect to its norm and the matrix Z is normal, Theorem 4.3.2 of Wilansky [13, pp. 61] gives the fact that the space $\mathcal{Z}_N$-Nakano sequence space is $BK-$ space. □

Theorem 2.3. *The sequence space $Ces(p)$ is linearly isomorphic to the space $\mathcal{Z}_N(p)$ i.e., $\ell(p) \cong \mathcal{Z}_N(p)$.*

Proof. We should show the existence of a linear bijection between the space $\mathcal{Z}_N(p)$ and $Ces(p)$. Consider the transformation Z define, with the notation of (2.2), from $\mathcal{Z}_N(p)$ to $Ces(p)$ by

$$\begin{aligned}
Z : \mathcal{Z}_N(p) &\longmapsto Ces(p) \\
x &\longmapsto Zx = y, \quad y = (y_i), \quad y_i = \alpha x_i + (1-\alpha)x_{i-1}\ , \ (i \in \mathbb{N})
\end{aligned}$$

The linearity of Z is clear. Further, it is trivial that $x = 0$ whenever $Zx = 0$ and hence Z is injective. Let $y \in Ces(p)$ and define the sequence $x = (x_i)$ by

$$x_i = M \sum_{j=0}^{i} (-1)^{i-j} N^{i-j} y_j, \quad (i \in \mathbb{N}), \quad \textit{where} \quad N = \frac{1-\alpha}{\alpha} \quad \textit{and} \quad M = \frac{1}{\alpha}.$$

Then, we have

$$\begin{aligned}
&\sum_n \left(\frac{1}{n+1} \sum_{i=0}^{n} |\alpha x_i + (1-\alpha) x_{i-1}| \right)^{p_n} \\
&= \sum_n \left(\frac{1}{n+1} \sum_{i=0}^{n} |\alpha M \sum_{j=0}^{i} (-1)^{i-j} N^{i-j} y_j + (1-\alpha) M \sum_{j=0}^{i-1} (-1)^{i-j} N^{i-j} y_j| \right)^{p_n} \\
&= \sum_n \left(\frac{1}{n+1} \sum_{i=0}^{n} |y_n| \right)^{p_n}
\end{aligned}$$

which says us that $x \in \mathcal{Z}_N(p)$. But also, we observe that

$$\begin{aligned}
\| x \|_{\mathcal{Z}} &= \inf \left\{ t > 0 : m(\frac{x}{t}) \leq 1 \right\} \\
&= \inf \left\{ t > 0 : \sum_n \left(\frac{1}{n+1} \sum_{i=0}^{n} |\frac{1}{t} (\alpha x_i + (1-\alpha) x_{i-1}|) \right)^{p_n} \leq 1 \right\} \\
&= \inf \left\{ t > 0 : \sum_n \left(\frac{1}{n+1} \sum_{i=0}^{n} |\frac{1}{t} (\alpha M \sum_{j=0}^{i} (-1)^{i-j} N^{i-j} y_j \right.\right. \\
&\left.\left. + \ (1-\alpha) M \sum_{j=0}^{i-1} (-1)^{i-j} N^{i-j} y_j|) \right)^{p_n} \leq 1 \right\} = \inf \left\{ t > 0 : m(\frac{y}{t}) \leq 1 \right\} \\
&= \| y \|_{Ces(p)} .
\end{aligned}$$

Consequently, we see from here that Z is surjective and is norm preserving. Hence, Z is linear bijection which therefore says us that the spaces $\mathcal{Z}_N$-Nakano sequence space and $\ell(p)$ are linearly isomorphic. □

Theorem 2.4. *For $x \in \mathcal{Z}_N(p)$ the modular m^* on $\mathcal{Z}_N(p)$ satisfies the following properties:*

1. *if $0 < t < 1$ then $t^K m^*(t^{-1}x) \leq m^*(x)$ and $m^*(tx) \leq t m^*(x)$,*
2. *if $t > 1$, then $m^*(x) \leq t^K m^*(t^{-1}x)$,*
3. *if $t \geq 1$, then $m^*(x) \leq t m^*(x) \leq m^*(tx)$.*

Proof. *1.* For $0 < t < 1$, we have

$$\begin{aligned} m^*(x) &= \sum_n \left(\frac{1}{n+1} \sum_{i=0}^n |\alpha x_i + (1-\alpha)x_{i-1}| \right)^{p_n} \\ &= \sum_n \left(\frac{t}{n+1} \sum_{i=0}^n |t^{-1}(\alpha x_i + (1-\alpha)x_{i-1})| \right)^{p_n} \\ &= \sum_n t^{p_n} \left(\frac{1}{n+1} \sum_{i=0}^n |t^{-1}(\alpha x_i + (1-\alpha)x_{i-1})| \right)^{p_n} \\ &\geq \sum_n t^K \left(\frac{1}{n+1} \sum_{i=0}^n |t^{-1}(\alpha x_i + (1-\alpha)x_{i-1})| \right)^{p_n} \\ &= t^K \sum_n \left(\frac{1}{n+1} \sum_{i=0}^n |t^{-1}(\alpha x_i + (1-\alpha)x_{i-1})| \right)^{p_n} = t^K m^*(t^{-1}x), \end{aligned}$$

and it implies by the convexity of m^* that $m^*(tx) \leq tm^*(x)$ hence *1* is satisfied.

2. Let $t > 1$ then,

$$\begin{aligned} m^*(x) &= \sum_n \left(\frac{1}{n+1} \sum_{i=0}^n |\alpha x_i + (1-\alpha)x_{i-1}| \right)^{p_n} \\ &= \sum_n t^{p_n} \left(\frac{1}{n+1} \sum_{i=0}^n |t^{-1}(\alpha x_i + (1-\alpha)x_{i-1})| \right)^{p_n} \\ &\leq t^K \sum_n \left(\frac{1}{n+1} \sum_{i=0}^n |t^{-1}(\alpha x_i + (1-\alpha)x_{i-1})| \right)^{p_n} = t^K m^*(t^{-1}x) \end{aligned}$$

so *2* is obtained.

3. It is obvious that *3* is satisfied by the convexity of m^*. □

Now, we give relationships between the Luxemburg norm and the modular m^* on the space Z_N- Nakano sequence space $\mathcal{Z}_N(p)$.

Theorem 2.5. *For any $x \in \mathcal{Z}_N(p)$, if $p = (p_n)$ is bounded, we have*

4. *if* $\| x \|_{\mathcal{Z}} < 1$ *then* $m^*(x) \leq \| x \|_{\mathcal{Z}}$
5. *if* $\| x \|_{\mathcal{Z}} > 1$ *then* $m^*(x) \geq \| x \|_{\mathcal{Z}}$
6. $\| x \|_{\mathcal{Z}} = 1$ *if and only if* $m^*(x) = 1$
7. $\| x \|_{\mathcal{Z}} < 1$ *if and only if* $m^*(x) < 1$
8. $\| x \|_{\mathcal{Z}} > 1$ *if and only if* $m^*(x) > 1$
9. *if* $0 < r < 1$ *and* $\| x \|_{\mathcal{Z}} > r$ *then* $m^*(x) > r^K$
10. *if* $r \geq 1$ *and* $\| x \|_{\mathcal{Z}} < r$ *then* $m^*(x) < r^K$

Proof. 4. Let $\epsilon > 0$ be such that $0 < \epsilon < 1 - \| x \|_{\mathcal{Z}} \Rightarrow \| x \|_{\mathcal{Z}} + \epsilon < 1$. From definition of $\| . \|_{\mathcal{Z}}$ there exists $t > 0$ such that $\| x \|_{\mathcal{Z}} + \epsilon > t$ and accordingly $\frac{\|x\|_{\mathcal{Z}}+\epsilon}{t} > 1$ and $m^*(t^{-1}x) \leq 1$. From Theorem 2.4 (*1.* and *3.*), we have

$$m^*(x) \leq m^*\left((\| x \|_{\mathcal{Z}} + \epsilon)t^{-1}x\right) \leq (\| x \|_{\mathcal{Z}} + \epsilon)m^*\left(t^{-1}x\right) \leq \| x \|_{\mathcal{Z}} + \epsilon$$

which implies that $m^*(x) \leq \| x \|_{\mathcal{Z}}$. So *4* is satisfied.

5. If $0 < \epsilon < (\| x \|_{\mathcal{Z}} - 1) \| x \|_{\mathcal{Z}}^{-1}$ then $1 < (1-\epsilon) \| x \|_{\mathcal{Z}} < \| x \|_{\mathcal{Z}}$ and by definition of $\| . \|_{\mathcal{Z}}$ with by part *1* of Theorem2.4 we have

$$1 < m^*(x[(1-\epsilon) \| x \|_{\mathcal{Z}}]^{-1}) \leq [(1-\epsilon) \| x \|_{\mathcal{Z}}]^{-1} m^*(x).$$

So $(1-\epsilon) \| x \|_{\mathcal{Z}} < m^*(x)$ for all $\epsilon \in (0, (\| x \|_{\mathcal{Z}} - 1) \| x \|_{\mathcal{Z}}^{-1})$. This implies that $\| x \|_{\mathcal{Z}} \leq m^*(x)$, hence *5* is obtained.

6. We have that $m^*(x) = 1$ implies that $\| x \|_{\mathcal{Z}} = 1$. Now assume that $\| x \|_{\mathcal{Z}} = 1$. By the definition of $\| x \|_{\mathcal{Z}}$ we have that for any $\epsilon > 0$ there exists $\mu > 0$ such that $1 + \epsilon > \mu > \| x \|_{\mathcal{Z}}$ and $m^*(x\mu^{-1}) \leq 1$. By part *P2* of Theorem 2.4, we have

$$m^*(x) \leq \mu^K m^*(x\mu^{-1}) \leq \mu^K < (1+\epsilon)^K.$$

By this way $(m^*(x))^{K^{-1}} < 1 + \epsilon$ for all $\epsilon > 0$, which implies $m^*(x) \leq 1$. If $m^*(x) < 1$, then we can choose $r \in (0,1)$ such that $m^*(x) < r^K < 1$. If we consider *1* of Theorem 2.4, we have $m^*(r^{-1}x) \leq (r^K)^{-1} m^*(x) < 1$ hence $\| x \|_{\mathcal{Z}} \leq r < 1$ which is a contradiction. Therefore $m^*(x) = 1$.
7. Proof is clear from *4.* and *6.*
8. Follows from *6.* and *7.*
9. Suppose $0 < r < 1$ and $\| x \|_{\mathcal{Z}} > r$. Then $\| xr^{-1} \|_{\mathcal{Z}} > 1$. By *5* we have $m^*(xr^{-1}) > 1$. Hence from *1* of Theorem 2.4, we obtain that $m^*(x) \geq r^K m^*(r^{-1}x) > r^K$.
10. Suppose that $r \geq 1$ and $\| x \|_{\mathcal{Z}} < r$. Then $\| xr^{-1} \|_{\mathcal{Z}} < 1$. From *7* we have $\| xr^{-1} \|_{\mathcal{Z}} < 1$. If $r = 1$, it is obvious that $m^*(x) < 1 = r^K$. If $r > 1$ then by part*2* of Theorem 2.4; we obtain that $m^*(x) \leq r^K m^*(r^{-1}x) < r^K$. □

Theorem 2.6. *Let (x_n) be a sequence in $\mathcal{Z}_N(p)$, where $p = (p_k)$ is bounded. Then;*
11. If $\| x_n \|_{\mathcal{Z}} \to 1$ as $n \to \infty$, then $m^*(x_n) \to 1$ as $n \to \infty$.
12. If $m^*(x_n) \to 0$ as $n \to \infty$, then $\| x_n \|_{\mathcal{Z}} \to 0$ as $n \to \infty$.

Proof. 11. Let's suppose that $\| x \|_{\mathcal{Z}} \to 1$ as $n \to \infty$ and let $\epsilon \in (0,1)$. Then there exists $N \in \mathbb{N}$ such that $1 - \epsilon < \| x_n \|_{\mathcal{Z}} < 1 + \epsilon$ for all $n \in \mathbb{N}$. By Theorem 2.5 we have $(1-\epsilon)^K < m^*(x_n) < (1+\epsilon)^K$ for all $n \geq N$ which implies $m^*(x_n) \to 1$ as $n \to \infty$.

12. Now, let us suppose that $\| x_n \|_{\mathcal{Z}} \nrightarrow 0$ as $n \to \infty$. Then there is an $\epsilon \in (0,1)$ and a subsequence (x_{n_k}) of (x_n) such that $\| x_{n_k} \|_{\mathcal{Z}} > \epsilon$ for all $k \in \mathbb{N}$. By part *9.* of Theorem 2.5 we have $m^*(x_{n_k}) > \epsilon^K$ for all $k \in \mathbb{N}$. This implies $m^*(x_{n_k}) \nrightarrow 0$ as $n \to \infty$. □

Now we shall give a lemma.

Lemma 2.7. *Let $x \in \mathcal{Z}_N(p)$ and $(x^k) \subseteq \mathcal{Z}_N(p)$. If $\lim_k m^*(x^k) = m^*(x)$ and $\lim_k x_i^k = x_i$ for all $i \in \mathbb{N}$ then $\lim_k x^k = x$ in $\mathcal{Z}_N(p)$, that is $\| x^k - x \|_{\mathcal{Z}} \to 0$ as $n \to \infty$.*

Proof. Let $\epsilon > 0$ be given. Since $m^*(x) = \sum_n \left(\frac{1}{n+1}\sum_{i=0}^n |\alpha x_i + (1-\alpha)x_{i-1}|\right)^{p_n} < \infty$ there is $n_0 \in \mathbb{N}$ such that

$$\sum_{n=n_0+1}^{\infty} \left(\frac{1}{n+1}\sum_{i=0}^{n} |\alpha x_i + (1-\alpha)x_{i-1}|\right)^{p_n} < \epsilon(2^{K+1}3)^{-1}. \tag{2.5}$$

Since

$$m^*(x^k) - \sum_{n=0}^{n_0} \left(\frac{1}{n+1}\sum_{i=0}^{n} |\alpha x_i + (1-\alpha)x_{i-1}|\right)^{p_n}$$
$$\to m^*(x) - \sum_{n=0}^{n_0} \left(\frac{1}{n+1}\sum_{i=0}^{n} |\alpha x_i + (1-\alpha)x_{i-1}|\right)^{p_n}$$

as $(k \to \infty)$ and $x_i^k \to x_i$ as $k \to \infty$ as for all $i \in \mathbb{N}$, there is $n_0 \in \mathbb{N}$ such that

$$m^*(x^k) - \sum_{n=0}^{n_0} \left(\frac{1}{n+1}\sum_{i=0}^{n} |\alpha x_i + (1-\alpha)x_{i-1}|\right)^{p_n} \tag{2.6}$$
$$< m^*(x) - \sum_{n=0}^{n_0} \left(\frac{1}{n+1}\sum_{i=0}^{n} |\alpha x_i + (1-\alpha)x_{i-1}|\right)^{p_n} + (2^{K+1}3)^{-1} \tag{2.7}$$

and

$$\sum_{n=0}^{n_0} \left(\frac{1}{n+1}\sum_{i=0}^{n} |\alpha(x_i^n - x_i) + (1-\alpha)(x_{i-1}^n - x_{i-1})|\right)^{p_n} < 3^{-1}\epsilon. \tag{2.8}$$

for all $k \geq k_0$. It follows from (2.5), (2.6) and (2.8) that for $k \geq k_0$

$$\begin{aligned}
m^*(x^n - x) &= \sum_n \left(\frac{1}{n+1} \sum_{i=0}^{n} |\alpha(x_i^n - x_i) + (1-\alpha)(x_{i-1}^n - x_{i-1})| \right)^{p_n} \\
&= \sum_{n=0}^{n_0} \left(\frac{1}{n+1} \sum_{i=0}^{n} |\alpha(x_i^n - x_i) + (1-\alpha)(x_{i-1}^n - x_{i-1})| \right)^{p_n} \\
&+ \sum_{n=n_0+1}^{\infty} \left(\frac{1}{n+1} \sum_{i=0}^{n} |\alpha(x_i^n - x_i) + (1-\alpha)(x_{i-1}^n - x_{i-1})| \right)^{p_n} \\
&< 3^{-1}\epsilon + 2^K \left[\sum_{n=n_0+1}^{\infty} \left(\frac{1}{n+1} \sum_{i=0}^{n} |\alpha x_i^n + (1-\alpha) x_{i-1}^n| \right)^{p_n} \right. \\
&+ \left. \sum_{n=n_0+1}^{\infty} \left(\frac{1}{n+1} \sum_{i=0}^{n} |\alpha x_i + (1-\alpha) x_{i-1}| \right)^{p_n} \right] \\
&= 3^{-1}\epsilon + 2^K \left[m(x^k) - \sum_{n=0}^{n_0} \left(\frac{1}{n+1} \sum_{i=0}^{n} |\alpha x_i^n + (1-\alpha) x_{i-1}^n| \right)^{p_n} \right. \\
&+ \left. \sum_{n=n_0+1}^{\infty} \left(\frac{1}{n+1} \sum_{i=0}^{n} |\alpha x_i + (1-\alpha) x_{i-1}| \right)^{p_n} \right] \\
&< 3^{-1}\epsilon + 2^K \left[m(x^k) - \sum_{n=0}^{n_0} \left(\frac{1}{n+1} \sum_{i=0}^{n} |\alpha x_i + (1-\alpha) x_{i-1}| \right)^{p_n} \right. \\
&+ \left. (2^K 3)^{-1}\epsilon + \sum_{n=n_0+1}^{\infty} \left(\frac{1}{n+1} \sum_{i=0}^{n} |\alpha x_i + (1-\alpha) x_{i-1}| \right)^{p_n} \right] \\
&= 3^{-1}\epsilon + 2^K \left[\sum_{n=n_0+1}^{\infty} \left(\frac{1}{n+1} \sum_{i=0}^{n} |\alpha x_i + (1-\alpha) x_{i-1}| \right)^{p_n} \right. \\
&+ \left. (2^K 3)^{-1}\epsilon + \sum_{n=n_0+1}^{\infty} \left(\frac{1}{n+1} \sum_{i=0}^{n} |\alpha x_i + (1-\alpha) x_{i-1}| \right)^{p_n} \right] \\
&= 3^{-1}\epsilon + 2^K \left[(2^K 3)^{-1}\epsilon + 2 \sum_{n=n_0+1}^{\infty} \left(\frac{1}{n+1} \sum_{i=0}^{n} |\alpha x_i + (1-\alpha) x_{i-1}| \right)^{p_n} \right] \\
&< 3^{-1}\epsilon + 3^{-1}\epsilon + 3^{-1}\epsilon = \epsilon
\end{aligned}$$

This show that $m^*(x^k - x) \to 0$ as $k \to \infty$. And consequently by part *8* of Theorem 2.6 we have that $\| x^k - x \|_{\mathcal{Z}} \to 0$ as $k \to \infty$. □

Theorem 2.8. *The $\mathcal{Z}_N(p)$ has the H-property (or Kadec-Klee property).*

Proof. Let $x \in S(\mathcal{Z}_N(p))$ and $(x^n) \subseteq \mathcal{Z}_N(p)$ be such that $\| x^n \|_{\mathcal{Z}} \to 1$ and $x^n \to x$ *weakly* as $n \to \infty$.

From Theorem2.1, we have $m^*(x) = 1$ so it follows from Theorem 2.4 that $m^*(x^n) \to m(x)$ as $n \to \infty$. Since the mapping $p_i : \mathcal{Z}_N(p) \longrightarrow \mathbb{R}$, defined by $p_i(y) = y_i$ is a continuous linear functional on $\mathcal{Z}_N(p)$ it follows that $x_i^n \to x_i$ as $n \to \infty$ for all $i \in \mathbb{N}$. Thus, by Lemma 2.7, we get $x^n \to x$ as $n \to \infty$. □

Theorem 2.9. *If $p = (p_n)$ is bounded then the $\mathcal{Z}_N(p)$ is rotund .*

Proof. Let $x \in S(\mathcal{Z}_N(p))$ and $y, z \in B(\mathcal{Z}_N(p))$ with $x = 2^{-1}(y + z)$. Since convexity of m^* and from Theorem 2.5 we have

$$1 = m^*(x) \leq 2^{-1}[m^*(y) + m^*(z)] \leq 2^{-1}(1 + 1),$$

so that $m^*(x) = 2^{-1}[m^*(y) + m^*(z)] = 1$. This suggest that

$$\left(\frac{1}{n+1}\sum_{k=0}^{n} |2^{-1}(\alpha(y_k + z_k) - (1-\alpha)(y_{k-1} + z_{k-1})|\right)^{p_n} \tag{2.9}$$

$$= 2^{-1}\left(\frac{1}{n+1}\sum_{k=0}^{n} |(\alpha y_k + (1-\alpha)y_{k-1}|\right)^{p_n} + 2^{-1}\left(\frac{1}{n+1}\sum_{k=0}^{n} |(\alpha z_k + (1-\alpha)z_{k-1}|\right)^{p_n}$$

for all $n \in \mathbb{N}$. We shall show that $y_n = z_n$ for all $n \in \mathbb{N}$. For $n = 0$ from (2.9) we have

$$|x_0|^{p_0} = 2^{-1}[|y_0|^{p_0} + |z_0|^{p_0}]. \tag{2.10}$$

Since the mapping $u \to |u|^{p_0}$ is strictly convex, it implies by (2.10) that $y_0 = z_0$. Now assume that $y_i = z_i$ for all $i = 1, 2, ..., n-1$. Then $y_i = z_i = x_i$ for all $i = 1, 2, ..., n-1$. From (2.9) we have

$$\left(\frac{1}{n+1}\sum_{k=0}^{n} |2^{-1}(\alpha(y_k + z_k) - (1-\alpha)(y_{k-1} + z_{k-1})|\right)^{p_n} \tag{2.11}$$

$$\begin{aligned} &= \left(2^{-1}[\frac{1}{n+1}\sum_{k=0}^{n} |\alpha y_k + (1-\alpha)y_{k-1}| + \frac{1}{n+1}\sum_{k=0}^{n} |\alpha z_k + (1-\alpha)z_{k-1}|]\right)^{p_n} \\ &= 2^{-1}\left(\frac{1}{n+1}\sum_{k=0}^{n} |\alpha z_k + (1-\alpha)z_{k-1}|\right)^{p_n} + 2^{-1}\left(\frac{1}{n+1}\sum_{k=0}^{n} |\alpha z_k + (1-\alpha)z_{k-1}|\right)^{p_n} \end{aligned}$$

By the convexity of the mapping $u \to |u|^{p_n}$ it implies that $\frac{1}{n+1}\sum_{k=0}^{n} |\alpha y_k + (1-\alpha)y_{k-1}| = \frac{1}{n+1}\sum_{k=0}^{n} |\alpha z_k + (1-\alpha)z_{k-1}|$. Since $y_n = z_n$ for all $i = 1, 2, ..., n-1$ we get that

$$|y_n| = |z_n|. \tag{2.12}$$

If $y_n = 0$, then we have $y_n = z_n = 0$. Suppose that $y_n \neq 0$. Then $z_n \neq 0$. If $y_n z_n < 0$ it follows from (2.12) that $y_n + z_n = 0$. This implies by (2.11) and (2.12)

$$\left(\frac{1}{n+1}\sum_{k=0}^{n-1}|\alpha x_k + (1-\alpha)x_{k-1}|\right)^{p_n}$$
$$= \left(\frac{1}{n+1}\sum_{k=0}^{n-1}|\alpha x_k + (1-\alpha)x_{k-1}| + |\alpha y_k + (1-\alpha)y_{k-1}|\right)^{p_n}$$

which is a contradiction. Thus, we have $y_n z_n > 0$. This implies that, by (2.10) $y_n = z_n$. Thus, by induction, we have $y_n = z_n$ for all $n \in \mathbb{N}$, so $y = z$. □

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