Int. Journal of Math. Analysis, Vol. 4, 2010, no. 27, 1341 - 1348

A Critical Look at Fixed Point Theorems for Occasionally Weakly Compatible Maps in Probabilistic Semi-Metric Spaces

K.P.R. Sastry[1], G.A. Naidu[2], P.V.S. Prasad[3], V. Madhavi Latha[4] and S.S.A. Sastri[5]

[1] 8-28-8/1, Tamil Street, Chinna Waltair, Visakhapatnam-530 017, India
kprsastry@hotmail.com

[2, 3, 4] Department of Mathematics, Andhra University
Visakhapatnam-530 003, India
drgolivean@yahoo.com
pvsprasad10@yahoo.in
vml_gsm@yahoo.co.in

[5] Department of Basic Science and Humanities, Coastal Institute of Technology and Management, Narapam, Vizianagaram- 535 183, India
sambharasas@yahoo.co.in

Abstract. In this paper, certain modifications of the results given in Harish Chandra and Arvind Bhatt [1] are shown.

Mathematics Subject Classification (2000): 47H10, 54H25

Keywords: Fixed points, compatible maps, weakly compatible maps, Occasionally weakly compatible maps, probabilistic Metric spaces

1. INTRODUCTION

Harish Chandra and Arvind Bhatt [1] proved a Lemma ([1], Lemma) which is not tenable as it is. We give an example to that effect and make suitable modification in the hypothesis of the Lemma to make the conclusion tenable.

Further we show that the example give in support of Theorem 3.1 of [1] does not satisfy the required condition. We establish this and provide an example in support of Theorem 3.1 of [1].

We follow the notations and definitions given in [1].

The following Lemma is given in [1].

Lemma 1.1:[1] Let (X, F) be a probabilistic semi- metric space *f* and *g* are self maps of X. Suppose *f* and *g* have a unique point of coincidence, $w = fx = gx$. Then w is the unique common fixed point of *f* and *g*.

The above lemma is not valid for *f* and *g* in general, in view of the following example, but it is clearly valid when *f* and *g* are occasionally weakly compatible.

Example 1.2: Let X = [0, 1] with usual metric. Then (X, F) is probabilistic metric space where $F_{xy}(t) = H(t - d(x, y))$ for all $x, y \in X$, H being the Heaviside function.

$$f(x) = \begin{cases} 1 - \frac{3}{2}x & if\ 0 \leq x \leq \frac{1}{2} \\ \frac{1}{2}(1-x) & if\ \frac{1}{2} < x \leq 1 \end{cases}$$

$$g(x) = \begin{cases} \frac{x}{2} & if\ 0 \leq x \leq \frac{1}{2} \\ \frac{3}{2}x - \frac{1}{2} & if\ \frac{1}{2} < x \leq 1 \end{cases}$$

Then *f* and *g* have unique coincidence point namely $f\frac{1}{2} = g\frac{1}{2} = \frac{1}{4}$.

f and *g* do not commute at ½ . so that the pair (*f*, *g*) is not occasionally weakly compatible. Further *f* have a unique fixed point and *g* has two fixed point. But *f* and *g* do not have common fixed point.

The following Theorem is proved in [1]

Theorem 1.3 ([1], Theorem 3.1): Suppose $\varphi : [0,1] \to [0,1]$ satisfies the condition $\varphi(q) > q$ for $0 \leq q < 1$. Let (X, F) be a probabilistic semi metric space. If f and g are occasionally weakly compatible self maps on X and

$$F_{fx,fy}(t) \geq \varphi[\min\{F_{gx,gy}(t), F_{gx,fy}(t), F_{gy,fx}(t), F_{gy,fy}(t)\}] \ldots (1)$$

for all $x,y \in X$ and $t > 0$, then f and g have a unique common fixed point.

The following example in given in [1] in support of the above theorem.

Example 1.4 ([1], Example3.1): Let X = [0, 1] and $\varphi : X \to X$ be defined as $\varphi(q) = \frac{1+q}{2}$. Then clearly $\varphi(q) > q$ for all $0 \leq q < 1$.
Define f and g by $f(x) = \frac{1+2x}{3}$ and $g(x) = \frac{1+4x}{5}$.
Then f and g satisfy all the conditions if Theorem 1.3 with respect to the distribution function F_{xy} defined as

$$F_{xy}(t) = \begin{cases} e^{-\frac{|x-y|}{t}} & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}$$

In this example f and g are occasionally weakly compatible maps, f, g satisfy the equation (1) and f and g have unique common fixed point 1.
However this example does not support Theorem 1.3 in view of the following:
We have $\varphi(q) = \frac{1+q}{2} \geq \frac{1}{2}$ for all $0 \leq q < 1$

And $F_{fx,fy}(t) = e^{\frac{-|fx-fy|}{t}}$

$= e^{\frac{-|a-b|}{t}}$ where $fx = a$ and $fy = b$

$\geq \varphi(q)$ where

$q = \varphi\left[\min\{F_{gx,gy}(t), F_{gx,fy}(t), F_{gy,fx}(t), F_{gy,fy}(t)\}\right]$

$\geq \frac{1}{2}$ $if\ a \neq b, for\ all\ t > 0$

(we can choose x and y such that $a \neq b$)

$\Rightarrow \frac{1}{e^{\frac{|a-b|}{t}}} \geq \frac{1}{2}$

$\Rightarrow 2 \geq e^{\frac{|a-b|}{t}}$

$\Rightarrow 2 \geq e$ if $t = |a - b|$, a contradiction.

Thus this example does not support the above Theorem1.3.

Now we provide the following eaample in support of Theorem 1.3.

Example 1.5: Let X = [0, 1] and φ: [0, 1] → [0, 1] be defined as

$$\varphi(q) = \begin{cases} \frac{1+q}{2} & if\ 0 \leq q < 1 \\ 0 & if\quad q = 1\,. \end{cases}$$

Then clearly $\varphi(q) > q$.
Define $f(x) = 0$ if $0 \leq x \leq 1$ and $g(x) = \begin{cases} 1 & if\ 0 \leq x < 1 \\ 0 & if\quad x = 1\,. \end{cases}$

Then f and g satisfy all the conditions of Theorem 1.3 with respect to the distribution functions F defined as $F_{xy}(t) = \begin{cases} e^{-\frac{|x-y|}{t}} & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}$

2. Main Results

Theorem1.3 is an inspiration to the following Theorem.

Theorem 2.1: Let a function φ be defined by φ : [0, 1] → [0, 1] satisfying the condition $\varphi(q) > q$ for all $q < 1$. Let (X, F) be a probabilistic semi metric space. If f and g are occasionally weakly compatible maps on X and
$F_{fx,fy}(t) \geq \varphi$ [min $\{F_{gx,gy}(t), F_{gx,fx}(t), F_{gx,fy}(t), F_{gy,fx}(t), F_{gy,fy}(t)\}$
for all $x,y \in X$ and $t > 0$, then f and g have a unique common fixed point.

Proof: Since f and g are occasionally weakly compatible maps, there exists a point u in X such that $f\,u = g\,u$ and $f\,g\,u = g\,f\,u$.
We prove that f and g have a unique common fixed point.
First we show that $f\,u$ is a common fixed point for f and g.

$$F_{fu,ffu}(t) \geq \varphi\,[\,min\,\{F_{gu,gfu}(t), F_{gu,fu}(t), F_{gu,ffu}(t), F_{gfu,fu}(t), F_{gfu,ffu}(t)\,\}\,]$$
$$= \varphi\,[min\,\{\,F_{fu,ffu}(t), F_{fu,fu}(t), F_{fu,ffu}(t), F_{ffu,fu}(t), F_{ffu,ffu}(t)\,\}\,]$$
$$> F_{fu,ffu}(t) \text{ if } F_{fu,ffu}(t) < 1$$

Hence $F_{fu,ffu}(t) > F_{fu,ffu}(t) \quad if\ F_{fu,ffu}(t) < 1$
which is a contradiction.
∴ $F_{fu,ffu}(t) = 1$ for every $t > 0$
so that $fu = ffu$ i.e. $f\,u$ is a fixed point of f.
But $ffu = fgu = gfu$ so that $gfu = ffu = fu$.
∴ fu is a common fixed point of f and g.
Uniqueness: Let u, v be two common fixed points of f and g.
i.e. $f\,u = g\,u = u$ and $f\,v = g\,v = v$. Then

$$F_{u,v}(t) \geq \varphi\,[\min\{F_{u,v}(t), F_{u,v}(t), F_{u,v}(t), F_{v,u}(t), F_{v,v}(t)\}\,]$$
$$> F_{u,v}(t) \quad \text{if} \quad F_{u,v}(t) < 1$$

so that $F_{u,v}(t) > F_{u,v}(t)$ if $F_{u.v}(t) < 1$
which is a contradiction.
∴ $F_{u,v}(t) = 1$ for every $t > 0$ and hence $u = v$.

We also have the following unique fixed point theorem for a pair of self maps on probabilistic semi metric spaces.

Theorem 2.2: Let a function φ is defined by $\varphi: [0,1] \to [0,1]$ satisfying the condition $\varphi(q) > q$ for all q< 1. Let (X, F) be a probabilistic semi metric space. Suppose *f* and *g* are occasionally weakly compatible maps on X and

$$F_{fx,fy}(t) \geq \varphi\left[\min\left\{F_{gx,fx}(t), F_{gx,fy}(t), F_{gy,fx}(t), F_{gy,fy}(t)\right\}\right]$$

for all $x, y \in X$ and $t > 0$.

Then *f* and *g* have a unique common fixed point.

Proof: Since *f* and *g* are occasionally weakly compatible maps, there exists a point $u \in$ X such that

fu = *gu* and *fgu* = *gfu*.

We prove that *f* and *g* have a unique common fixed point.

$$F_{fu,ffu}(t) \geq \varphi[\min\{F_{gu,fu}(t), F_{gu,ffu}(t), F_{gfu,fu}(t), F_{gfu,ffu}(t)\}]$$
$$= \varphi[\min\left\{F_{fu,fu}(t), F_{fu,ffu}(t), F_{ffu,fu}(t), F_{ffu,ffu}(t)\right\}]$$
$$> F_{fu,ffu}(t) \quad if\ F_{fu,ffu}(t) < 1$$

which is a contradiction.

$\therefore F_{fu,ffu}(t) = 1$ for every $t > 0$

$$\therefore fu = ffu$$

$\therefore fu$ is a fixed point of f.

But $ffu = fgu = gfu$.

$\therefore fu$ is a common fixed point of *f* and *g*.

Uniqueness: Let *u* and *v* be two common fixed points of *f* and *g*.

i.e. *fu* = *gu* = *u* and *fv* = *gv* = *v*

Then $F_{u,v}(t) \geq \varphi\left[\min\left\{F_{u,v}(t), F_{u,v}(t), F_{v,u}(t), F_{v,v}(t)\right\}\right]$

$$> F_{u,v}(t) \quad if\ \ F_{u,v}(t) < 1$$

$\therefore$ $F_{u,v}(t) > F_{u,v}(t)$ if $F_{u,v}(t) < 1$, a contradiction.

$\therefore$ $F_{u,v}(t) = 1$ for every $t > 0$.

Hence $u = v$

$\therefore$ *f* and *g* have unique common fixed point.

The following Theorem is given in [1].

Theorem 2.3 ([1], Theorem 3.2): Let (X, F) be a probabilistic semi-metric space. If *f* and *g* are occasionally weakly compatible maps in X and

$$F_{fx,fy}(t) \geq F_{gx,gy}\left(\tfrac{t}{a}\right) + \text{Min}\left\{F_{fx,gx}\left(\tfrac{t}{b}\right), F_{fy,gy}\left(\tfrac{t}{b}\right)\right\} + \text{Min}\left\{F_{gx,gy}\left(\tfrac{t}{c}\right), F_{gx,fx}\left(\tfrac{t}{c}\right), F_{gy,fy}\left(\tfrac{t}{c}\right)\right\} \ldots (2)$$

for all $x,y \in X$ with $fx \neq fy$ and $t > 0$ where $0 < a < 1$, $0 < b < 1$

and $0 < c < 1$. Then f and g have a unique common fixed point.

In Theorem 2.3, If f and g are occasionally weakly compatible maps, then we show that condition (2) does not hold, and hence Theorem 2.3 becomes void.

Proof: Since f and g are occasionally weakly compatible maps, there exists a point u in X such that $fu = gu$, $fgu = gfu$.

For any $t > 0$, consider

$$F_{fu,ffu}(t) \geq F_{gu,gfu}(\tfrac{t}{a}) + \text{Min}\ \{F_{fu,gu}(\tfrac{t}{b}), F_{ffu,gfu}(\tfrac{t}{b})\} + \text{Min}\ \{F_{gu,gfu}(\tfrac{t}{c}), F_{gu,fu}(\tfrac{t}{c}), F_{gfu,ffu}(\tfrac{t}{c})\}$$

$$= F_{fu,ffu}(\tfrac{t}{a}) + \text{Min}\ \{F_{fu,fu}(\tfrac{t}{b}), F_{ffu,ffu}(\tfrac{t}{b})\} + \text{Min}\ \{F_{fu,ffu}(\tfrac{t}{c}), F_{fu,fu}(\tfrac{t}{c}), F_{ffu,ffu}(\tfrac{t}{c})\}$$

$$= F_{fu,ffu}(\tfrac{t}{a}) + \text{Min}\ \{1, 1\} + \text{Min}\ \{ F_{fu,ffu}(\tfrac{t}{c}), 1,1 \}$$

$$= F_{fu,ffu}(\tfrac{t}{a}) + 1 + F_{fu,ffu}(\tfrac{t}{c}) \geq 1$$

$\therefore 1 \geq F_{fu,ffu}(t) \geq F_{fu,ffu}(\tfrac{t}{a}) + 1 + F_{fu,ffu}(\tfrac{t}{c})$

$\Rightarrow 0 \geq F_{fu,ffu}(\tfrac{t}{a}) + F_{fu,ffu}(\tfrac{t}{c})$

$\Rightarrow F_{fu,ffu}(\tfrac{t}{a}) + F_{fu,ffu}(\tfrac{t}{c}) = 0$

$\Rightarrow F_{fu,ffu}(\tfrac{t}{a}) = 0$ and $F_{fu,ffu}(\tfrac{t}{c}) = 0$ for every $t > 0$

$\Rightarrow \frac{t}{a} = 0$ *and* $\frac{t}{c} = 0$, which is a contradiction.

Hence condition (2) does not hold.

Consequently Theorem 2.3 is void.

Now we give below, a modification of Theorem 2.3.

Theorem 2.4: Let (X, F) be a probabilistic semi-metric space. If f and g be self maps in X and

$$F_{fx,fy}(t) \geq F_{gx,gy}(\tfrac{t}{a}) + \text{Min}\ \{F_{fx,gx}(\tfrac{t}{b}), F_{fy,gy}(\tfrac{t}{b})\} + \text{Min}\ \{F_{gx,gy}(\tfrac{t}{c}), F_{gx,fx}(\tfrac{t}{c}), F_{gy,fy}(\tfrac{t}{c})\} \ldots (3)$$

for all $x,y \in X$ with $fx \neq fy$ and $t > 0$ where $0 < a < 1$, $0 < b < 1$ and $0 < c < 1$, then f and g have at most one coincidence point.

Proof: We prove that f and g have at most one point of coincidence,

Suppose $fx = gx = w$ for some $x \in$ X and $fy = gy = w'$ for some $y \in$ X

assume that $w \neq w'$. Then by (3)

$$F_{fx,fy}(t) \geq F_{gx,gy}(\tfrac{t}{a}) + \text{Min}\ \{F_{fx,gx}(\tfrac{t}{b}), F_{fy,gy}(\tfrac{t}{b})\} + \text{Min}\ \{F_{gx,gy}(\tfrac{t}{c}), F_{gx,fx}(\tfrac{t}{c}), F_{gy,fy}(\tfrac{t}{c})\}$$

$\Rightarrow F_{fx,fy}(t) \geq F_{fx,fy}(\tfrac{t}{a}) + \text{Min}\ \{1, 1\} + \text{Min}\ \{F_{gx,gy}(\tfrac{t}{c}), 1, 1\}$

$= F_{fx,fy}\left(\frac{t}{a}\right) + 1 + F_{fx,fy}\left(\frac{t}{c}\right)$
$\geq F_{fx,fy}(t) + 1 + F_{fx,fy}(t)$
($\because 0 < a < 1 \Rightarrow \frac{1}{a} > 1 \Rightarrow \frac{t}{a} > t$ and F is increasing)

Hence $0 \geq 1 + F_{fx,fy}(t) \geq 1$, a contradiction.
$\therefore fx = fy$ i.e. $w = w'$

Therefore *f* and *g* have at most one coincidence point.
However Theorem 2.3 is true when the two minima in condition (2) are dropped. In fact we have the following Theorem which is another modification of Theorem 2.3.

Theorem 2.5: Let (X, F) be a probabilistic semi metric space. Suppose *f* and *g* are occasionally weakly compatible maps and

$$F_{fx,fy}(t) \geq F_{gx,gy}\left(\frac{t}{a}\right) \quad \dots (4)$$

for all $x, y \in X$ with $fx \neq fy$ and $t > 0$ where $0 < a < 1$.
Then *f* and *g* have a unique common fixed point.

Proof: Since f and g are occasionally weakly compatible maps,
there exists u in X such that $fu = gu, fgu = gfu$.
Suppose x,y are common fixed points of *f* and *g*. Let $t > 0$.
Then $F_{x,y}(t) \geq F_{x,y}\left(\frac{t}{a}\right)$.
Since $t < \frac{t}{a}$, we have $F_{x,y}(t) \leq F_{x,y}\left(\frac{t}{a}\right)$ ……(5).
Hence $F_{x,y}(t) = F_{x,y}\left(\frac{t}{a}\right)$
Suppose $t < s < \frac{t}{a}$, then $F_{x,y}(t) \leq F_{x,y}(s) \leq F_{x,y}\left(\frac{t}{a}\right) = F_{x,y}(t)$
so that $F_{x,y}(t) = F_{x,y}(s)$
$\therefore F_{x,y}(t)$ is a constant function in $[\, t, \frac{t}{a}\,]$
Replacing t by $\frac{t}{a}$ in (5), we get

$$F_{x,y}\left(\frac{t}{a}\right) \geq F_{x,y}\left(\frac{t}{a^2}\right)$$

Since $\frac{t}{a} < \frac{t}{a^2}$, we have $F_{x,y}\left(\frac{t}{a}\right) \leq F_{x,y}\left(\frac{t}{a^2}\right)$
$\therefore F_{x,y}(t) = F_{x,y}\left(\frac{t}{a}\right) = F_{x,y}\left(\frac{t}{a^2}\right)$
$\therefore$ For any $p \in [\frac{t}{a}, \frac{t}{a^2}]$, we have
$F_{x,y}\left(\frac{t}{a}\right) \leq F_{x,y}(p) \leq F_{x,y}\left(\frac{t}{a^2}\right) = F_{x,y}\left(\frac{t}{a}\right)$
Hence $F_{x,y}\left(\frac{t}{a}\right) = F_{x,y}(p)$
$\therefore F_{x,y}(p)$ is a constant function in $[\frac{t}{a}, \frac{t}{a^2}]$
Continuing in this way follows that $F_{x,y}(p)$ is a constant function in $[\, t, \infty)$.
This being true for $t > 0$, it follows that $F_{x,y}(p)$ is a constant function in $(0, \infty)$.
Since $\sup_{t \in R} F_{x,y}(t) = 1$, now follows that $F_{x,y}(t) = 1$ for all $t > 0$
$\therefore F_{x,y}(t) = 1$ for all $t > 0$.

Consequently $x = y$.
Hence f and g have a unique fixed common fixed point.

References

1. **Harish Chandra and Arvind Bhatt:** Fixed point Theorems for occasionally weakly compatible maps in Probabilistic Semi-Metric space. Int. Journal of Math. Analysis, Vol. 3, 2009, no.12, 563-570.

Received: January, 2010