Int. Journal of Math. Analysis, Vol. 4, 2010, no. 25, 1217 - 1224

Characterizations for Strictly Singular and Strictly Discontinuous Operators on Locally Convex Spaces

C. Ganesa Moorthy

Department of Mathematics, Alagappa University, Karaikudi - 630 003, India
ganesamoorthyc@gmail.com

C.T. Ramasamy

Department of Mathematics, Alagappa University, Karaikudi - 630 003, India
ctrams83@gmail.com

Abstract

A characterization of strictly singular operators on a locally convex space is derived. It is proved by using this characterization that the collection of all strictly singular operators on a locally convex space form an ideal in the algebra of all continuous linear operators on a locally convex space. A new discontinuous operator called strictly discontinuous operator is introduced, and a characterization is obtained for this type of operators. This characterization is applied to prove that composition two strictly discontinuous operators is a strictly discontinuous operator.

Mathematics Subject Classification: 46B28, 46A32, 47L05, 47L60

Keywords: Strictly singular operators, precompact operators

1 Introduction

The concept of a strictly singular operator was introduced by Kato [3], in his treatment of perturbation theory. He defined strictly singular operators on Banach spaces. The concept of a strictly singular operator was extended to locally convex spaces by D. van Dulst [1]. D. van Dulst obtained a characterization of strictly singular operators on Ptak spaces. A characterization is obtained in the present article for strictly singular operators on a general locally convex space. This characterization enables us to prove that sum of two strictly singular operators on a locally convex space into a locally convex space is again a strictly singular operator. It follows from the characterization that composition of a continuous operator and a strictly singular operator on a locally convex space is a strictly singular operator. This is the first part of this article. The second part is concerned with discontinuous "inverses" of strictly singular operators. They are called in this article as strictly discontinuous operators. This is a new class of operators. Sum of two unbounded operators need to be unbounded. For example, consider the relation $T+(-T)=0$, when T is an unbounded operator. Similarly composition of two unbounded operators need not be an unbounded operator. For example, consider $T_1, T_2 : c_{00} \to c_{00}$ defined by $T_1(x_1, x_2, x_3, x_4, \cdots) = (x_1, \frac{x_2}{2^2}, 3x_3, \frac{x_4}{4^2}, \cdots)$; $T_2(x_1, x_2, x_3, x_4, \cdots) = (x_1, 2x_2, \frac{x_3}{3^2}, 4x_4, \cdots)$, where the space c_{00} of all finite scalar sequences is endowed with the supremum norm $\|.\|_\infty$. Although T_1 and T_2 are unbounded, $T_1 \circ T_2$ and $T_2 \circ T_1$ are bounded precompact operators where the range spaces are of infinite dimension. However, it is known that composition of two (unbounded) Fredholm operators is a Fredholm operator (see: [2, p. 103]). We shall show in the second part that composition of two strictly discontinuous operators is again a strictly discontinuous operator, by using a characterization. Continuity and boundedness of a linear operator between normed spaces are equivalent. However, it is different in the case of operators on locally convex spaces. For example, if τ is the weak topology on an infinite dimensional normed space $(X, \|.\|)$, then the identity operator from the locally convex space (X, τ) onto $(X, \|.\|)$ is a bounded operator but not a continuous operator. We rename strictly discontinuous operator into strictly unbounded operator, when they are defined on normed spaces.

The following result is due to van Dulst [1]

Theorem 1.1 *[1] Suppose $T : X \to Y$ is a continuous operator, where X and Y are locally convex spaces. The following are equivalent*

(i) T is precompact

(ii) There exists a $0-$neighbourhood U in X with the property that for every $0-$neighbourhood V in Y there is a linear subspace M of X of the form $M = \bigcap_{i=1}^{\infty} \mathcal{N}(x_i')$, with $x_i' \in X'$ and x_i' bounded on U $(i = 1, 2, \cdots, n)$ such that $T(U \cap M) \subset V$. Here $\mathcal{N}(x_i')$ is the null space of x_i'

(iii) There exists a $0-$neighbourhood U in X with the property that for every $0-$neighbourhood V in Y there is a linear subspace M having finite deficiency in X such that $T(U \cap M) \subset V$

(i)$\Leftrightarrow$ (ii) was proved by van Dulst [1]; and the continuity of T implies (i)$\Leftrightarrow$ (iii) as it was observed by Goldberg [2, Theorem III.2.3].

Let $(X, (p_i)_{i\in I})$, $(Y, (q_j)_{j\in J})$ and $(Z, (r_k)_{k\in K})$ be locally convex spaces, I, J and K be directed sets, and p_i, q_j and r_k be seminorms increasing with them (see: [4]). Let T be a linear operator having domain $\mathcal{D}(T)$, a subspace of X, and range $\mathcal{R}(T)$, a subspace of Y. To a given subspace M of X, a given $i \in I$ and $j \in J$, define

$$\gamma_{i,j,M}(T) = \gamma_{i,j}(T_M) = \sup\{q_j(Tx) : p_i(x) \leq 1,\ x \in M\}$$

where T_M is the restriction of T to M. We reserve the notation $\gamma_{i,j}$ or $\gamma_{i,j,M}$ even if domains, co-domains and operators are changed.

2 Characterization and Properties of Strictly singular operators

A continuous linear operator from a locally convex space into a locally convex space is said to be strictly singular, if its restriction to any infinite dimensional subspace is not a topological isomorphism.

In this section, we shall take $\mathcal{D}(T) = X$.

Theorem 2.1 *Suppose T is continuous. Then the following are equivalent.*

(i) T is strictly singular

(ii) Given $\epsilon > 0$, $j \in J$ and an infinite dimensional subspace M of X, there exists $i_0 \in I$ (which depends only on M), and there exists an infinite dimensional subspace N of M such that $\gamma_{i,j,N}(T) = \gamma_{i,j}(T_N) \leq \epsilon$, for all $i \geq i_0$.

(iii) Given $\epsilon > 0$, $j \in J$ and an infinite dimensional subspace M of X, there exists $i_0 \in I$ (which depends only on M), and there exists an infinite dimensional subspace N of M such that $\gamma_{i,j,N}(T) = \gamma_{i,j}(T_N) \leq \epsilon$, for all $i \geq i_0$ and T_N is precompact.

(iv) Given an infinite dimensional subspace M of X, there exists an infinite dimensional subspace N of M such that T_N is precompact.

Proof: (i) $\Rightarrow$ (iii): Suppose T is strictly singular. Fix $\epsilon > 0$ and $j \in J$. Let M be an infinite dimensional subspace of X. Then T_M is strictly singular. So there is an $i_0 \in I$ such that for any $k > 0$, $p_{i_o}(x) \leq k q_j(Tx)$ is not satisfied for all $x \in M$. Then we can find sequences $x_1, x_2, \cdots \in M$ and $x'_1, x'_2 \cdots \in X'$ such that

$$p_{i_0}(x_k) = \sup\{|x'_k(x)| : x \in X, p_{i_0}(x) \leq 1\} = x'_k(x_k) = 1 \qquad (2.1)$$

and $q_j(T_M x_k) \leq \epsilon/4^k$, for all k; and

$$x_k \in \bigcap_{i=1}^{k-1} \mathcal{N}(x'_i) \text{ or equivalently } x'_i(x_k) = 0 \text{ for } i < k \qquad (2.2)$$

Here $\mathcal{N}(x_i')$ is the null space of x_i'. Suppose $\alpha_1 x_1 + \cdots + \alpha_n x_n = 0$. Then $x_1'(\alpha_1 x_1 + \cdots + \alpha_n x_n) = 0$, and by (2.1) and (2.2), we have $\alpha_1 = 0$. Also $x_2'(\alpha_2 x_2 + \cdots + \alpha_n x_n) = 0$; so $\alpha_2 = 0$. Similarly we can show that $\alpha_3 = \alpha_4 = \cdots = \alpha_n = 0$. This proves that the set $\{x_k\}$ is linearly independent. Write $N = span\{x_1, x_2 \cdots\}$. Then N is an infinite dimensional subspace of M. Suppose $x = \alpha_1 x_1 + \cdots + \alpha_n x_n$. Then from (2.1) and (2.2), we have $|\alpha_1| = |x_1'(x)| \leq p_{i_o}(x)$. In fact

$$|\alpha_k| \leq 2^{k-1} p_{i_0}(x), \quad k = 1, 2, \cdots n \tag{2.3}$$

For suppose (2.3) is true for $k \leq t < n$. Then from (2.1) and (2.2), we have

$$x_{t+1}'(x) = \sum_{i=1}^{t} \alpha_i x_{t+1}'(x_i) + \alpha_{t+1}$$

This implies that,

$$|\alpha_{t+1}| \leq |x_{t+1}'(x)| + \sum_{i=1}^{t} |\alpha_j||x_{t+1}'(x_i)| \leq p_{i_o}(x) + \sum_{i=1}^{t} 2^{i-1} p_{i_0}(x) = 2^t p_{i_0}(x)$$

Thus (2.3) follows by induction.

Also

$$q_j(Tx) \leq \sum_{i=1}^{n} |\alpha_i| q_j(Tx_i) \leq \sum_{i=1}^{n} 2^{i-1} 4^{-i} \epsilon p_{i_0}(x) \leq \epsilon p_{i_o}(x)$$

Thus $\gamma_{i_0,j}(T_N) \leq \epsilon$, and hence $\gamma_{i,j}(T_N) \leq \epsilon$, for all $i \geq i_0$. This proves (ii). If $x = \alpha_m x_m + \cdots + \alpha_n x_n$, then the bound $q_j(Tx) \leq \sum\limits_{i=m}^{n} 2^{i-1} 4^{-i} \epsilon p_{i_0}(x)$ and Theorem 1.1 imply that T_N is precompact. This proves (iii).

(ii) $\Rightarrow$ (i): Suppose (ii) is true. If T is not strictly singular, then there is an infinite dimensional subspace M of X such that T_M is an topological isomorphism. We find i_0 for M as in (ii). Then there is a $j \in J$ and there is a $k > 0$ such that $p_{i_0}(x) \leq k q_j(Tx)$, for all $x \in M$. This contradicts the conclusion part of (ii) with $\epsilon = 1/2k$. Hence T is strictly singular. Other implications are trivial.

Proposition 2.2 *Suppose $T_1 : X \to Y$ and $T_2 : X \to Y$ are strictly singular. Then $T = (T_1 + T_2) : X \to Y$ and $\alpha T_1 : X \to Y$ are strictly singular, for any scalar α.*

Proof: Let $\epsilon > 0$, $j \in J$, and an infinite dimensional subspace M be given. By theorem 2.1, we can find an $i_1 \in I$ and infinite dimensional subspace N_1 of M such that $\gamma_{i,j}(T_{1N_1}) \leq \epsilon/2$, for all $i \geq i_1$. For this N_1, there is an $i_2 \in I$ and an infinite dimensional subspace N of N_1 such that $\gamma_{i,j}(T_{2N}) \leq \epsilon/2$, for all $i \geq i_2$. Choose i_0 such that $i_0 \geq i_1$ and $i_0 \geq i_2$. Then we have $\gamma_{i,j}(T_N) \leq \epsilon$, for all $i \geq i_0$. This proves that T is strictly singular.

Proposition 2.3 *Suppose $T : X \to Y$ is strictly singular and $S : Z \to X$ is continuous. Then $T \circ S$ is strictly singular.*

Proof: Without loss of generality, let us assume that nr_l is included in $(r_k)_{k\in K}$ for any $r_l \in (r_k)_{k\in K}$ and for any positive integer n. Fix $\epsilon > 0$, $j \in J$ and an infinite dimensional subspace M of Z. As T is strictly singular, we can find $i_0 \in I$ and an infinite dimensional subspace U of $S(M)$ such that $\gamma_{i,j}(T_U) \leq \epsilon$, for all $i \geq i_0$, provided $\dim S(M) = \infty$. For this i_0, we can find k_0 such that $p_{i_0}(Sz) \leq r_k(z)$ for all $z \in Z$, and for all $k \geq k_0$ and hence $\gamma_{k,j}((T \circ S)_N) \leq \epsilon$, for all $k \geq k_0$, where $N = S^{-1}(U)$. If $\dim S(M) < \infty$, we take $N = M \cap \mathcal{N}(S)$, where $\mathcal{N}(S)$ is the null space of S, so that $(T \circ S)_N$ is zero. This completes the proof.

Proposition 2.4 *Suppose $T : X \to Y$ is strictly singular and $S : Y \to Z$ is continuous. Then $S \circ T$ is strictly singular.*

Proof: Fix $\epsilon > 0$, $k \in K$ and an infinite dimensional subspace M of X. As S is continuous, there are a $j_0 \in J$ and a positive number d such that $r_k(Sy) \leq dq_j(y)$, for all $y \in Y, j \geq j_0$. As T is strictly singular, for this $j_0 \in J$ we can find $i_0 \in I$ and an infinite dimensional subspace N of M such that $\gamma_{i,j_0}(T_N) \leq \epsilon/d$, for all $i \geq i_0$. Thus we have $\gamma_{i,k}((S \circ T)_N) \leq \epsilon$, for all $i \geq i_0$. This completes the proof.

3 Definition and Characterization of strictly discontinuous operators

Definition 3.1 *A linear operator $T : \mathcal{D}(T) \to Y$ with $\dim \mathcal{R}(T) = \infty$ is said to be a Strictly discontinuous operator if T is not continuous on any infinite dimensional subspace M contained in its domain $\mathcal{D}(T)$for which $\dim T(M) = \infty$.*

From the definition it follows that, if T is a strictly discontinuous operator on $\mathcal{D}(T)$ and $T_1 : X \to Y$ is a continuous linear operator then $T + T_1$ is strictly discontinuous on $\mathcal{D}(T) \subset X$.

Theorem 3.2 *Suppose T is a linear operator with $\dim \mathcal{R}(T) = \infty$. Then the following are equivalent.*

(i) T is strictly discontinuous

(ii) Given $\epsilon > 0, i \in I$ and an infinite dimensional subspace M of $\mathcal{D}(T)$, there is a $j_0 \in J$ (which depends only on M), and there is an infinite dimensional subspace N of M such that $p_i(x) \leq \epsilon q_j(Tx)$, for all $j \geq j_0$, $x \in N$

(iii) Given an infinite dimensional subspace M of $\mathcal{D}(T)$ satisfying $\dim T(M) = \infty$, there is a infinite dimensional subspace N of M such that T is 1-1 on N and $T^{-1} : T(N) \to N$ is precompact.

(iv) Given $\epsilon > 0, i \in I$ and an infinite dimensional subspace M of $\mathcal{D}(T)$, there is a $j_0 \in J$ (which depends only on M), and there is an infinite dimensional subspace N of M such that T is 1-1 on N, and $p_i(x) \leq \epsilon q_j(Tx)$, for all $j \geq j_0$, $x \in N$.

(v) Given $\epsilon > 0, i \in I$ and an infinite dimensional subspace M of $\mathcal{D}(T)$, there is a $j_0 \in J$ (which depends only on M), and there is an infinite dimensional subspace N of M such that T is 1-1 on N, $p_i(x) \leq \epsilon q_j(Tx)$, for all $j \geq j_0$, $x \in N$ and $T^{-1} : T(N) \to N$ is precompact.

Proof: Suppose T is a strictly discontinuous operator. Fix $\epsilon > 0$ and $i \in I$. Let M be an infinite dimensional subspace of X. Then T_M is not continuous. So there is a $j_0 \in J$ such that for any $k > 0$, $p_i(x) \geq kq_{j_o}(Tx)$ is not satisfied for all $x \in M$. Then we can find sequences $x_1, x_2, \cdots \in M$ and $y'_1, y'_2 \cdots \in Y'$ such that

$$q_{j_0}(Tx_k) = \sup\{|y'_k(y)| : y \in Y, q_{j_0}(y) \leq 1\} = y'_k(Tx_k) = 1 \tag{3.1}$$

and $p_i(x_k) \leq \epsilon/4^k$, for all k; and

$$x_k \in \bigcap_{i=1}^{k-1} \mathcal{N}(y'_i \circ T) \text{ or equivalently } y'_i(Tx_k) = 0 \text{ for } i < k \tag{3.2}$$

Suppose $\alpha_1 Tx_1 + \cdots + \alpha_n Tx_n = 0$. Then $y'_1(\alpha_1 Tx_1 + \cdots + \alpha_n Tx_n) = 0$, and by (3.1) and (3.2), we have $\alpha_1 = 0$. Also $y'_2(\alpha_2 Tx_2 + \cdots + \alpha_n Tx_n) = 0$; so $\alpha_2 = 0$. Similarly we can show that $\alpha_3 = \alpha_4 = \cdots = \alpha_n = 0$. This proves that the set $\{Tx_1, Tx_2, \cdots\}$ is linearly independent. Hence $\{x_1, x_2, \cdots\}$ is linearly independent. Write $N = span\{x_1, x_2, \cdots\}$. Then N is an infinite dimensional subspace of M such that T is 1-1 on N. Suppose $x = \alpha_1 x_1 + \cdots + \alpha_m x_m$. Then from (3.1) and (3.2), we have $|\alpha_1| = |y'_1(Tx)| \leq q_{j_o}(Tx)$. In fact

$$|\alpha_k| \leq 2^{k-1} q_{j_0}(Tx), \quad k = 1, 2, \cdots m \tag{3.3}$$

For suppose (3.3) is true for $k \leq t < m$. Then from (3.1) and (3.2), we have

$$y'_{t+1}(Tx) = \sum_{k=1}^{t} \alpha_i y'_{t+1}(Tx_k) + \alpha_{t+1}$$

This implies that

$$|\alpha_{t+1}| \leq |y'_{t+1}(Tx)| + \sum_{k=1}^{t} |\alpha_t||y'_{t+1}(Tx_i)| \leq q_{j_o}(Tx) + \sum_{k=1}^{t} 2^{k-1} q_{j_0}(Tx) = 2^t q_{j_0}(Tx)$$

Thus (3.3) follows by induction.

Also

$$p_i(x) \leq \sum_{k=1}^{m} |\alpha_k| p_i(x_k) \leq \sum_{k=1}^{m} 2^{k-1} 4^{-k} \epsilon q_{j_0}(Tx) \leq \epsilon q_{j_o}(Tx)$$

Thus $p_i(x) \leq \epsilon q_j(Tx)$, for all $j \geq j_0$ and $x \in N$. If $x = \alpha_m x_m + \cdots + \alpha_n x_n$, then the bound $p_i(x) \leq \sum_{k=m}^{n} 2^{k-1} 4^{-k} \epsilon q_{j_0}(Tx)$ and Theorem 1.1 imply that $T^{-1} : T(N) \to N$ is precompact. This proves (v). Other implications are trivial.

Corollary 3.3 *Suppose $T : \mathcal{D}(T) \to Y$ and $S : \mathcal{D}(S) \to Z$ are strictly discontinuous operators with $\dim \mathcal{R}(T) = \infty$ where $\mathcal{D}(S) \subset Y$ and $\mathcal{D}(T) \subset X$. Then $S \circ T$ is a strictly discontinuous operator.*

Proof: Let M be an infinite dimensional subspace of $\mathcal{D}(S \circ T)$ with $\dim(S \circ T)(M) = \infty$. Then there is an infinite dimensional subspace N_Y of $T(M)$ such that S is 1-1 on N_Y and $S^{-1} : S(N_Y) \to N_Y$ is precompact. Note that $T^{-1}(N_Y)$ is also an infinite dimensional subspace of M, and so there is an infinite dimensional subspace N_X of $T^{-1}(N_Y)$ such that T is 1-1 on N_X and $T^{-1} : T(N_X) \to N_X$ is precompact. Hence $(S \circ T)^{-1} : (S \circ T)(N_X) \to N_X$ is precompact. This proves $S \circ T$ is a strictly discontinuous operator.

The concept of strictly discontinuous operators is new even for normed spaces. We rename the strictly discontinuous operators as strictly unbounded operators when they are defined on normed spaces, and we record the following corollaries.

Corollary 3.4 *Suppose $T : \mathcal{D}(T) \to Y$ is a linear operator with $\dim \mathcal{R}(T) = \infty$, where X and Y are normed spaces, and $\mathcal{D}(T) \subset X$. Then the following four statements are equivalent.*

(i) T is a strictly unbounded operator.

(ii) For every infinite dimensional subspace $M \subset \mathcal{D}(T)$ satisfying $\dim T(M) = \infty$ and for every $\epsilon > 0$, there is an infinite dimensional subspace $N \subset M$ such that $\|x\| \leq \epsilon \|Tx\|$ for all $x \in N$.

(iii) For every infinite dimensional subspace $M \subset \mathcal{D}(T)$ satisfying $\dim T(M) = \infty$ and for every $\epsilon > 0$, there is an infinite dimensional subspace $N \subset M$ such that T is 1-1 on N and $\|x\| \leq \epsilon \|Tx\|$ for all $x \in N$.

(iv) Given an infinite dimensional subspace M of $\mathcal{D}(T)$ satisfying $\dim T(M) = \infty$, there is an infinite dimensional subspace N of M such that T is 1-1 on N and $T^{-1} : T(N) \to N$ is precompact.

(v) *For every infinite dimensional subspace* $M \subset \mathcal{D}(T)$ *satisfying* $\dim T(M) = \infty$ *and for every* $\epsilon > 0$, *there is an infinite dimensional subspace* $N \subset M$ *such that* T *is 1-1 on* N; $\|x\| \leq \epsilon \|Tx\|$ *for all* $x \in N$ *and* $T^{-1} : T(N) \to N$ *is precompact.*

Corollary 3.5 *Suppose* $T : \mathcal{D}(T) \to Y$ *and* $S : \mathcal{D}(S) \to Z$ *are strictly unbounded operators with* $\dim \mathcal{R}(T_2 \circ T_1) = \infty$, *where* $\mathcal{D}(T) \subset X$ *and* $\mathcal{D}(S) \subset Y$ *and when* X, Y *and* Z *are normed spaces. Then* $S \circ T$ *is also a strictly unbounded operator*

The identity operator mentioned before theorem 1.1 is not strictly discontinuous even though it is bounded.

Example 3.6 *Let* $T : \mathcal{D}(T) \to Y$ *be a 1-1 strictly singular operator when* X *and* Y *are normed spaces. Then* $T^{-1} : T(X) \to X$ *is a strictly unbounded operator.*

Example 3.7 *The differential operator* $T : \mathcal{D}(T) \to Y$ *defined by* $Tf = f'$ *is a strictly unbounded operator, where* $X = Y = C[0,1]$ *and* $\mathcal{D}(T) = C^1[0,1]$ *are endowed with the supremum norm.*

The operator T given in [2, Ex. II.2.7] is unbounded to the extent that there is no non-zero $y' \in Y'$ such that $y' \circ T$ is continuous. This operator T is not a strictly unbounded operator. Thus strictly unbounded operators are more unbounded than ordinary unbounded operators.

Acknowledgement: The second author like to thank DAE, India, for their financial support (in the form of fellowship)

References

[1] D. van Dulst, *Perturabtion theory and strictly singular operators in locally convex spaces*, Studia math. 38 (1970), 341 - 372.

[2] S. Goldberg, *Unbounded linear operators, theory and applications*, McGraw-Hill, 1966.

[3] T. Kato, *Perturbations theory for nullity, deficiency and other quantities of linear operators*, Journ. d'Anal. Math. 6 (1958), 273 - 322.

[4] W. Rudin, *Functional Analysis*, second ed., McGraw-Hill Inc.., New York, 1991.

Received: November, 2009