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# Strong Convergence Theorems According to a New Iterative Scheme with Errors for Mapping Nonself I-Asymptotically Quasi-Nonexpansive Types

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**Abstract**

In this paper, a new iterative scheme with errors for mapping nonself I-asymptotically quasi-nonexpansive types and nonself I-asymptotically quasi-nonexpansive types in Banach space is defined. The results obtained in this paper extend and improve upon those recently announced by S.S.Yao and L.Wang [ Yao & Wang, (2008) : 'Strong Convergence Theorems for Nonself I-Asymptotically Quasi-Nonexpansive Mappings,' published in Applied Mathematical Sciences 19(2008 : 919-928) ] among others.

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## 1. Introduction

Let X be a real Banach space and let C be a nonempty subset of X, $P : X \to C$ be a nonexpansive retraction of X onto C. A nonself mapping $T : C \to X$ is called asymptotically nonexpansive [1] if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that for each $n \in \mathbb{N}$,

$$\| T(PT)^{n-1}x - T(PT)^{n-1}y \| \le k_n \| x - y \|, \text{ for all } x, y \in C.$$

T is said to be uniformly L- Lipschitzian if there exists a constant $L > 0$ such that

$$\| T(PT)^{n-1}x - T(PT)^{n-1}y \| \le L \| x - y \|, \text{ for all } x, y \in C.$$

$T : C \to X$ is completely continuous [2] if for all bounded sequences $\{x_n\} \subset C$ there exists a convergent subsequence of $\{Tx_n\}$.

Recalling that a Banach space $X$ is called uniformly convex [5] if, for every $0 < \varepsilon \le 2$, there exists $\delta = \delta(\varepsilon) > 0$ such that $\left\| \frac{x+y}{2} \right\| \le 1 - \delta$ for every x, $y \in S_X$ and $\| x - y \| \ge \varepsilon$, $S_X = \{x \in X : \|x\| = 1\}$.

Let $T, I : C \to C$, then T is called I-nonexpansive on C [4] if $\| Tx - Ty \| \le \| Ix - Iy \|$ for all $x, y \in C$. T is called I-asymptotically nonexpansive on C if there exists a sequence $\{\lambda_k\} \subset [0, \infty)$ with $\lambda_k \to 0$ as $k \to \infty$ such that

$$\| T^k x - T^k y \| \le (\lambda_k + 1)\| I^k x - I^k y \|, \text{ for all } x, y \in C \text{ and } k = 1, 2, 3, \ldots$$

T is called I-asymptotically quasi-nonexpansive on C [7] if there exists a sequence $\{\lambda_k\} \subset [0, \infty)$ with $\lambda_k \to 0$ as $k \to \infty$ such that

$$\| T^k x - f \| \le (\lambda_k + 1)\| I^k x - f \|, \text{ for all } x \in C \text{ and } k \ge 0 \text{ and for every}$$

$f \in F(T) \cap F(I)$, where $\phi \ne F(T) \cap F(I)$ be the set of all common fixed points of T and I.

Let $T, I : C \to X$, then T is called nonself I-asymptotically quasi-nonexpansive [6] if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that for each $n \in \mathbb{N}$,

$$\| T(PT)^{n-1}x - f \| \le k_n\|I(PI)^{n-1}x - f\|, \text{ for all } x \in C \text{ and } f \in F(T) \cap F(I),$$

where P is a retraction from X onto C.

In 2004, N.Shahzad [4] introduced the concept of I-nonexpansive mapping in Banach space. In 2007, S.Temir and O.Gul [7] defined I-asymptotically quasi-nonexpansive mapping and studied the weak convergence theorems for I-asymptotically quasi-nonexpansive mapping in Hilbert space.

More recently, S.S.Yao and L.Wang [6] defined nonself I-asymptotically quasi-nonexpansive mapping and proved some strong convergence theorems for such mapping in uniformly convex Banach spaces.

The purpose of this paper is to introduce the concept of nonself I-asymptotically quasi-nonexpansive types and strong convergence theorems, and define a new iterative scheme with errors which modified the iterative scheme of S.S.Yao and L.Wang [6].

## 2. Preliminaries

Let $C$ be a nonempty subset of a Banach space X. A subset $C$ is called retract of $X$ if there exists continuous mapping $P : X \to C$ such that $Px = x$ for all $x \in C$. It is well known that every closed convex subset of a uniformly convex Banach space is a retract. A mapping $P : X \to C$ is called a retraction if $P^2 = P$. Note that if a mapping P is a retraction, then $Pz = z$ for all z in the range of P.

Let X be a real Banach space and let C be a nonempty closed convex subset of X. Let $P : X \to C$ be a retraction of X onto C and let $I : C \to X$ be a nonself mapping and $T : C \to X$ be a nonself I-asymptotically quasi-nonexpansive type as defined by definition 3.1.

**Algorithm 1.** For a given $x_1 \in C$, we compute the sequence $\{x_n\}$ by the iterative scheme

$$\begin{aligned} x_{n+1} &= P(a_n I(PI)^{n-1} y_n + (1 - a_n - b_n)x_n + b_n u_n) \\ y_n &= P(c_n T(PT)^{n-1} x_n + (1 - c_n - d_n)x_n + d_n v_n), \quad n \geq 1, \end{aligned} \tag{2.1}$$

where $\{u_n\}$, $\{v_n\}$ are bounded sequences in C and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$, $\{a_n + b_n\}$ and $\{c_n + d_n\}$ are appropriate sequences in [0, 1].

If $b_n \equiv 0$ and $d_n \equiv 0$, then (2.1) is reduced to the iterative scheme defined by S.S.Yao and L.Wang [6], as follows:

**Algorithm 2.** For a given $x_1 \in C$, we compute the sequence $\{x_n\}$ by the iterative scheme

$$\begin{aligned} x_{n+1} &= P(a_n I(PI)^{n-1} y_n + (1 - a_n)x_n) \\ y_n &= P(c_n T(PT)^{n-1} x_n + (1 - c_n)x_n), \quad n \geq 1, \end{aligned} \tag{2.2}$$

where $\{a_n\}$, $\{c_n\}$ are appropriate sequences in [0, 1].

Recalling a well-known concept, and the following essential lemmas, in order to prove our main results:

**Lemma 2.1** [3]. Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \quad \forall\, n = 1, 2, \ldots$$

It $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_n a_n$ exists.

**Lemma 2.2** [2]. Let X be a real uniformly convex Banach space and $0 \leq p \leq t_n \leq q < 1$ for all positive integers $n \geq 1$. Also suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of X such that $\limsup_n \|x_n\| \leq r$, $\limsup_n \|y_n\| \leq r$ and $\limsup_n \|t_n x_n + (1 - t_n)y_n\| = r$ hold for some $r \geq 0$, then $\lim_n \|x_n - y_n\| = 0$.

## 3. Main Results

In this section, we provide proof of a convergence theorem for a new iterative scheme with errors for a mapping of nonself I-asymptotically quasi-nonexpansive types. In providing such proof of our main results, the following definition and lemmas are required:

**Definition 3.1.** Let C be a nonempty closed convex subset of real Banach space X. $T : C \to X$ be a nonself I-asymptotically quasi-nonexpansive mapping, $I : C \to X$ be a nonself mapping, $\phi \neq F(T) \cap F(I)$ be the set of all common fixed points of T and I.

T is called a nonself I-asymptotically quasi-nonexpansive type mapping if T is uniformly continuous, and

$$\limsup_{n} \{ \sup_{x \in C} ( \| T(PT)^{n-1} x - q \| - \| I(PI)^{n-1} x - q \| ) \} \leq 0,$$

for all $q \in F(T) \cap F(I)$, where P is a retraction from X onto C.

**Lemma 3.1.** Let C be a nonempty subset of a real Banach space X. Let $T : C \to X$ be a nonself I-asymptotically quasi-nonexpansive type, $I : C \to X$ be a nonself asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty)$, $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, P be a retraction from X onto C. Put

$$G_n = \max \{0, \sup_{x \in C} ( \| T(PT)^{n-1} x - q \| - \| I(PI)^{n-1} x - q \| ) \},$$

$\forall n \geq 1$, $\forall q \in F(T) \cap F(I)$, so that $\sum_{n=1}^{\infty} G_n < \infty$.

Suppose that sequence $\{x_n\}$ is generated by (2.1) with $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} d_n < \infty$.

If $F(T) \cap F(I) \neq \phi$, then $\lim_{n} \| x_n - q \|$ exists for any $q \in F(T) \cap F(I)$.

**Proof.** Setting $k_n = 1 + r_n$. Since $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, so $\sum_{n=1}^{\infty} r_n < \infty$.

Let $q \in F(T) \cap F(I)$, and

$$M_1 = \sup\{ \| u_n - q \| : n \geq 1\},$$

$$M_2 = \sup\{ \| v_n - q \| : n \geq 1\}.$$

Using (2.1) with $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} d_n < \infty$, we have

$$\begin{aligned}
\| x_{n+1} - q \| &= \| P(a_n I(PI)^{n-1} y_n + (1 - a_n - b_n) x_n + b_n u_n) - q \| \\
&\leq a_n \| I(PI)^{n-1} y_n - q \| + (1 - a_n - b_n) \| x_n - q \| + b_n \| u_n - q \| \\
&\leq a_n k_n \| y_n - q \| + (1 - a_n - b_n) \| x_n - q \| + b_n \| u_n - q \| \\
&= a_n k_n \| P(c_n T(PT)^{n-1} x_n + (1 - c_n - d_n) x_n + d_n v_n) - q \| + (1 - a_n - b_n) \| x_n - q \| \\
&\quad + b_n \| u_n - q \| \\
&\leq a_n k_n \| c_n (T(PT)^{n-1} x_n - q) + (1 - c_n - d_n)(x_n - q) + d_n (v_n - q) \| \\
&\quad + (1 - a_n - b_n) \| x_n - q \| + b_n M_1 \\
&\leq a_n k_n c_n (\| T(PT)^{n-1} x_n - q \| + a_n k_n (1 - c_n - d_n) \| x_n - q \| + a_n k_n d_n \| v_n - q \| \\
&\quad + (1 - a_n - b_n) \| x_n - q \| + b_n M_1 \\
&= a_n k_n c_n (\| T(PT)^{n-1} x_n - q \| - \| I(PI)^{n-1} x_n - q \| + a_n k_n c_n \| I(PI)^{n-1} x_n - q \| \\
&\quad + a_n k_n (1 - c_n - d_n) \| x_n - q \| + a_n k_n d_n \| v_n - q \| + (1 - a_n - b_n) \| x_n - q \| + b_n M_1
\end{aligned}$$

$$\le a_nk_nc_n \sup_{x\in C}\{\| T(PT)^{n-1}x - q \| - \| I(PI)^{n-1}x - q \|\} + a_nk_nk_nc_n\| x_n - q \|$$
$$+ a_nk_n(1 - c_n - d_n)\| x_n - q \| + a_nk_nd_nM_2 + (1 - a_n - b_n)\| x_n - q \| + b_nM_1$$
$$\le a_nk_nc_nG_n + [a_nc_n k_n^2 + a_nk_n(1 - c_n - d_n) + (1 - a_n - b_n)]\|x_n - q\| + a_nk_nd_nM_2 + b_nM_1$$
$$= [a_nc_n(1 + r_n)^2 + a_n(1 + r_n) - a_nc_n(1 + r_n) - a_nd_n(1 + r_n) + 1 - a_n - b_n]\|x_n - q\| + a_nc_n(1 + r_n)G_n + a_nd_n(1 + r_n)M_2 + b_nM_1$$
$$= [a_nc_nr_n + a_nc_n r_n^2 + a_nr_n - a_nd_n - a_nd_nr_n + 1 - b_n]\| x_n - q \| + a_nc_n(1 + r_n)G_n + a_nd_n(1 + r_n)M_2 + b_nM_1$$
$$\le [1 + r_n + r_n + r_n^2]\| x_n - q \| + (1 + r_n)G_n + (1 + r_n)d_nM_2 + b_nM_1$$
$$= [1 + (2r_n + r_n^2)]\| x_n - q \| + s_n, \tag{3.1}$$

where $s_n = (1 + r_n)G_n + (1 + r_n)d_nM_2 + b_nM_1$.

Since $\sum_{n=1}^{\infty} r_n < \infty$, $\sum_{n=1}^{\infty} G_n < \infty$, $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} d_n < \infty$, we see that $\sum_{n=1}^{\infty}(2r_n + r_n^2) < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$.

If follows from Lemma 2.1 that $\lim_n \| x_n - q \|$ exists. This completes the proof. #

**Lemma 3.2.** Let $X$ be a uniformly convex Banach space. Let $C$, $T$, $I$ and $\{x_n\}$ be same as in Lemma 3.1. Put

$$G_n = \max \{0, \sup_{x\in C}(\| T(PT)^{n-1}x - q \| - \| I(PI)^{n-1}x - q \|)\}, \ \forall n \ge 1,$$

$\forall q \in F(T) \cap F(I)$, so that $\sum_{n=1}^{\infty} G_n < \infty$.

If $T$ is uniformly $L$-Lipschitzian for some $L > 0$ and $F(T) \cap F(I) \ne \phi$, then $\lim_n \| Tx_n - x_n \| = \lim_n \| Ix_n - x_n \| = 0$.

**Proof.** By Lemma 3.1, for any $q \in F(T) \cap F(I)$, $\lim_n \| x_n - q \|$ exists, then $\{x_n\}$ is bounded. Assume $\lim_n \| x_n - q \| = t \ge 0$. Let

$$M_1 = \sup\{ \| u_n - q \| : n \ge 1 \} \text{ and}$$
$$M_2 = \sup\{ \| v_n - q \| : n \ge 1 \}.$$

Using (2.1) with $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} d_n < \infty$, we have

$$\| y_n - q \| = \| P(c_nT(PT)^{n-1}x_n + (1 - c_n - d_n)x_n + d_nv_n) - q \|$$
$$\le c_n \| T(PT)^{n-1}x_n - q \| + (1 - c_n - d_n)\| x_n - q \| + d_n\| v_n - q \|$$
$$= c_n(\|T(PT)^{n-1}x_n - q\| - \| I(PI)^{n-1}x_n - q \|) + c_n\| I(PI)^{n-1}x_n - q \| + (1 - c_n - d_n)\| x_n - q \| + d_nM_2$$
$$\le c_n \sup_{x\in C}\{\| T(PT)^{n-1}x - q \| - \| I(PI)^{n-1}x - q \|) + c_nk_n\| x_n - q \|$$

$$
\begin{aligned}
&+ (1 - c_n - d_n)\| x_n - q \| + d_n M_2 \\
\leq\ & c_n G_n + c_n(1 + r_n)\| x_n - q \| + (1 - c_n - d_n)\| x_n - q \| + d_n M_2 \\
=\ & c_n G_n + (c_n r_n + 1 - d_n)\| x_n - q \| + d_n M_2 \\
\leq\ & (1 + r_n)\| x_n - q \| + G_n + d_n M_2 \\
=\ & (1 + r_n)\| x_n - q \| + e_n, \qquad (3.2)
\end{aligned}
$$

where $e_n = G_n + d_n M_2$. Since $\sum_{n=1}^{\infty} G_n < \infty$ and $\sum_{n=1}^{\infty} d_n < \infty$, so that $\sum_{n=1}^{\infty} e_n < \infty$.

Taking $\limsup_n$ on both sides in above inequality, we obtain

$$\limsup_n \| y_n - q \| \leq t. \qquad (3.3)$$

Since $\| I(PI)^{n-1} y_n - q \| \leq (1 + r_n)\| y_n - q \|$.

Taking $\limsup_n$ on both sides in above inequality and using (3.3), we have

$$\limsup_n \| I(PI)^{n-1} y_n - q \| \leq t.$$

Since $\lim_n \| x_{n+1} - q \| = t$, then

$$
\begin{aligned}
t &= \lim_n \| P(a_n I(PI)^{n-1} y_n + (1 - a_n - b_n) x_n + b_n u_n) - q \| \\
&= \lim_n \| a_n (I(PI)^{n-1} y_n - q) + (1 - a_n - b_n)(x_n - q) + b_n (u_n - q) \| \\
&\leq \lim_n [\| a_n (I(PI)^{n-1} y_n - q) + (1 - a_n)(x_n - q) \|] - \lim_n b_n \| x_n - q \| + \lim_n b_n M_1.
\end{aligned}
$$

Since $\sum_{n=1}^{\infty} b_n < \infty$, so that $\lim_n b_n \| x_n - q \| = 0$ and $\lim_n b_n M_1 = 0$, we have

$$t \leq \lim_n [\| a_n (I(PI)^{n-1} y_n - q) + (1 - a_n)(x_n - q) \|].$$

Similarly, for proof (3.1), we have

$$
\begin{aligned}
&\lim_n \| a_n (I(PI)^{n-1} y_n - q) + (1 - a_n)(x_n - q) \| \\
&\quad \leq \lim_n \| x_n - q \| + \lim_n (2r + r_n^2) \| x_n - q \| + \lim_n (1 + r_n) G_n + \lim_n (1 + r_n) d_n M_2.
\end{aligned}
$$

Since $\sum_{n=1}^{\infty} r_n < \infty$, $\sum_{n=1}^{\infty} G_n < \infty$ and $\sum_{n=1}^{\infty} d_n < \infty$, we see that $\lim_n (2r + r_n^2) \| x_n - q \| = 0$, $\lim_n (1 + r_n) G_n = 0$ and $\lim_n (1 + r_n) d_n M_2 = 0$. We have

$$t \leq \lim_n \| a_n (I(PI)^{n-1} y_n - q) + (1 - a_n)(x_n - q) \| \leq \lim_n \| x_n - q \| = t$$

so that $\lim_n \| a_n (I(PI)^{n-1} y_n - q) + (1 - a_n)(x_n - q) \| = t$.

It follows from Lemma 2.2 that,

$$\lim_n \| I(PI)^{n-1} y_n - x_n \| = 0. \qquad (3.4)$$

Next,
$$
\begin{aligned}
\| x_n - q \| &\leq \| x_n - I(PI)^{n-1} y_n \| + \| I(PI)^{n-1} y_n - q \| \\
&\leq \| x_n - I(PI)^{n-1} y_n \| + (1 + r_n) \| y_n - q \|
\end{aligned}
$$

gives that $t = \lim_n \| x_n - q \| \leq \liminf_n \| y_n - q \|$. By (3.3), we have

$$\lim_n \| y_n - q \| = t.$$

So that $t = \lim_n \| P(c_nT(PT)^{n-1}x_n + (1 - c_n - d_n)x_n + d_nv_n) - q \|$

$$\leq \lim_n (\| c_n(T(PT)^{n-1}x_n - q) + (1 - c_n)(x_n - q) \| - \lim_n d_n\| x_n - q \| + \lim_n d_n\| v_n - q \|$$

$$\leq \lim_n (\| c_n(T(PT)^{n-1}x_n - q) + (1 - c_n)(x_n - q) \|$$

Similarly, for proof (3.2), we have

$$t \leq \lim_n (\| c_n(T(PT)^{n-1}x_n - q) + (1 - c_n)(x_n - q) \| \leq \lim_n \| x_n - q \| = t.$$

So that $\lim_n (\| c_n(T(PT)^{n-1}x_n - q) + (1 - c_n)(x_n - q) \| = t.$ (3.5)

Next, $\|T(PT)^{n-1}x_n - q\| = \|T(PT)^{n-1}x_n - q\| - \|I(PI)^{n-1}x_n - q\| + \| I(PI)^{n-1}x_n - q \|$

$$\leq \sup_{x\in C} (\|T(PT)^{n-1}x - q\| - \|I(PI)^{n-1}x - q\|) + \|I(PI)^{n-1}x_n - q\|$$

$$\leq G_n + (1 + r_n)\| x_n - q \|$$

$$= G_n + \| x_n - q \| + r_n\| x_n - q \|.$$

Since $\sum_{n=1}^{\infty} G_n < \infty$, $\sum_{n=1}^{\infty} r_n < \infty$ and $\lim \sup_n \| x_n - q \| \leq t$,

we have $\lim \sup_n \| T(PT)^{n-1}x_n - q \| \leq t.$ (3.6)

By (3.5), (3.6), $\lim \sup_n \| x_n - q \| \leq t$ and Lemma 2.2, we have

$$\lim_n \| T(PT)^{n-1}x_n - x_n \| = 0. \tag{3.7}$$

Also, $\| I(PI)^{n-1}x_n - x_n \| \leq \| I(PI)^{n-1}x_n - I(PI)^{n-1}y_n \| + \| I(PI)^{n-1}y_n - x_n \|$

$$\leq (1 + r_n)\| x_n - y_n \| + \| I(PI)^{n-1}y_n - x_n \|$$

$$= (1 + r_n)\|c_n(x_n - T(PT)^{n-1}x_n)\| + d_n(\|x_n - q\| + \|q - v_n\|) + \| I(PI)^{n-1}y_n - x_n \|$$

$$\leq c_n(1 + r_n)\| x_n - T(PT)^{n-1}x_n \| + d_n(\| x_n - q \| + M_2) + \| I(PI)^{n-1}y_n - x_n \|.$$

Thus by (3.4), (3.7) and $\sum_{n=1}^{\infty} d_n < \infty$, we have

$$\lim_n \| I(PI)^{n-1}x_n - x_n \| = 0. \tag{3.8}$$

Since $\| I(PI)^{n-1}x_n - T(PT)^{n-1}x_n \| = \| I(PI)^{n-1}x_n - x_n + x_n - T(PT)^{n-1}x_n \|$

$$\leq \| I(PI)^{n-1}x_n - x_n \| + \| x_n - T(PT)^{n-1}x_n \|.$$

It follows from (3.7) and (3.8) that

$$\lim_n \| I(PI)^{n-1}x_n - T(PT)^{n-1}x_n \| = 0. \tag{3.9}$$

In addition, $\| x_{n+1} - x_n \| \leq a_n \| I(PI)^{n-1}y_n - x_n \| + b_n\| u_n - x_n \|$

$$\leq a_n \| I(PI)^{n-1}y_n - x_n \| + b_n (\| u_n - q \| + \| q - x_n \|)$$

$$\leq a_n \| I(PI)^{n-1}y_n - x_n \| + b_n (M_1 + \| q - x_n \|)$$

Thus by (3.4) and $\sum_{n=1}^{\infty} b_n < \infty$, we have

$$\lim_{n} \| x_{n+1} - x_n \| = 0. \tag{3.10}$$

Since $\| I(PI)^{n-1}y_n - x_{n+1} \| \le \| I(PI)^{n-1}y_n - x_n \| + \| x_n - x_{n+1} \|$,
by (3.4) and (3.10), we have

$$\lim_{n} \| I(PI)^{n-1}y_n - x_{n+1} \| = 0. \tag{3.11}$$

So,
$$\begin{aligned}\| x_{n+1} - y_n \| &\le \| x_{n+1} - I(PI)^{n-1}y_n \| + \| I(PI)^{n-1}y_n - y_n \| \\ &\le \| x_{n+1} - I(PI)^{n-1}y_n \| + \| I(PI)^{n-1}y_n - x_n \| + \| x_n - y_n \| \\ &= \| x_{n+1} - I(PI)^{n-1}y_n \| + \| I(PI)^{n-1}y_n - x_n \| + \| y_n - x_n \| \\ &= \| x_{n+1} - I(PI)^{n-1}y_n \| + \| I(PI)^{n-1}y_n - x_n \| \\ &\quad + \| P(c_nT(PT)^{n-1}x_n + (1 - c_n - d_n)x_n + d_nv_n) - x_n \| \\ &\le \| x_{n+1} - I(PI)^{n-1}y_n \| + \| I(PI)^{n-1}y_n - x_n \| \\ &\quad + c_n\| T(PT)^{n-1}x_n - x_n \| + d_n\| v_n - x_n \|.\end{aligned}$$

Using (3.4), (3.7), (3.11) and $\sum_{n=1}^{\infty} d_n < \infty$, we have

$$\lim_{n} \| x_{n+1} - y_n \| = 0. \tag{3.12}$$

$$\begin{aligned}\| Ix_n - x_n \| &= \| Ix_n - I(PI)^{n-1}y_{n-1} + I(PI)^{n-1}y_{n-1} - I(PI)^{n-1}x_n + I(PI)^{n-1}x_n - x_n \| \\ &\le \|Ix_n - I(PI)^{n-1}y_{n-1}\| + \|I(PI)^{n-1}y_{n-1} - I(PI)^{n-1}x_n\| + \|I(PI)^{n-1}x_n - x_n\| \\ &\le (1 + r_1)\|x_n - I(PI)^{n-2}y_{n-1}\| + (1 + r_n)\|y_{n-1} - x_n\| + \|I(PI)^{n-1}x_n - x_n\|.\end{aligned}$$

It follows from (3.8), (3.11) and (3.12), we obtain

$$\lim_{n} \| Ix_n - x_n \| = 0. \tag{3.13}$$

Since $T$ is uniformly $L$-Lipschitzian for some $L > 0$,

$$\begin{aligned}\| Tx_n - x_n \| &= \| Tx_n - T(PT)^{n-1}x_n + T(PT)^{n-1}x_n - x_n \| \\ &\le \| Tx_n - T(PT)^{n-1}x_n \| + \| T(PT)^{n-1}x_n - x_n \| \\ &\le L\| x_n - T(PT)^{n-2}x_n \| + \| T(PT)^{n-1}x_n - x_n \| \\ &\le L[\, \| x_n - x_{n-1} \| + \| x_{n-1} - T(PT)^{n-2}x_{n-1} \| \\ &\quad + \| T(PT)^{n-2}x_{n-1} - T(PT)^{n-2}x_n \| \,] + \| T(PT)^{n-1}x_n - x_n \| \\ &\le L\|x_{n-1} - T(PT)^{n-2}x_{n-1}\| + (L^2 + L)\|x_n - x_{n-1}\| + \|T(PT)^{n-1}x_n - x_n\|.\end{aligned}$$

Using (3.7) and (3.10), we have

$$\lim_{n} \| Tx_n - x_n \| = 0. \tag{3.14}$$

This completes the proof. #

**Theorem 3.3.** Let $X, C, T, I$ and $\{x_n\}$ be same as in Lemma 3.2. Put

$$G_n = \max \{0, \sup_{x \in C}(\| T(PT)^{n-1}x - q \| - \| I(PI)^{n-1}x - q \|)\}, \ \forall\, n \ge 1,$$

$\forall\, q \in F(T) \cap F(I)$, so that $\sum_{n=1}^{\infty} G_n < \infty$.

If $I$ is completely continuous and $F(T) \cap F(I) \neq \phi$, then $\{x_n\}$ is converged strongly to a common fixed point of $T$ and $I$.

**Proof.** From Lemma 3.1, we know that $\lim_{n} \| x_n - q \|$ exists for any $q \in F(T) \cap F(I)$, then $\{x_n\}$ is bounded. By Lemma 3.2, we have

$$\lim_{n} \| Tx_n - x_n \| = 0 \text{ and } \lim_{n} \| Ix_n - x_n \| = 0. \quad (3.15)$$

Suppose that $I$ is completely continuous, and noting that $\{x_n\}$ is bounded: We conclude that subsequence $\{Ix_{n_j}\}$ of $\{Ix_n\}$ exists, such that $\{Ix_{n_j}\}$ converges. Therefore, from (3.15), $\{x_{n_j}\}$ is converged. Let $x_{n_j} \to r$ as $j \to \infty$. From the continuity of $P, T, I$ and (3.15), we have $r = Tr = Ir$. Thus, $\{x_n\}$ is converged strongly to a common fixed point $r$ of $T$ and $I$. This completes the proof. #

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