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Second Order F-Pseudolinearity in Multiobjective Nonlinear Programming

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Abstract

A new class of functions namely, second order F-pseudolinear functions which is weaker than second order F-convex functions, is introduced. Sufficient optimality conditions for properly efficiency and mixed type duality theorems for multiobjective nonlinear programming problems are established under the assumptions of second order F-pseudolinearity and second order F-quasiconvexity.

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1 Introduction

A second order dual for a non linear programming problem was introduced by Mangasarian [8] and established duality results for nonlinear programming problems. Mond [9] proved duality theorems under the assumption of second order convexity. The study of the second duality is significant due to the computational advantage over first order duality as it provides tighter bounds for the value of the objective functions when approximations are used. Under various generalized convexity conditions, many researchers [1,9,10,11,14,16,17] have obtained second order optimality conditions and duality theorems for nonlinear programming problems. Zhang and Mond [17] obtained various duality results under the assumption of second order (F, ρ)-convexity as an extension of (F, ρ)-convexity. Aghezzaf [1] introduced a second order mixed type duality for vector programming problems and established various duality results under the assumptions of second order (F, ρ)-convex and its related functions.

Chew and Choo [4] introduced pseudolinear functions and obtained optimality conditions for a feasible point to be an efficient solution for multiobjective programming problems involving pseudolinear functions. Suneja and Shila Devi Gupta [13] introduced η-pseudolinear functions which is the generalization of pseudolinear functions and obtained optimality conditions and duality results for multiobjective programming problems. Pandian [12] introduced F-pseudolinear functions, as a generalization of pseudolinear functions and η-pseudolinear functions and obtained optimality conditions and various duality results for multiobjective programming problems.

In this paper, we introduce a new class of functions namely, second order F-pseudolinear functions which is weaker then second order F-convex functions and prove some of its properties. Further, we obtain second order sufficient optimality conditions for a feasible point to be a properly efficient solution of a multiobjective programming problem and second order mixed type duality results for multiobjective programming problems under the assumptions of second order F-pseudolinearity and second order F-quasiconvexity.

2 Preliminaries

Throughout this paper, the conventions for vectors in R^n will be followed the notation of Mangasarian [7]. Let X be an open convex subset of R^n and R_+ denote the set of all positive real numbers and $e = (1,1,...,1) \in R^k$. Let F be a sublinear function defined by $F: XxXxR^n \to R^n$ and $p(x,u)$, $\psi(x,u)$, $q(x,u)$, $r(x,u)$, $\alpha(x,u)$ and $\beta(x,u)$ be vector valued functions defined from XxX to R^n. Let us assume that $h: X \to R$, $f: X \to R^k$ and $g: R^n \to R^m$ where $f = (f_1,..., f_k)$ and $g = (g_1,..., g_m)$ are twice differentiable function on X, $\lambda = (\lambda_1, \lambda_2,..., \lambda_k) \in R^k$ and $\mu = (\mu_1, \mu_2,..., \mu_m) \in R^m$.

We need the following definitions which can be found in [2,6,10,11].

Definition 1: A function $F: XxXxR^n \to R$ is said to be sublinear on R^n if for each $x, u \in X$,

(i) $F(x,u;a+b) \le F(x,u;a) + F(x,u;b)$, for all $a,b \in R^n$ and

(ii) $F(x,u;\alpha a) = \alpha F(x,u;a)$, for all $\alpha \ge 0$ in R and $a \in R^n$

Note 1: From (ii), it follows that $F(x,u;0) = 0$.

Definition 2: The function h is said to be

(i) second order F-convex with respect to vector functions $p(x,u)$, $q(x,u)$ and

$r(x,u)$ at $u \in X$ if for all $x \in X$,

$$h(x) - h(u) + \frac{1}{2} q(x,u)\nabla^2 h(u) r(x,u) \geq F(x,u;\nabla h(u) + \nabla^2 h(u) p(x,u)).$$

(ii) second order F-pseudoconvex with respect to vector functions $p(x,u)$, $q(x,u)$ and $r(x,u)$ at $u \in X$ if for all $x \in X$,

$$F(x,u;\nabla h(u) + \nabla^2 h(u) p(x,u)) \geq 0 \Rightarrow h(x) - h(u) + \frac{1}{2} q(x,u)\nabla^2 h(u) r(x,u) \geq 0$$

and

(iii) second order F-quasiconvex with respect to vector functions $p(x,u)$, $q(x,u)$ and $r(x,u)$ at $u \in X$ if for all $x \in X$,

$$h(x) \leq h(u) - \frac{1}{2} q(x,u)\nabla^2 h(u) r(x,u) \Rightarrow F(x,u;\nabla h(u) + \nabla^2 h(u) p(x,u)) \leq 0.$$

Consider the following multiobjective nonlinear programming problem

(VP) Minimize $f(x) = (f_1(x), f_2(x), \ldots, f_k(x))$
subject to $g(x) \leqq 0$, $x \in X$.

We need the following definitions and the results which can be found in [5,12].

Definition 3: A feasible point $x^\circ \in X$ is said to be

(i) an efficient solution for (VP) if there exist no other point $x \in P$ such that $f(x) \leq f(x^\circ)$.

(ii) a properly efficient solution for (VP) if it is an efficient and if there exists a scalar $M > 0$ such that, for each i, for each feasible point x for which $f_r(x) < f_r(x^\circ)$, we have

$$f_i(x^\circ) - f_i(x) \leq M(f_r(x) - f_r(x^\circ))$$

for some r such that $f_r(x) > f_r(x^\circ)$.

Lemma 1: Let $\lambda^\circ > 0$ in R^k be fixed with $\lambda^{\circ t} e = 1$. If x° is an optimal solution of (VP_{λ°) where (VP_{λ°) Minimize $\lambda^{\circ t} f(x)$

subject to $g(x) \leqq 0$, $x \in X$,

then x° is a properly efficient solution for (VP).

We need the following necessary optimality conditions for proving strong duality theorem which can be found in Chankong and Haimes [3] and Pandian [12].

Theorem 1: (Necessary optimality conditions). Assume that x° is an efficient solution for (VP) and a constraint qualification is satisfied at x° for each $(VP_r(x^\circ)), r = 1,2,\ldots,k$ where

$(VP_r(x^\circ))$ Minimize $f_r(x)$

subject to $f_i(x) \leq f_i(x^\circ)$, for all $i \neq r$

$g(x) \leqq 0$, $x \in X$.

Then, there exist scalars $\lambda^\circ > 0$ in R^k with $\lambda^{\circ t} e = 1$ and $\mu^\circ \geqq 0$ in R^m such that $(x^\circ, \lambda^\circ, \mu^\circ)$ satisfies

$$\nabla(\lambda^{\circ t} f(x^\circ) + \mu^{\circ t} g(x^\circ)) = 0 \tag{1}$$

$$\mu_j^\circ g_j(x^\circ) = 0, \ j = 1,2,\dots,m. \tag{2}$$

3 Second order F-pseudolinear functions

We, now define a new class of functions namely, second order F-pseudolinear functions which is weaker than second order F-convex functions as follows.

Let us assume that *q(x,u)* and *r(x,u)* be vector valued functions defined from XxX to R^n and be partially continuous at $x = u$ with $q(u,u) \neq 0$ and $r(u,u) \neq 0$.

Definition 4:. The function h is said to be second order F-pseudolinear at $u \in X$ with respect to vector functions $p(x,u)$, $q(x,u)$ and $r(x,u)$ if both h and $-h$ are second order F-pseudoconvex with respect to the same vector functions $p(x,u)$, $q(x,u)$ and $r(x,u)$ at $u \in X$.

The function h is said to be second order F-pseudolinear on X with respect to vector functions $p(x,u)$, $q(x,u)$ and $r(x,u)$ if h is second order F-pseudolinear at each point $u \in X$ with respect to the vector functions $p(x,u)$, $q(x,u)$ and $r(x,u)$.

Remark 1: Every second order F-convex function with respect to $p(x,u), q(x,u)$ *and* $r(x,u)$ is second order F-pseudolinear with respect to the same $p(x,u), q(x,u)$ *and* $r(x,u)$, but the converse is not true. This is demonstrated by the following example

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Example 1: Let $X = (0,\infty)$. Define the functions $h : X \to R$, $F : XxXxR \to R$ and $p(x,u), q(x,u)$ and $r(x,u) : XxX \to R$ as follows:

$$h(x) = \frac{3}{x} - \frac{x}{3}, \quad F(x,u;a) = \left(h(x) - h(u) + \frac{3}{u^3} \right) |a| , \quad p(x,u) = \frac{u^3}{6}\left(\frac{1}{3} - \frac{3}{x^2} \right),$$

$$q(x,u) = 2((x+u)^2 + 1) \quad \text{and} \quad r(x,u) = \frac{u^2}{3((x+u)^2 + 1)}$$

where x and $u \in R$.

The function h is second order F-pseudolinear with respect to $p(x,u)$, $q(x,u)$ and $r(x,u)$ at each $u \in X$. But the function h is not second order F-convex with respect to $p(x,u)$, $q(x,u)$ and $r(x,u)$ at $u = 1$ because for $x = 2$ and $u = 1$,

$$h(x) - h(u) + \frac{1}{2} q(x,u)\nabla^2 h(u) r(x,u) < F(x,u;\nabla h(u) + \nabla^2 h(u) p(x,u)) .$$

We need the following proposition which can be used in the propositions of second order F-pseudolinear functions.

Proposition 1 : Every second order pseudo F-convex with respect to $p(x,u)$, $q(x,u)$ and $r(x,u)$ is a second order F-quasiconvex function with respect to the same $p(x,u)$, $q(x,u)$ and $r(x,u)$. .

Proof: Suppose that h is second order F-pseudoconvex with respect to $p(x,u)$, $q(x,u)$ and $r(x,u)$ at u and $h(x) \le h(u) - \frac{1}{2} q(x,u)\nabla^2 h(u) r(x,u)$, for all $x \in X$.

Assume that $F(x,u;\nabla h(u) + \nabla^2 h(u) p(x,u)) > 0$ for some $x \in X$.

By second order F-pseudoconvexity of h , we have

$$h(x) \ge h(u) - \frac{1}{2} q(x,u)\nabla^2 h(u) r(x,u).$$

Thus, $h(x) = h(u) - \frac{1}{2} q(x,u)\nabla^2 h(u) r(x,u).$

Since h is continuous at u and $q(x,u)$ and $r(x,u)$ are continuous at u and strictly positive, we have $\nabla^2 h(u) = 0$ and $\nabla h(u) = 0$.

By sublinearity of F, we can conclude that $F(x,u;\nabla h(u) + \nabla^2 h(u) p(x,u)) = 0$ which contradicts the assumptions.

Therefore, $F(x,u;\nabla h(u) + \nabla^2 h(u) p(x,u)) \le 0$, for all x.

Thus, h is second order F-quasiconvex with respect to $p(x,u)$, $q(x,u)$ and $r(x,u)$. Hence the result.

We, now derive some properties of second order F-pseudolinear functions.

Proposition 2 : Let h be second order F-pseudolinear with respect to $p(x,u)$, $q(x,u)$ and $r(x,u)$. Then, for all $x,u \in \mathrm{X}$,

$$h(x)-h(u)+\frac{1}{2}q(x,u)\nabla^2 h(u)r(x,u)=0 \;\Leftrightarrow\; \mathrm{F}(x,u;\nabla h(u)+\nabla^2 h(u)p(x,u))=0\,.$$

Proof: Suppose that $h(x)=h(u)-\frac{1}{2}q(x,u)\nabla^2 h(u)r(x,u)$.

By the second order F-pseudolinearity of h and the Proposition 1., we can conclude that $F(x,u;\nabla h(u)+\nabla^2 h(u)p(x,u))=0$.

Suppose that $F(x,u;\nabla h(u)+\nabla^2 h(u)p(x,u))=0$.

By the sublinearity of F and the second order F-pseudolinearity of h, we can conclude that

$$h(x)=h(u)-\frac{1}{2}q(x,u)\nabla^2 h(u)r(x,u)\,.$$

Hence the proposition.

Proposition 3: Let h be a twice differentiable function on X. The following statements are all equivalent to one another.

(I) h is second order F-pseudolinear on X with respect to $p(x,u)$, $q(x,u)$ and $r(x,u)$.

(II) There exist functions $\phi : XxX \to R_+/\{0\}$ and p,q *and* $r : XxX \to R^n$ such that for all $x,\, u \in \mathrm{X}$,

$$h(x)-h(u)+\frac{1}{2}q(x,u)\nabla^2 h(u)r(x,u) \;=\; \phi(x,u)\,F(x,u;\nabla h(u)+\nabla^2 h(u)p(x,u))\,.$$

Proof: To prove (I)$\Rightarrow$(II)

Case (i): Suppose that $F(x,u;\nabla h(u)+\nabla^2 h(u)p(x,u))=0$.

We define $\phi(x,u)=1$, for all $x,\, u\in \mathrm{X}$.

From the hypothesis (I) and by the Proposition 2., the hypothesis (II) follows.

Case (ii): Suppose that $F(x,u;\nabla h(u)+\nabla^2 h(u)p(x,u))\neq 0$.

We define

$$\phi(x,u)=\frac{h(x)-h(u)+\frac{1}{2}q(x,u)\nabla^2 h(u)r(x,u)}{F(x,u;\nabla h(u)+\nabla^2 h(u)p(x,u))}$$

From the hypothesis (I) and by the Proposition 2., we have $\phi(x,u)>0$, for all x and u.

Thus, there exist a function $\phi : XxX \to R_+/\{0\}$ such that

$$h(x)-h(u)+\frac{1}{2}q(x,u)\nabla^2 h(u)r(x,u) \;=\; \phi(x,u)\;F(x,u;\nabla h(u)+\nabla^2 h(u)p(x,u))$$

To prove (II)$\Rightarrow$(I)

Since $\phi(x,u)>0$ and by the hypothesis (II) and the sublinearity of F, we can easily show that h is second order F-pseudolinear on X with respect to $p(x,u)$, $q(x,u)$ and $r(x,u)$ on X.

Hence the proposition.

Note 2: From the Proposition 3., we can say that h is second order F-pseudolinear on X with respect to $p(x,u)$, $q(x,u)$ and $r(x,u)$ if there exists a function $\phi : XxX \to R_+/\{0\}$ such that

$$h(x) - h(u) + \frac{1}{2}q(x,u)\nabla^2 h(u)r(x,u) = \phi(x,u)\, F(x,u;\nabla h(u) + \nabla^2 h(u)\, p(x,u))$$

In otherwords, we say that h is second order F-pseudolinear on X with respect to $\phi(x,u)$, $p(x,u)$, $q(x,u)$ and $r(x,u)$.

Remark 2: From the definition we can conclude that every second order F-pseudolinear function with respect to $\phi(x,u)$, $p(x,u)$, $q(x,u)$ and $r(x,u)$ is not second order F-convex function with respect to $p(x,u)$, $q(x,u)$ and $r(x,u)$, but is second order $\overline{F}$ -convex function with respect to $p(x,u)$, $q(x,u)$ and $r(x,u)$ where $\overline{F}(x,u;z) = \phi(x,u)F(x,u;z)$.

Remark 3: From the definition we can conclude that if the function h is F-pseudolinear function [12] at $u \in X$ with respect to $\phi(x,u)$ with $\nabla^2 h(u) = 0$, is second order F-pseudolinear function with respect to the same $\phi(x,u)$ any functions $p(x,u)$, $q(x,u)$ and $r(x,u)$.

Now, we prove the sum of the non-negative scalar multiplication of second order F-pseudolinear functions is also second order F-pseudolinear function.

Theorem 2: Let f be a function from $X \to R^n$ and σ be a non-negative vector in R^n. If each component of f , f_i, i =1,2,…,k is second order F-pseudolinear on X with respect to $\phi(x,u)$, $p(x,u)$, $q(x,u)$ and $r(x,u)$ on X , then $\sigma^t f$ is second order F-pseudolinear on X with respect to the same $\phi(x,u)$, $p(x,u)$, $q(x,u)$ and $r(x,u)$.

Proof: Since each f_i , i = 1,2,…, k is second order F-pseudolinear with respect to $\phi(x,u)$, $p(x,u)$, $q(x,u)$ and $r(x,u)$ and $\sigma_i \geq 0$, for all i, i =1,2,…,k and by the sublinearity of F, it follows that

$$\sigma^t f(x) - \sigma^t f(u) + \frac{1}{2}q(x,u)\nabla^2(\sigma^t f(u))r(x,u) =$$

$$\phi(x,u)\, F(x,u;\nabla(\sigma^t f(u)) + \nabla^2(\sigma^t f(u))p(x,u))$$

Thus, $\sigma^t f$ is second order F-pseudolinear with respect to the same $\phi(x,u)$, $p(x,u)$, $q(x,u)$ and $r(x,u)$.
Hence the theorem.

4 Sufficient optimality conditions

We, now prove the sufficient optimality conditions for a feasible solution of (VP) to be a properly efficient solution of (VP) under second order F-pseudolinearity and second order F-quasiconvexity assumptions on the functions involved.

Theorem 3: (**Sufficient Optimality Conditions**). Let x° be feasible for (VP) and there exist scalars $\lambda^\circ > 0$ in R^k with $\lambda^{\circ t} e = 1$, $\mu_j^\circ \geqq 0$, $j \in I(x^\circ) = \{j : g_j(x^\circ) = 0\}$ and a vector function $p : XxX \to R^n$ such that

$$\nabla(\lambda^{\circ t} f(x^\circ) + \sum_{j \in I(x^\circ)} \mu_j^\circ g_j(x^\circ)) + \nabla^2(\lambda^{\circ t} f(x^\circ) + \sum_{j \in I(x^\circ)} \mu_j^\circ g_j(x^\circ)) p(x, x^\circ) = 0. \quad (3)$$

Assume that each f_i, $i = 1,2,\ldots,k$ is second order F-pseudolinear at x° with respect to $\phi_i(x, x^\circ)$, $p(x,x^\circ)$, $q(x,x^\circ)$ and $r(x,x^\circ)$ with $q(x, x^\circ)\nabla^2 f_i(x^\circ) r(x, x^\circ) \geq 0$ and $\sum_{j \in I(x^\circ)} \mu_j^\circ g_j$ is second order F-quasiconvex at x° with respect to vector functions $p(x,x^\circ)$, $\alpha(x,x^\circ)$ and $\beta(x,x^\circ)$ with $\alpha(x, x^\circ) \sum_{j \in I(x^\circ)} \mu_j^\circ \nabla^2 g_j(x^\circ) \beta(x, x^\circ) \geq 0$.

Then, x° is a properly efficient solution for the problem (VP).

Proof: Let x be feasible for (VP)

Since $\mu_j^\circ \geqq 0$, $j \in I(x^\circ)$ and $\sum_{j \in I(x^\circ)} \mu_j^\circ g_j$ is second order F-quasiconvex at x° and also, by the sublinearity of F, we have

$$F(x, x^\circ; \sum_{j \in I(x^\circ)} \mu_j^\circ \nabla g_j(x^\circ) + \sum_{j \in I(x^\circ)} \mu_j^\circ \nabla^2 g_j(x^\circ) p(x, x^\circ)) \leq 0 \quad (4)$$

Suppose that x° is not an efficient solution for (VP)

Then, there exist a feasible x for (VP) such that $f(x) \leq f(x^\circ)$.

Since each f_i, $i = 1,2,\ldots,k$ is second order F-pseudolinear function with respect to with respect to $\phi_i(x, x^\circ)$, $p(x,x^\circ)$, $q(x,x^\circ)$ and $r(x,x^\circ)$ and $q(x, x^\circ)\nabla^2 f_i(x^\circ) r(x, x^\circ) \geq 0$ and $\phi_i(x, x^\circ) > 0$, $i = 1,2,\ldots,k$, it follows that ,

$$F(x, x^\circ; \nabla f_i(x^\circ) + \nabla^2 f_i(x^\circ) p(x, x^\circ)) \leq 0 \text{ , for all i. and}$$

$$F(x, x^\circ; \nabla f_i(x^\circ) + \nabla^2 f_i(x^\circ) p(x, x^\circ)) < 0 \text{, for some r, } 1 \leq r \leq k.$$

Since $\lambda^\circ > 0$ in R^k with $\lambda^{\circ t} e = 1$ and by the sublinearity of F, we have

$$F(x, x^\circ; \lambda^{\circ t} \nabla f(x^\circ) + \nabla^2 \lambda^{\circ t} f(x^\circ) p(x, x^\circ)) < 0.$$

Now, from (3) and by the sublinearity of F, we have

$$F(x, x^\circ; \sum_{j \in I(x^\circ)} \mu_j^\circ \nabla g_j(x^\circ) + \sum_{j \in I(x^\circ)} \mu_j^\circ \nabla^2 g_j(x^\circ) p(x, x^\circ)) > 0.$$

which contradicts (4). Therefore, x° is an efficient solution for (VP).

Suppose that x° is not a properly efficient solution for (VP)
Then, for every M > 0 , there exists a point x of (VP) and an index i such that

$$f_i(x^{\circ}) - f_i(x) > M[f_r(x) - f_r(x^{\circ})]$$

for r satisfying $f_r(x) - f_r(x^{\circ}) > 0$ whenever $f_i(x^{\circ}) - f_i(x) > 0$.

This means that $f_i(x^{\circ}) - f_i(x)$ can be made arbitrarily large . Since each f_i, i =1,2,…,k is second order F-pseudolinear function at x° with $q(x,x^{\circ})\nabla^2 f_i(x^{\circ})r(x,x^{\circ}) \geq 0$ and $\phi_i(x,x^{\circ}) > 0$, i = 1,2,…,k. it follows that $-F(x,x^{\circ};\nabla f_i(x^{\circ}) + \nabla^2 f_i(x^{\circ})p(x,x^{\circ}))$ can be made arbitrarily large and hence for $\lambda_i > 0$, i = 1,2,...,k , we have

$$F(x,x^{\circ};\lambda^{\circ t}\nabla f(x^{\circ}) + \nabla^2\lambda^{\circ t} f(x^{\circ})p(x,x^{\circ})) < 0\,.$$

Now, from (3) and by the sublinearity of F, we have

$$F(x,x^{\circ};\sum_{j\in I(x^{\circ})}\mu_j^{\circ}\nabla g_j(x^{\circ}) + \sum_{j\in I(x^{\circ})}\mu_j^{\circ}\nabla^2 g_j(x^{\circ})p(x,x^{\circ})) > 0\,.$$

which contradicts (4). Therefore, x° is a properly efficient solution for (P).
Hence the theorem.

5 Mixed type duality theorems

Let J_1 be the subset of M = $\{1,2,\cdots,m\}$ and $J_2 = M / J_1$. We consider the following second order mixed type dual (VMD) for the problem (VP).

(VMD) Maximize $f(u) + \mu_{J_1} g_{J_1}(u) - \frac{1}{2}q(x,u)\nabla^2(f(u) + \mu_{J_1} g_{J_1}(u))r(x,u)$

subject to

$$\nabla[\lambda^t f(u) + \mu^t g(u)] + \nabla^2[\lambda^t f(u) + \mu^t g(u)]p(x,u) = 0 \tag{5}$$

$$\mu_{J_2} g_{J_2}(u) - \frac{1}{2}\alpha(x,u)\nabla^2\mu_{J_2} g_{J_2}\beta(x,u) \geq 0$$

$$\lambda > 0\,,\quad \mu \geqq 0\,,\ \lambda^t e = 1$$

where $u \in X$, $\mu \in R^m$, $\mu_{J_1} g_{J_1}(u) = \sum_{j\in J_1}\mu_j g_j(u)$ and $\mu_{J_2} g_{J_2}(u) = \sum_{j\in J_2}\mu_j g_j(u)$.

Note 3: We get the second order Mond-Weir dual [17] for $J_1 = \Phi$ and the second order Mangasarian dual [9] for $J_2 = \Phi$ in (VMD) respectively.

We, now prove the various duality results between (VP) and (VMD) under the assumptions of second order F-pseudolinearity and second order F-quasiconvexity..

Theorem 4: (**Weak Duality**). Let x be feasible for (VP) and $(u,\lambda,\mu,p,q,r,\alpha,\beta)$ be feasible for (VMD). Assume that each i =1,2,…,k, $f_i + \mu_{J_1} g_{J_1}$ is second order

F-pseudolinear at u with respect to $\phi(x,u)$, $p(x,u)$, $q(x,u)$ and $r(x,u)$ with $q(x,u)\nabla^2(\lambda^t f(u)+\mu_{J_1} g_{J_1}(u))r(x,u) \geq 0$ and $\mu_{J_2} g_{J_2}$ is second order F-quasiconvex at u with respect to vector functions $p(x,u)$, $\alpha(x,u)$ and $\beta(x,u)$.
Then, $\lambda^t f(x) \geq \lambda^t f(u)+\mu_{J_1} g_{J_1}(u)-\frac{1}{2}q(x,u)\nabla^2(\lambda^t f(u)+\mu_{J_1} g_{J_1}(u))r(x,u)$

Proof: Since each i =1,2,…,k, $f_i+\mu_{J_1} g_{J_1}$ is second order F-pseudolinear at u with respect to $\phi(x,u)$, $p(x,u)$, $q(x,u)$ and $r(x,u)$ and $\lambda > 0$ with $\lambda^t e = 1$ and by Theorem 2, $\lambda^t f+\mu_{J_1} g_{J_1}$ is second order F-pseudolinear at u with respect to $\phi(x,u)$, $p(x,u)$, $q(x,u)$ and $r(x,u)$.
Now, since x is feasible for (VP) and $(u,\lambda,\mu,p,q,r,\alpha,\beta)$ is feasible for (VMD) and $\mu_{J_2} g_{J_2}$ is second order F-quasiconvex at u with respect to vector functions $p(x,u)$, $\alpha(x,u)$ and $\beta(x,u)$ and the sublinearity of *F*, we have

$$F(x,u;\nabla(\mu_{J_2} g_{J_2}(u))+\nabla^2(\mu_{J_2} g_{J_2}(u))p(x,u)) \leq 0$$

From (5) and by the sublinearity of F, it follows that

$$F(x,u;\nabla(\lambda^t f(u)+\mu_{J_1} g_{J_1}(u))+\nabla^2(\lambda^t f(u)+\mu_{J_1} g_{J_1}(u))p(x,u)) \geq 0.$$

Since $\lambda^t f+\mu_{J_1} g_{J_1}$ is second order F-pseudolinear at u with respect to $\phi(x,u)$, $p(x,u)$, $q(x,u)$ and $r(x,u)$ and $q(x,u)\nabla^2(\lambda^t f(u)+\mu_{J_1} g_{J_1}(u))r(x,u) \geq 0$, it follows that

$$\lambda^t f(x) \geq \lambda^t f(u)+\mu_{J_1} g_{J_1}(u)-\frac{1}{2}q(x,u)\nabla^2(\lambda^t f(u)+\mu_{J_1} g_{J_1}(u))r(x,u) \quad .$$

Hence the theorem.

Theorem 5: Let the conditions of the weak duality theorem 4. be satisfied for all feasible solutions $(x,u,\lambda,\mu,p,q,r,\alpha,\beta)$. If x° is feasible for (VP) and $(u^\circ,\lambda^\circ,\mu^\circ,p,q,r,\alpha,\beta)$) is feasible for (VMD) such that

$$\lambda^{\circ t} f(x^\circ) = \lambda^{\circ t} f(u^\circ) - \frac{1}{2}q(x^\circ,u^\circ)\nabla^2\lambda^{\circ t} f(u^\circ)r(x^\circ,u^\circ), \qquad (6)$$

then x° and $(u^\circ,\lambda^\circ,\mu^\circ,p,q,r,\alpha,\beta)$ are properly efficient solutions for (VP) and (VMD) respectively.

Proof : By the weak duality theorem 4. and from (6) and also, by the Lemma 1, we can conclude that x° and $(u^\circ,\lambda^\circ,\mu^\circ,p,q,r,\alpha,\beta)$ are properly efficient solutions for (VP) and (VMD) respectively.

Hence the theorem.

Theorem 6:(Strong Duality). Let x° be an efficient solution for (VP) and assume that g satisfies a constrained qualification [7] at x°. Then there exist $\lambda^\circ \in R^k$, $\mu^\circ \in R^m$ and vector functions $\overline{p}, \overline{q}, \overline{r}, \overline{\alpha}$ and $\overline{\beta}$ such that the vector functions $\overline{p}(x,x^\circ)=0$, $\overline{q}(x,x^\circ)=0$ or $\overline{r}(x,x^\circ)=0$, $\overline{\alpha}(x,x^\circ)=0$ or $\overline{\beta}(x,x^\circ)=0$ and $(x^\circ,\lambda^\circ,\mu^\circ,\overline{p},\ \overline{q},\ \overline{r},\ \overline{\alpha},\ \overline{\beta})$ is a feasible solution for (VMD) and also, the objective values of (VP) at x° and (VMD) at $(x^\circ,\lambda^\circ,\mu^\circ,\overline{p},\ \overline{q},\ \overline{r},\ \overline{\alpha},\ \overline{\beta})$ are equal. If the weak duality theorem 4 between the problems (VP) and (VMD) holds for all feasible solutions $(x,u,\lambda,\mu,p,q,r,\alpha,\beta)$, then $(x^\circ,\lambda^\circ,\mu^\circ,\overline{p},\ \overline{q},\ \overline{r},\ \overline{\alpha},\ \overline{\beta})$ is a properly efficient solution for (VMD).

Proof: By the Theorem 1. , there exist $\lambda^\circ \in R^k$, $\mu^\circ \in R^m$ such that (1) and (2) are satisfied. Therefore, $(x^\circ,\lambda^\circ,\mu^\circ,\overline{p},\ \overline{q},\ \overline{r},\ \overline{\alpha},\ \overline{\beta})$ with the vector functions $\overline{p}(x,x^\circ)=0$, $\overline{q}(x,x^\circ)=0$ or $\overline{r}(x,x^\circ)=0$ and $\overline{\alpha}(x,x^\circ)=0$ or $\overline{\beta}(x,x^\circ)=0$, is a feasible solution for (VMD). Clearly, the objective values of (VP) at x° and (VMD) at $(x^\circ,\lambda^\circ,\mu^\circ,\overline{p},\ \overline{q},\ \overline{r},\ \overline{\alpha},\ \overline{\beta})$ are the same. By the weak duality theorem 4, we can conclude that $(x^\circ,\lambda^\circ,\mu^\circ,\overline{p},\ \overline{q},\ \overline{r},\ \overline{\alpha},\ \overline{\beta})$ is a properly efficient solution for (VMD).
Hence the theorem.

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