Int. Journal of Math. Analysis, Vol. 4, 2010, no. 26, 1271 - 1284

Properties of Certain Analytic Functions Associated with Two Boundary Points

Hitoshi Shiraishi and Shigeyoshi Owa

Department of Mathematics, Kinki University
Higashi-Osaka, Osaka 577-8502, Japan
shiraishi@math.kindai.ac.jp
owa@math.kindai.ac.jp

Abstract

For analytic functions $f(z)$ in the closed unit disk $\overline{\mathbb{U}}$, two boundary points z_1 and z_2 such that $\alpha = (f'(z_1)+f'(z_2))/2 \in f'(\mathbb{U})$ are considered. The object of the present paper is to discuss some interesting conditions for $f(z)$ to be $|f'(z)-1| < \rho|1-\alpha|$ in $\mathbb{U}$ with some examples.

Mathematics Subject Classification: Primary 30C45

Keywords: analytic, univalent, Jack's lemma

1 Introduction

Let $\mathcal{A}_n$ denote the class of functions

$$f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \dots \qquad (n = 1, 2, 3, \dots)$$

that are analytic in the closed unit disk $\overline{\mathbb{U}} = \{z \in \mathbb{C} : |z| \leqq 1\}$ and $\mathcal{A} = \mathcal{A}_1$. Also, the open unit disk is denoted by $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

For two boundary points z_1 and z_2, let us consider

$$\alpha = \frac{f'(z_1) + f'(z_2)}{2} \in f'(\mathbb{U}).$$

For such α $(\alpha \neq 1)$, if $f(z) \in \mathcal{A}_n$ satisfies

$$\left| \frac{f'(z) - \alpha}{1 - \alpha} - 1 \right| < \rho \qquad (z \in \mathbb{U})$$

for some real $\rho > 0$, then

$$|f'(z) - 1| < \rho|1 - \alpha| \qquad (z \in \mathbb{U}).$$

Therefore, if $0 < \rho|1-\alpha| < 1$, then $f(z)$ is close-to-convex (univalent) in $\mathbb{U}$. In the present paper, we use such a technique for $f(z)$.

The basic tool in proving our results is the following lemma due to Jack [1] (also, due to Miller and Mocanu [2]).

Lemma 1. *Let the function $w(z)$ defined by*

$$w(z) = a_n z^n + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \dots \qquad (n = 1, 2, 3, \dots)$$

be analytic in $\mathbb{U}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r$ at a point $z_0 \in \mathbb{U}$, then there exists a real number $k \geqq n$ such that

$$\frac{z_0 w'(z_0)}{w(z_0)} = k$$

and

$$\operatorname{Re}\left(\frac{z_0 w''(z_0)}{w'(z_0)}\right) + 1 \geqq k.$$

2 Main results

Applying Lemma 1, we drive the following results.

Theorem 1. *If $f(z) \in \mathcal{A}_n$ satisfies*

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{|1-\alpha|n\rho}{1+|1-\alpha|\rho} \qquad (z \in \mathbb{U})$$

for some complex $\alpha = \dfrac{f'(z_1) + f'(z_2)}{2} \in f'(\mathbb{U})$ and $\alpha \neq 1$ such that $z_1, z_2 \in \partial\mathbb{U}$, and for some real $\rho > 1$, then

$$|f'(z) - 1| < \rho|1-\alpha| \qquad (z \in \mathbb{U}).$$

Proof. Let us define $w(z)$ by

$$\begin{aligned} w(z) &= \frac{f'(z) - \alpha}{1-\alpha} - 1 \qquad (z \in \mathbb{U}) \qquad (1) \\ &= \frac{(n+1)a_{n+1}}{1-\alpha} z^n + \frac{(n+2)a_{n+2}}{1-\alpha} z^{n+1} + \dots . \end{aligned}$$

Then, clearly, $w(z)$ is analytic in $\mathbb{U}$ and $w(0) = 0$. Differentiating both sides in (1), we obtain

$$\frac{zf''(z)}{f'(z)} = \frac{(1-\alpha)zw'(z)}{(1-\alpha)w(z)+1},$$

and therefore,

$$\left|\frac{zf''(z)}{f'(z)}\right| = \left|\frac{(1-\alpha)zw'(z)}{(1-\alpha)w(z)+1}\right| < \frac{|1-\alpha|n\rho}{1+|1-\alpha|\rho} \qquad (z \in \mathbb{U}).$$

If there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leqq |z_0|} |w(z)| = |w(z_0)| = \rho,$$

then Lemma 1 gives us that $w(z_0) = \rho e^{i\theta}$ and $z_0 w'(z_0) = kw(z_0)$ $(k \geqq n)$.

For such a point z_0, we have

$$\begin{aligned}
\left|\frac{z_0 f''(z_0)}{f'(z_0)}\right| &= \left|\frac{(1-\alpha)z_0 w'(z_0)}{(1-\alpha)w(z_0)+1}\right| \\
&= \left|\frac{(1-\alpha)kw(z_0)}{(1-\alpha)w(z_0)+1}\right| \\
&= \frac{|(1-\alpha)|k\rho}{|(1-\alpha)\rho e^{i\theta}+1|} \\
&= \frac{|(1-\alpha)|k\rho}{\sqrt{(1+|1-\alpha|\rho\cos\phi)^2 + (|1-\alpha|\rho\sin\phi)^2}} \qquad (\phi = \theta + \arg(1-\alpha)) \\
&= \frac{|(1-\alpha)|k\rho}{\sqrt{1+|1-\alpha|^2\rho^2 + 2|1-\alpha|\rho\cos\phi}} \\
&\geqq \frac{|1-\alpha|k\rho}{1+|1-\alpha|\rho} \\
&\geqq \frac{|1-\alpha|n\rho}{1+|1-\alpha|\rho}.
\end{aligned}$$

This contradicts our condition in the theorem. Therefore, there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)| = \rho$. This means that $|w(z)| < \rho$ for all $z \in \mathbb{U}$. It follows that

$$\left|\frac{f'(z)-\alpha}{1-\alpha} - 1\right| < \rho \qquad (z \in \mathbb{U})$$

so that $|f'(z) - 1| < \rho|1-\alpha|$ in $\mathbb{U}$. □

Example 1. Let us consider a function

$$f(z) = z + a_{n+1}z^{n+1} \qquad (z \in \mathbb{U})$$

with $|a_{n+1}| < \dfrac{1}{2(n+1)}$. Differenciating the function $f(z)$, we obtain

$$\frac{zf''(z)}{f'(z)} = \frac{n(n+1)a_{n+1}z^n}{1+(n+1)a_{n+1}z^n},$$

and therefore,

$$\left|\frac{zf''(z)}{f'(z)}\right| = \left|\frac{n(n+1)a_{n+1}z^n}{1+(n+1)a_{n+1}z^n}\right| < \frac{n(n+1)|a_{n+1}|}{1-(n+1)|a_{n+1}|} \qquad (z \in \mathbb{U}).$$

We choose two boundary points z_1 and z_2 such that $f'(z_1) = 1+(n+1)|a_{n+1}|$ and $f'(z_2) = 1+(n+1)|a_{n+1}|i$. Then we know that $z_1 = e^{-i\frac{\arg(a_{n+1})}{n}}$ and $z_2 = e^{i\frac{\pi - 2\arg(a_{n+1})}{2n}}$. For such z_1 and z_2, we obtain

$$\alpha = \frac{f'(z_1)+f'(z_2)}{2} = 1 + \frac{(n+1)|a_{n+1}|(1+i)}{2}$$

so that

$$1-\alpha = -\frac{(n+1)|a_{n+1}|(1+i)}{2}.$$

Now, we consider some $\rho > 1$ such that

$$\frac{n(n+1)|a_{n+1}|}{1-(n+1)|a_{n+1}|} \leqq \frac{|1-\alpha|n\rho}{1+|1-\alpha|\rho} = \frac{n(n+1)|a_{n+1}|\rho}{\sqrt{2}+(n+1)|a_{n+1}|\rho}.$$

This gives us that

$$\rho \geqq \frac{\sqrt{2}}{1-2(n+1)|a_{n+1}|}.$$

For such α and ρ, we know that $f(z)$ satisfies

$$|f'(z)-1| < (n+1)|a_{n+1}| \leqq \frac{(n+1)|a_{n+1}|}{1-2(n+1)|a_{n+1}|} \leqq \rho|1-\alpha|.$$

If we defined

$$\max_{z\in\mathbb{U}} |f'(z)-\alpha| = M_\alpha, \tag{2}$$

then we can have

$$\begin{aligned}|f'(z)-1| &< \rho|1-\alpha| \\ &= \rho|f'(0)-\alpha| \\ &\leqq \rho M_\alpha \qquad (z \in \mathbb{U}).\end{aligned}$$

So, we get

Corollary 1. *If $f(z) \in \mathcal{A}_n$ satisfies*

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{|1-\alpha|n\rho}{1+|1-\alpha|\rho} \qquad (z \in \mathbb{U})$$

for some complex $\alpha = \dfrac{f'(z_1)+f'(z_2)}{2} \in f'(\mathbb{U})$ and $\alpha \neq 1$ such that $z_1, z_2 \in \partial\mathbb{U}$, and for some real $\rho > 1$, then

$$|f'(z)-1| < \rho M_\alpha \qquad (z \in \mathbb{U}).$$

Putting

$$\alpha = \frac{f'(z_1)+f'(z_2)+\ldots+f'(z_m)}{m} \tag{3}$$

for $z_1, z_2, \ldots, z_m \in \partial\mathbb{U}$, we obtain

Corollary 2. *If $f(z) \in \mathcal{A}_n$ satisfies*

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{|1-\alpha|n\rho}{1+|1-\alpha|\rho} \qquad (z \in \mathbb{U})$$

for some complex $\alpha = \dfrac{f'(z_1)+f'(z_2)+\ldots+f'(z_m)}{m} \in f'(\mathbb{U})$ such that $z_1, z_2, \ldots, z_m \in \partial\mathbb{U}$ and $\alpha \neq 1$, and for some real $\rho > 1$, then

$$|f'(z)-1| < \rho|1-\alpha| \qquad (z \in \mathbb{U}).$$

We also derive

Theorem 2. *If $f(z) \in \mathcal{A}_n$ satisfies*

$$\left| zf''(z) - \frac{zf''(z)}{f'(z)} \right| < \frac{|1-\alpha|^2 n\rho^2}{1+|1-\alpha|\rho} \qquad (z \in \mathbb{U})$$

for some complex $\alpha = \dfrac{f'(z_1)+f'(z_2)}{2} \in f'(\mathbb{U})$ *and* $\alpha \neq 1$ *such that* $z_1, z_2 \in \partial\mathbb{U}$, *and for some real* $\rho > 1$, *then*

$$|f'(z) - 1| < \rho|1-\alpha| \qquad (z \in \mathbb{U}).$$

Proof. Define $w(z)$ by

$$\begin{aligned} w(z) &= \frac{f'(z)-\alpha}{1-\alpha} - 1 \qquad (z \in \mathbb{U}) \qquad (4) \\ &= \frac{(n+1)a_{n+1}}{1-\alpha} z^n + \frac{(n+2)a_{n+2}}{1-\alpha} z^{n+1} + \dots . \end{aligned}$$

Evidently, $w(z)$ is analytic in $\mathbb{U}$ and $w(0) = 0$. Differentiating (4) logarithmically and simplyfing, we have

$$\begin{aligned} zf''(z) - \frac{zf''(z)}{f'(z)} &= zf''(z)\left(1 - \frac{1}{f'(z)}\right) \\ &= \frac{(1-\alpha)^2 zw'(z)w(z)}{(1-\alpha)w(z)+1}, \end{aligned}$$

and hence,

$$\left| zf''(z) - \frac{zf''(z)}{f'(z)} \right| = \left| \frac{(1-\alpha)^2 zw'(z)w(z)}{(1-\alpha)w(z)+1} \right| < \frac{|1-\alpha|^2 n\rho^2}{1+|1-\alpha|\rho} \qquad (z \in \mathbb{U}).$$

If there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leqq |z_0|} |w(z)| = |w(z_0)| = \rho,$$

then Lemma 1 gives us that $w(z_0) = \rho e^{i\theta}$ and $z_0 w'(z_0) = kw(z_0)$ $(k \geqq n)$.

For such a point z_0, we have

$$\begin{aligned}\left|z_0 f''(z_0) - \frac{z_0 f''(z_0)}{f'(z_0)}\right| &= \left|\frac{(1-\alpha)^2 z_0 w'(z_0) w(z_0)}{(1-\alpha)w(z_0)+1}\right| \\ &= \frac{|1-\alpha|^2\rho^2 k}{|(1-\alpha)\rho e^{i\theta}+1|} \\ &= \frac{|1-\alpha|^2\rho^2 k}{\sqrt{(1+|1-\alpha|\rho\cos\phi)^2 + (|1-\alpha|\rho\sin\phi)^2}} \qquad (\phi = \theta + \arg(1-\alpha)) \\ &= \frac{|1-\alpha|^2\rho^2 k}{\sqrt{1+|1-\alpha|^2\rho^2 + 2|1-\alpha|\rho\cos\phi}} \\ &\geqq \frac{|1-\alpha|^2 k\rho^2}{1+|1-\alpha|\rho} \\ &\geqq \frac{|1-\alpha|^2 n\rho^2}{1+|1-\alpha|\rho}\end{aligned}$$

This contradicts our condition in the theorem. Therefore, there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)| = \rho$. This means that $|w(z)| < \rho$ for all $z \in \mathbb{U}$. This implies that

$$\left|\frac{f'(z)-\alpha}{1-\alpha} - 1\right| < \rho \qquad (z \in \mathbb{U}).$$

□

Example 2. We consider a function $f(z)$ given by

$$f(z) = z + a_{n+1}z^{n+1} \qquad (z \in \mathbb{U})$$

with $|a_{n+1}| < \dfrac{1}{n+1}$. Differenciating $f(z)$, we obtain that

$$zf''(z) - \frac{zf''(z)}{f'(z)} = \frac{n(n+1)^2 a_{n+1}^2 z^{2n}}{1+(n+1)a_{n+1}z^n},$$

that is, that

$$\begin{aligned}\left|zf''(z) - \frac{zf''(z)}{f'(z)}\right| &= \left|\frac{n(n+1)^2 a_{n+1}^2 z^{2n}}{1+(n+1)a_{n+1}z^n}\right| \\ &< \frac{n(n+1)^2|a_{n+1}|^2}{1-(n+1)|a_{n+1}|} \qquad (z \in \mathbb{U}).\end{aligned}$$

Choosing same points z_1 and z_2 in Example 1, we see that

$$1-\alpha = -\frac{(n+1)|a_{n+1}|(1+i)}{2}.$$

If we consider some $\rho > 1$ such that

$$\frac{n(n+1)^2|a_{n+1}|^2}{1-(n+1)|a_{n+1}|} \leqq \frac{|1-\alpha|^2 n\rho^2}{1+|1-\alpha|\rho},$$

we have that

$$\rho \geqq \frac{\sqrt{2}}{1-(n+1)|a_{n+1}|}.$$

For such α and ρ, we know that

$$|f'(z)-1| < (n+1)|a_{n+1}| \leqq \frac{(n+1)|a_{n+1}|}{1-(n+1)|a_{n+1}|} \leqq \rho|1-\alpha|.$$

Further, we obtain

Corollary 3. *If $f(z) \in \mathcal{A}_n$ satisfies*

$$\left| zf''(z) - \frac{zf''(z)}{f'(z)} \right| < \frac{|1-\alpha|^2 n\rho^2}{1+|1-\alpha|\rho} \qquad (z \in \mathbb{U})$$

for some complex $\alpha = \dfrac{f'(z_1)+f'(z_2)}{2} \in f'(\mathbb{U})$ and $\alpha \neq 1$ such that $z_1, z_2 \in \partial\mathbb{U}$, and for some real $\rho > 1$, then

$$|f'(z)-1| < \rho M_\alpha \qquad (z \in \mathbb{U}).$$

Corollary 4. *If $f(z) \in \mathcal{A}_n$ satisfies*

$$\left| zf''(z) - \frac{zf''(z)}{f'(z)} \right| < \frac{|1-\alpha|^2 n\rho^2}{1+|1-\alpha|\rho} \qquad (z \in \mathbb{U})$$

for some complex $\alpha = \dfrac{f'(z_1)+f'(z_2)+\ldots+f'(z_m)}{m} \in f'(\mathbb{U})$ such that $z_1, z_2, \ldots, z_m \in \partial\mathbb{U}$ and $\alpha \neq 1$, and for some real $\rho > 1$, then

$$|f'(z)-1| < \rho|1-\alpha| \qquad (z \in \mathbb{U}).$$

Further, we discuss a new application for Lemma 1.

Theorem 3. *If $f(z) \in \mathcal{A}_n$ satisfies*

$$\operatorname{Re}\left(\frac{z(zf''(z))'}{f'(z)-1}\right) < n^2 \qquad (z \in \mathbb{U})$$

for some complex $\alpha = \dfrac{f'(z_1)+f'(z_2)}{2} \in f'(\mathbb{U})$ *and* $\alpha \neq 1$ *such that* $z_1, z_2 \in \partial\mathbb{U}$, *and for some real* $\rho > 1$, *then*

$$|f'(z)-1| < \rho|1-\alpha| \qquad (z \in \mathbb{U}).$$

Proof. Defining the function $w(z)$ by

$$\begin{aligned} w(z) &= \frac{f'(z)-\alpha}{1-\alpha} - 1 \qquad (z \in \mathbb{U}) \\ &= \frac{(n+1)a_{n+1}}{1-\alpha} z^n + \frac{(n+2)a_{n+2}}{1-\alpha} z^{n+1} + \dots, \end{aligned}$$

we have that $w(z)$ is analytic in $\mathbb{U}$ with $w(0)=0$. Since,

$$\frac{z(zf''(z))'}{f'(z)-1} = \frac{zw'(z)}{w(z)} + \frac{z^2w''(z)}{w(z)},$$

we obtain that

$$\begin{aligned} \operatorname{Re}\left(\frac{z(zf''(z))'}{f'(z)-1}\right) &= \operatorname{Re}\left(\frac{zw'(z)}{w(z)} + \frac{z^2w''(z)}{w(z)}\right) \\ &= \operatorname{Re}\left(\frac{zw'(z)}{w(z)}\left(1+\frac{zw''(z)}{w'(z)}\right)\right) < n^2 \qquad (z \in \mathbb{U}). \end{aligned}$$

If there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leqq |z_0|} |w(z)| = |w(z_0)| = \rho,$$

then Lemma 1 gives us that $w(z_0) = \rho e^{i\theta}$ and $z_0 w'(z_0) = k w(z_0)$ $(k \geqq n)$.

Thus we have

$$\begin{aligned} \operatorname{Re}\left(\frac{z_0(z_0 f''(z_0))'}{f'(z_0)-1}\right) &= \operatorname{Re}\left(\frac{z_0w'(z_0)}{w(z_0)}\left(1+\frac{z_0w''(z_0)}{w'(z_0)}\right)\right) \\ &= k\left(1+\operatorname{Re}\left(\frac{z_0w''(z_0)}{w'(z_0)}\right)\right) \\ &= k^2 \\ &\geqq n^2. \end{aligned}$$

This contradicts our condition in the theorem. Therefore, there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)| = \rho$. This means that $|w(z)| < \rho$ for all $z \in \mathbb{U}$. □

We also have the following corollaries.

Corollary 5. *If* $f(z) \in \mathcal{A}_n$ *satisfies*

$$\operatorname{Re}\left(\frac{z(zf''(z))'}{f'(z)-1}\right) < n^2 \qquad (z \in \mathbb{U})$$

for some complex $\alpha = \dfrac{f'(z_1)+f'(z_2)}{2} \in f'(\mathbb{U})$ *and* $\alpha \neq 1$ *such that* $z_1, z_2 \in \partial\mathbb{U}$, *and for some real* $\rho > 1$, *then*

$$|f'(z)-1| < \rho M_\alpha \qquad (z \in \mathbb{U}).$$

Corollary 6. *If* $f(z) \in \mathcal{A}_n$ *satisfies*

$$\operatorname{Re}\left(\frac{z(zf''(z))'}{f'(z)-1}\right) < n^2 \qquad (z \in \mathbb{U})$$

for some complex $\alpha = \dfrac{f'(z_1)+f'(z_2)+\ldots+f'(z_m)}{m} \in f'(\mathbb{U})$ *such that* $z_1, z_2, \ldots, z_m \in \partial\mathbb{U}$ *and* $\alpha \neq 1$, *and for some real* $\rho > 1$, *then*

$$|f'(z)-1| < \rho|1-\alpha| \qquad (z \in \mathbb{U}).$$

Next our result is contained in

Theorem 4. *If* $f(z) \in \mathcal{A}_n$ *satisfies*

$$\frac{zf''(z)}{f'(z)-1} \neq k \qquad (z \in \mathbb{U})$$

for some complex $\alpha = \dfrac{f'(z_1)+f'(z_2)}{2} \in f'(\mathbb{U})$ *and* $\alpha \neq 1$ *such that* $z_1, z_2 \in \partial\mathbb{U}$, *some real* $\rho > 1$, *and for all real* $k \geqq n$, *then*

$$|f'(z)-1| < \rho|1-\alpha| \qquad (z \in \mathbb{U}).$$

Proof. Let us define the function $w(z)$ by

$$w(z) = \frac{f'(z) - \alpha}{1 - \alpha} - 1 \qquad (z \in \mathbb{U}) \tag{5}$$
$$= \frac{(n+1)a_{n+1}}{1-\alpha} z^n + \frac{(n+2)a_{n+2}}{1-\alpha} z^{n+1} + \dots .$$

Clearly, $w(z)$ is analytic in $\mathbb{U}$ with $w(0) = 0$. We want to prove that $|w(z)| < \rho$ in $\mathbb{U}$. Note that

$$\frac{zf''(z)}{f'(z) - 1} = \frac{zw'(z)}{w(z)} \qquad (z \in \mathbb{U}).$$

If there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leqq |z_0|} |w(z)| = |w(z_0)| = \rho,$$

then Lemma 1 gives us that $w(z_0) = \rho e^{i\theta}$ and $z_0 w'(z_0) = k w(z_0)$ $(k \geqq n)$.

Thus we have

$$\frac{z_0 f''(z_0)}{f'(z_0) - 1} = \frac{z_0 w'(z_0)}{w(z_0)}$$
$$= k$$

This contradicts the condition in the theorem. Therefore, there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)| = \rho$. This means that $|w(z)| < \rho$ for all $z \in \mathbb{U}$. We conclude that $|f'(z) - 1| < \rho |1 - \alpha|$ in $\mathbb{U}$. □

Corollary 7. *If $f(z) \in \mathcal{A}_n$ satisfies*

$$\frac{zf''(z)}{f'(z) - 1} \neq k \qquad (z \in \mathbb{U})$$

for some complex $\alpha = \dfrac{f'(z_1) + f'(z_2)}{2} \in f'(\mathbb{U})$ and $\alpha \neq 1$ such that $z_1, z_2 \in \partial\mathbb{U}$, some real $\rho > 1$, and for all real $k \geqq n$, then

$$|f'(z) - 1| < \rho M_\alpha \qquad (z \in \mathbb{U}).$$

Corollary 8. *If $f(z) \in \mathcal{A}_n$ satisfies*

$$\frac{zf''(z)}{f'(z) - 1} \neq k \qquad (z \in \mathbb{U})$$

for some complex $\alpha = \dfrac{f'(z_1) + f'(z_2) + \ldots + f'(z_m)}{m} \in f'(\mathbb{U})$ *such that* $z_1, z_2, \ldots, z_m \in \partial\mathbb{U}$, *some real* $\rho > 1$, *and for all real* $k \geqq n$, *then*

$$|f'(z) - 1| < \rho|1 - \alpha| \qquad (z \in \mathbb{U}).$$

Finally, we derive

Theorem 5. *If* $f(z) \in \mathcal{A}_n$ *satisfies*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \frac{|1-\beta|n\rho}{1+|1-\beta|\rho} \qquad (z \in \mathbb{U})$$

for some complex $\beta = \dfrac{F(z_1) + F(z_2)}{2} \in F(\mathbb{U})$ *such that* $z_1, z_2 \in \partial\mathbb{U}$ *and* $\beta \neq 1$, *and for some real* $\rho > 1$, *where* $F(z) = \dfrac{f(z)}{z}$, *then*

$$\left|\frac{f(z)}{z} - 1\right| < \rho|1 - \beta| \qquad (z \in \mathbb{U}).$$

Proof. Let us define $w(z)$ by

$$\begin{aligned} w(z) &= \frac{\frac{f(z)}{z} - \beta}{1-\beta} - 1 \qquad (z \in \mathbb{U}) \\ &= \frac{a_{n+1}}{1-\beta} z^n + \frac{a_{n+2}}{1-\beta} z^{n+1} + \ldots . \end{aligned} \tag{6}$$

Then, we have that $w(z)$ is analytic in $\mathbb{U}$ and $w(0) = 0$. Differenciating (6) in both side logarithmically and simplifying, we obtain

$$\frac{zf'(z)}{f(z)} = \frac{(1-\beta)zw'(z)}{(1-\beta)w(z)+1},$$

and hence,

$$\left|\frac{zf'(z)}{f(z)}\right| = \left|\frac{(1-\beta)zw'(z)}{(1-\beta)w(z)+1}\right| < \frac{|1-\beta|n\rho}{1+|1-\beta|\rho} \qquad (z \in \mathbb{U}).$$

Using the same process of the proof in Theorem 1, we complete the proof of the theorem. □

Example 3. We consider a function

$$f(z) = z + a_{n+1}z^{n+1} \qquad (z \in \mathbb{U})$$

with $|a_{n+1}| < \dfrac{1}{2}$. Differenciating the function, we obtain

$$\frac{zf'(z)}{f(z)} - 1 = \frac{na_{n+1}z^n}{1 + a_{n+1}z^n},$$

and therefore,

$$\begin{aligned}\left|\frac{zf'(z)}{f(z)} - 1\right| &= \left|\frac{na_{n+1}z^n}{1 + a_{n+1}z^n}\right| \\ &< \frac{n|a_{n+1}|}{1 - |a_{n+1}|} \qquad (z \in \mathbb{U}).\end{aligned}$$

Consider two boundary points $z_1 = e^{-i\frac{\arg(a_{n+1})}{n}}$ and $z_2 = e^{i\frac{\pi - 2\arg(a_{n+1})}{2n}}$. Then, since $F(z_1) = 1 + |a_{n+1}|$ and $F(z_2) = 1 + |a_{n+1}|i$, we see that

$$1 - \beta = -\frac{|a_{n+1}|(1+i)}{2}.$$

For such β, we consider some ρ such that

$$\frac{n|a_{n+1}|}{1 - |a_{n+1}|} \leqq \frac{|1-\beta|n\rho}{1 + |1-\beta|\rho}.$$

This gives us that

$$\rho \geqq \frac{\sqrt{2}}{1 - 2|a_{n+1}|}.$$

Therefore, we have that

$$\left|\frac{f(z)}{z} - 1\right| < |a_{n+1}| \leqq \frac{|a_{n+1}|}{1 - 2|a_{n+1}|} \leqq \rho|1-\beta|.$$

Defining M_β by

$$\max_{z \in \mathbb{U}} \left|\frac{f(z)}{z} - \beta\right| = M_\beta,$$

we have the following corollary.

Corollary 9. *If $f(z) \in \mathcal{A}_n$ satisfies*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \frac{|1-\beta|n\rho}{1+|1-\beta|\rho} \qquad (z \in \mathbb{U})$$

for some complex $\beta = \dfrac{F(z_1)+F(z_2)}{2} \in F(\mathbb{U})$ *such that* $z_1, z_2 \in \partial\mathbb{U}$ *and* $\beta \neq 1$, *and for some real* $\rho > 1$, *where* $F(z) = \dfrac{f(z)}{z}$, *then*

$$\left|\frac{f(z)}{z} - 1\right| < \rho M_\beta \qquad (z \in \mathbb{U}).$$

Also considering

$$\beta = \frac{\frac{f(z_1)}{z_1} + \frac{f(z_2)}{z_2} + \ldots + \frac{f(z_m)}{z_m}}{m}$$

for $z_1, z_2, \ldots, z_m \in \partial\mathbb{U}$, we obtain

Corollary 10. *If $f(z) \in \mathcal{A}_n$ satisfies*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \frac{|1-\beta|n\rho}{1+|1-\beta|\rho} \qquad (z \in \mathbb{U})$$

for some complex $\beta = \dfrac{F(z_1)+F(z_2)+\ldots+F(z_m)}{m} \in F(\mathbb{U})$ *such that* $z_1, z_2, \ldots, z_m \in \partial\mathbb{U}$ *and* $\beta \neq 1$, *and for some real* $\rho > 1$ *where* $F(z) = \dfrac{f(z)}{z}$, *then*

$$\left|\frac{f(z)}{z} - 1\right| < \rho|1-\beta| \qquad (z \in \mathbb{U}).$$

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Received: January, 2010