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Some Common Fixed Point Theorems in Intuitionistic Fuzzy Metric Spaces

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Abstract. This paper is three fold. Firstly, we shall discuss the notions of different variants of R-weakly commuting mappings (R-weakly commuting mappings of type (A_g), R-weakly commuting mapping of type (A_f) and R-weakly commuting mapping of type (P)) and E.A.property in intuitionistic fuzzy metric spaces and then provide various examples to reflect upon the distinctiveness among different variants of R-weakly commuting mappings, see for detail in [14] . Secondly, we prove common fixed point theorem using control function and E.A. property along with weakly compatible maps. At the end, we prove a common fixed point theorem for weakly compatible maps in intuitionistic fuzzy metric spaces which generalizes the result of Alaca, Turkoglu and Yildiz [3].Our results extend, generalize and fuzzify several fixed point theorems on metric spaces, Menger spaces, uniform spaces and fuzzy metric spaces.

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1. Introduction

It proved a turning point in the development of fuzzy mathematics when the notion of fuzzy set was introduced by Zadeh [26]. The fuzzy set theory has applications in neural network theory, stability theory, mathematical programming, modeling theory, engineering sciences, medical sciences, image processing, control theory, communication etc. There are many view points of the notion of the metric space in fuzzy topology. Atanassov [4] introduced and studied the concept of intuitionistic fuzzy sets. Intuitionistic fuzzy sets as a generalization of fuzzy sets can be useful in situations when description of a problem by a (fuzzy) linguistic variable, given in terms of a membership function only, seems too rough. For example, in decision making problems, particularly in the case of medial diagnosis, sales analysis, new product marketing etc. there is a fair chance of the existence of a non-null hesitation part at each moment of evaluation of an unknown object i.e, In other words, the application of intuitionistic fuzzy sets instead of fuzzy sets means the introduction of another degree of freedom into a set description. By employing intuitionistic fuzzy sets in databases we can express a hesitation concerning examined objects. Coker [6] introduced the concept of intuitionistic fuzzy topological spaces. Alace et al. [2] proved the well-known fixed point theorems of Banach [5] in the setting of intuitionistic fuzzy metric spaces. Later on, Turkoglu et al. [24] proved Jungck's [11] common fixed point theorem in the setting of intuitionistic fuzzy metric space. Turkoglu et al. [24] further formulated the notions of weakly commuting and R-weakly commuting mappings in intuitionistic fuzzy metric spaces and proved the intuitionistic fuzzy version of Pant's theorem [17]. Gregori et al. [9], Saadati and Park [21] studied the concept of intuitionistic fuzzy metric space and its applications. No wonder that intuitionistic fuzzy fixed point theory has become an area of interest for specialists in fixed point theory as intuitionistic fuzzy mathematics has covered new possibilities for fixed point theorists. Recently, many authors have also studied the fixed point theory in fuzzy and intuitionistic fuzzy metric spaces (see [8], [10], [18], [19], [25]).

2. Preliminaries

We begin by briefly recalling some definitions and notions from fixed point theory literature that we will use in the sequel.

The concepts of triangular norms (t-norm) and triangular conorms (t-conorm) were originally introduced by Schweizer and Sklar [22] **.**

Definition 2.1[22] A binary operation $*$: $[0,1]\times[0,1] \to [0,1]$ is continuous t-norm if $*$ is satisfying the following conditions:

(i) $*$ is commutative and associative;

(ii) $*$ is continuous;

(iii) $a * 1 = a$ for all $a\in[0, 1]$;

(iv) $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d\in[0, 1]$.

Definition 2.2[22] A binary operation $\Diamond$: $[0,1]\times[0,1] \to [0,1]$ is continuous t-conorm if $\Diamond$ is satisfying the following conditions:

(i) $\Diamond$ is commutative and associative;

(ii) $\Diamond$ is continuous;

(iii) $a \Diamond 0 = a$ for all $a\in[0, 1]$;

(iv) $a \Diamond b \le c \Diamond d$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d\in[0, 1]$.

Definition 2.3[2] A 5-tuple $(X, M, N, *, \Diamond)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, $\Diamond$ is a continuous t-conorm and M, N are fuzzy sets on $X^2\times[0, \infty)$ satisfying the following conditions:

(i) $M(x, y, t) + N(x, y, t) \le 1$ for all $x, y\in X$ and $t > 0$;

(ii) $M(x, y, 0) = 0$ for all $x, y\in X$;

(iii) $M(x, y, t) = 1$ for all $x, y\in X$ and $t > 0$ if and only if $x = y$;

(iv) $M(x, y, t) = M(y, x, t)$ for all $x, y\in X$ and $t > 0$;

(v) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$ for all $x, y, z \in X$ and $s, t > 0$;

(vi) for all $x, y\in X$, $M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous;

(vii) $\lim_{t\to\infty}M(x, y, t) = 1$ for all $x, y\in X$ and $t > 0$;

(viii) $N(x, y, 0) = 1$ for all $x, y\in X$;

(ix) $N(x, y, t) = 0$ for all $x, y\in X$ and $t > 0$ if and only if $x = y$;

(x) $N(x, y, t) = N(y, x, t)$ for all $x, y\in X$ and $t > 0$;

(xi) $N(x, y, t) \Diamond N(y, z, s) \ge N(x, z, t + s)$ for all $x, y, z\in X$ and $s, t > 0$;

(xii) for all $x, y\in X$, $N(x, y, .) : [0, \infty)\to[0, 1]$ is right continuous;

(xiii) $\lim_{t\to\infty}N(x, y, t) = 0$ for all x, y in X.

(M, N) is called an intuitionistic fuzzy metric on X. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

Remark 2.2[1] An intuitionistic fuzzy metric spaces with continuous t-norm $*$ and continuous t-conorm $\Diamond$ defined by $a*a \ge a$ and $(1-a) \Diamond (1-a) \le (1-a)$ for all $a\in[0, 1]$. Then for all $x, y\in X$, $M(x, y, *)$ is non-decreasing and $N(x, y, \Diamond)$ is non-increasing.

Alaca, Turkoglu and Yildiz [2] introduced the following notions:

Definition 2.4. Let (X, M, N, *, ◊) be an intuitionistic fuzzy metric space. Then

(a) a sequence $\{x_n\}$ in X is said to be Cauchy sequence if, for all t > 0 and p > 0, $\lim_{n\to\infty}M(x_{n+p}, x_n, t) = 1$, $\lim_{n\to\infty}N(x_{n+p}, x_n, t) = 0$.

(b) a sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if, for all t > 0, $\lim_{n\to\infty}M(x_n, x, t) = 1$, $\lim_{n\to\infty}N(x_n, x, t) = 0$.

Since * and ◊ are continuous, the limit is uniquely determined from (v) and (xi) of Definition 2.3, respectively. An intuitionistic fuzzy metric space (X, M, N, *, ◊) is said to be complete if and only if every Cauchy sequence in X is convergent.

3. Fixed points results using control functions and E.A. property

The notion of weak commutativity [23] and compatibility [12] gives a new direction to the study of common fixed point theorems for the mappings satisfying some contractive type conditions. Consequently, the recent literature of fixed point theory has witnessed the evolution of several weak conditions of commutativity such as: Compatible mappings of type (A), Compatible mappings of type (B), Compatible mappings of type (P), Compatible mappings of type (C), R-weakly commuting mappings and several others whose lucid survey and illustration are available in Murthy [16]. In what follows, we choose to utilize the most natural of these weak conditions, namely, 'weak compatibility' due to Jungck.

Turkoglu, Alaca and Yildiz [25] introduced the notions of compatible mappings in intuitionistic fuzzy metric space, akin to the concept of compatible mappings introduced by Jungck [12] in metric spaces as follows:

Definition 3.1. A pair of self-mappings (f, g) of an intuitionistic fuzzy metric space (X,M,N,*, ◊) is said to be compatible if $\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1$ and $\lim_{n\to\infty}N(fgx_n, gfx_n, t) = 0$ for every t > 0, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty}fx_n = \lim_{n\to\infty}gx_n = z$ for some $z \in X$.

Definition 3.2. A pair of self-mappings (f, g) of an intuitionistic fuzzy metric space (X,M,N,*, ◊) is said to be non-compatible if $\lim_{n\to\infty} M(fgx_n, gfx_n, t) \neq 1$ or non-existent and $\lim_{n\to\infty}N(fgx_n, gfx_n, t) \neq 0$ or non-existent for every t > 0, whenever $\{x_n\}$ is a sequence in X such that

$\lim_{n\to\infty}fx_n = \lim_{n\to\infty}gx_n = z$ for some $z \in X$.

In 1998, Jungck and Rhoades [13] introduced the concept of weakly compatible maps as follows:

Definition 3.3. Two self maps f and g are said to be weakly compatible if they commute at coincidence points.

Turkoglu et al. [24] first formulated the definition of weakly commuting and R-weakly commuting mappings in intuitionistic fuzzy metric spaces and then proved the intuitionistic fuzzy version of Pant's theorem [17].

Definition 3.4[24] A pair of self-mappings (f,g) of an intuitionistic fuzzy metric space (X, M, N, *, ◊) is said to be

(i) weakly commuting if $M(fgx, gfx, t) \geq M(fx, gx, t)$ and $N(fgx, gfx, t) \leq N(fx, gx, t)$ for all $x \in X$ and $t > 0$.

(ii) R-weakly commuting if there exists some $R > 0$ such that

$$M(fgx, gfx, t) \geq M(fx, gx, t/R)$$

and $N(fgx, gfx, t) \leq N(fx, gx, t/R)$ for all $x \in X$ and $t > 0$.

Now, there arises a natural question: "How fixed point theorems can be improved to the setting of non-complete metric spaces and without continuity of f and g over the whole space X?" We give the partial answer. It seems that fixed point theorems can be improved in two ways: either imposing certain restrictions on the space X or by replacing the notion of R-weakly commutativity of mappings with certain improved notion.

In 1997, Pathak, Cho and Kang [20] introduced the of R-weakly commuting mappings of type (A_f), R-weakly commuting mapping of type (A_g). Now in a similar mode, Kumar and Vats [14] introduced the notions of R-weakly commuting mapping of type (A_g), R-weakly commuting mapping of type (A_f) and R-weakly commuting mapping of type (P) in intuitionistic fuzzy metric spaces and provide various examples to reflect upon the distinctiveness among them.

Definition 3.5. A pair of self-mappings (f, g) of an intuitionistic fuzzy metric space (X,M,N,*, ◊) is said to be

(i) R-weakly commuting mappings of type (A_g) if there exists some $R > 0$ such that

$$M(gfx, ffx, t) \geq M(fx, gx, t/R)$$

and $N(gfx, ffx, t) \leq N(fx, gx, t/R)$ for all $x \in X$ and $t > 0$.

(ii) R-weakly commuting mappings of type (A_f) if there exists some $R > 0$ such that

$$M(fgx, ggx, t) \geq M(fx, gx, t/R)$$

and $N(fgx, ggx, t) \leq N(fx, gx, t/R)$ for all $x \in X$ and $t > 0$.

(iii) R-weakly commuting mappings of type (P) if there exists some $R > 0$ such that

$$M(ffx, ggx, t) \geq M(fx, gx, t/R)$$

and $N(ffx, ggx, t) \leq N(fx, gx, t/R)$ for all $x \in X$ and $t > 0$.

Aamri and Moutawakil(M. Aamri and D.E. Moutawakil, Some new common fixed point theorems under strict contractive conditions, J. Math. Anal. Appl., **270**(2002), 181-188.) generalized the concept of non compatiblity in metric spaces by defining the notion of E.A. property and proved common fixed point theorems using this property.

In a similar mode we state E.A. property in intuitionistic fuzzy metric spaces .

Definition 3.6. A pair of self mappings (f , g) of a intuitionistic fuzzy metric space (X, M, N, *, ◊) is said to satisfy the E.A. property if there exist a sequence $\{x_n\}$ in X such that

$\lim_{n\to\infty} M(fx_n, gx_n, t) = 1$ $\lim_{n\to\infty} N(fx_n, gx_n, t) = 0$.

Example 3.1. Let X =[0,∞). Consider (X, M, N, *, ◊) be an intuitionistic fuzzy metric space as in example 2.1. Define T, S : X → [0,∞) by $Tx = \frac{x}{5}$ and $Sx = \frac{2x}{5}$ for all x in X.Consider. $x_n = 1/n$ .Now, $\lim_{n\to\infty} M(Sx_n, Tx_n, t) = 1$ and $\lim_{n\to\infty} N(Sx_n, Tx_n, t) = 0$.Therefore, S and T satisfy property E.A.

Now we state two Lemmas which are useful in proving our main results.

Lemma 3.1[1]. Let (X, M, N, *, ◊) be an intuitionistic fuzzy metric space and for all $x, y \in X$, $t>0$ and if for a number $k \in (0, 1)$, $M(x, y, kt) \geq M(x, y, t)$ and $N(x, y, kt) \leq N(x, y, t)$, then $x = y$.

Lemma 3.2 [1]. Let (X, M, N, *, ◊) be an intuitionistic fuzzy metric space and $\{y_n\}$ be a sequence in X. If there exists a number $k \in (0, 1)$ such that

$$M(y_{n+2}, y_{n+1}, kt) \geq M(y_{n+1}, y_n, t) \text{ and } N(y_{n+2}, y_{n+1}, kt) \leq N(y_{n+1}, y_n, t)$$

for all t > 0 and n = 1, 2, … , then $\{y_n\}$ is a Cauchy sequence in X.

Now we prove M. Imdad and J.Ali (Mathematical communications 11(2006), 153-163)

Theorems in the setting of intuitionistic fuzzy metric spaces.

Let A, B, S and T be self maps of a complete intuitionistic fuzzy metric spaces (X, M, N, *, ◊) with continuous t-norm * and continuous t-conorm ◊ defined by $a*a \geq a$ and $(1-a) \Diamond (1-a) \leq (1-a)$ for all $a \in [0, 1]$ satisfying the following conditions:

(3.1) $A(X) \subset T(X)$ and $B(X) \subset S(X)$,

(3.2) $M(Ax, By, t) \geq \varphi(\min\{M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t)\})$

and $N(Ax, By, t) \leq \psi(\max\{N(Sx, Ty, t), N(Sx, Ax, t), N(By, Ty, t)\})$

for all $x, y \in X$, where $\varphi, \psi : [0, 1] \to [0, 1]$ is a continuous function such that $\varphi(s) > s$ and $\psi(s) < s$ for each $0 < s < 1$ with $M(x,y, t) > 0$ ($x,y \in X, t > 0$) .

Then for any arbitrary point $x_0 \in X$, by (3.1), we can choose a point $x_1 \in X$ such that $Ax_0 = Tx_1$ and for this point x_1, there exists a point $x_2 \in X$ such that $Sx_2 = Bx_1$ and so on. Continuing in this way, we can construct a sequence $\{y_n\}$ in X such that

(3.3) $y_{2n} = Tx_{2n+1} = Ax_{2n}$, $y_{2n+1} = Sx_{2n+2} = Bx_{2n+1}$ for n = 0, 1, 2 . . .

Before proving our main result, we first prove the following Lemma.

Lemma 3.3. Let A, B, S and T be self maps of a complete intuitionistic fuzzy metric spaces (X, M, N, *, ◊) with continuous t-norm * and continuous t-conorm ◊ defined by $a*a \geq a$ and $(1-a) \Diamond (1-a) \leq (1-a)$ for all $a \in [0, 1]$ satisfying the conditions (3.1) and (3.2). Then the sequence $\{y_n\}$ defined by (3.3) is a Cauchy sequence in X.

Proof. For t > 0,

$$\begin{aligned} M(y_{2n}, y_{2n+1}, t) &= M(Ax_{2n}, Bx_{2n+1}, t) \\ &\geq \varphi(\min\{M(Sx_{2n}, Tx_{2n+1}, t), M(Sx_{2n}, Ax_{2n}, t), M(Tx_{2n+1}, Bx_{2n+1}, t)\}) \\ &= \varphi(\min\{M(y_{2n-1}, y_{2n}, t), M(y_{2n-1}, y_{2n}, t), M(y_{2n}, y_{2n+1}, t)\}) \end{aligned}$$

$$(3.4)\qquad > \begin{cases} M(y_{2n-1}, y_{2n}, t), & \text{if } M(y_{2n-1}, y_{2n}, t) < M(y_{2n}, y_{2n+1}, t) \\ M(y_{2n}, y_{2n+1}, t), & \text{if } M(y_{2n-1}, y_{2n}, t) \geq M(y_{2n}, y_{2n+1}, t), \end{cases} \quad \text{and}$$

$$\begin{aligned} N(y_{2n}, y_{2n+1}, t) &= N(Ax_{2n}, Bx_{2n+1}, t) \\ &\leq \psi\,(\max\{N(Sx_{2n}, Tx_{2n+1}, t), N(Sx_{2n}, Ax_{2n}, t), N(Tx_{2n+1}, Bx_{2n+1}, t)\}) \\ &= \psi\,(\max\{N(y_{2n-1}, y_{2n}, t), N(y_{2n-1}, y_{2n}, t), N(y_{2n}, y_{2n+1}, t)\}) \end{aligned}$$

$$(3.5)\qquad < \begin{cases} N(y_{2n-1}, y_{2n}, t), & \text{if } N(y_{2n-1}, y_{2n}, t) > N(y_{2n}, y_{2n+1}, t) \\ N(y_{2n}, y_{2n+1}, t), & \text{if } N(y_{2n-1}, y_{2n}, t) \leq N(y_{2n}, y_{2n+1}, t), \end{cases}$$

as $\varphi(s) > s$ and $\psi(s) < s$ for $0 < s < 1$. Thus $\{M(y_{2n}, y_{2n+1}, t), n \geq 0\}$ is an increasing sequence of positive real numbers in [0, 1] and therefore tends to a limit $l \leq 1$. We assert that $l = 1$. If not, $l < 1$ which on letting $n\to\infty$ in (3.4) one gets $l \geq \varphi(l) > l$ a contradiction yielding thereby $l = 1$. Therefore for every $n \in I^+$, using analogous arguments one can show that $\{M(y_{2n+1}, y_{2n+2}, t), n \geq 0\}$ is a increasing sequence of positive real numbers in [0, 1] which tends to a limit $l = 1$. Also $\{N(y_{2n}, y_{2n+1}, t), n \geq 0\}$ is an decreasing sequence of positive real numbers in [0, 1] and therefore tends to a limit $k \leq 0$. We assert that $k = 0$. If not, $k > 0$ which on letting $n\to\infty$ in (3.5) one gets $k \leq \psi\,(k) < k$ a contradiction yielding thereby $k = 0$. Therefore for every $n \in N$, using analogous arguments one can show that $\{N(y_{2n+1}, y_{2n+2}, t), n \geq 0\}$ is a decreasing sequence of positive real numbers in [0, 1] which tends to a limit $k = 0$. Therefore for every $n \in I^+$

$$M(y_n, y_{n+1}, t) > M(y_{n-1}, y_n, t) \text{ and } \lim_{n\to\infty} M(y_n, y_{n+1}, t) = 1 \text{ and}$$
$$N(y_n, y_{n+1}, t) < N(y_{n-1}, y_n, t) \text{ and } \lim_{n\to\infty} N(y_n, y_{n+1}, t) = 0.$$

Now for any positive integer p, we obtain

$$M(y_n, y_{n+p}, t) \geq M(y_n, y_{n+1}, t/p) * \ldots * M(y_{n+p-1}, y_{n+p}, t/p) \text{ and}$$
$$N(y_n, y_{n+p}, t) \leq N(y_n, y_{n+1}, t/p) \lozenge \ldots \lozenge N(y_{n+p-1}, y_{n+p}, t/p).$$

Since $\lim_{n\to\infty} M(y_n, y_{n+1}, t) = 1$ and $\lim_{n\to\infty} N(y_n, y_{n+1}, t) = 0$ for $t > 0$, it follows that

$$\lim_{n\to\infty} M(y_n, y_{n+p}, t) \geq 1 * 1 * \ldots * 1 = 1 \text{ and}$$
$$\lim_{n\to\infty} N(y_n, y_{n+p}, t) \leq 0 \lozenge 0 \lozenge \ldots \lozenge 0 = 0,$$ which shows that $\{y_n\}$ is a Cauchy sequence in X.

Now we prove our main result as follows:

Theorem 3.1. Let A, B, S and T be self maps of a intuitionistic fuzzy metric spaces $(X, M, N, *, \lozenge)$ with continuous t-norm * and continuous t-conorm $\lozenge$ defined by $a*a \geq a$ and $(1-a) \lozenge (1-a) \leq (1-a)$ for all $a \in [0, 1]$ satisfying (3.1), (3.2), (3.3) and the following condition:

(3.6) one of A(X), B(X), S(X) and T (X) is a complete subspace of X, then

(i) A and S have a point of coincidence,

(ii) B and T have a point of coincidence.

Moreover, if the pairs (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point.

Proof. From Lemma 3.3, $\{y_n\}$ is a Cauchy sequence in X. Now suppose that S(X) is a complete subspace of X, then the subsequence $y_{2n+1} = Sx_{2n+2}$ must get a limit in

S(X). Call it to be u and $v \in S^{-1}u$. Then Sv = u. As $\{y_n\}$ is a Cauchy sequence containing a convergent subsequence $\{y_{2n+1}\}$, therefore the sequence $\{y_n\}$ also converges implying thereby the convergence of $\{y_{2n}\}$ being a subsequence of the convergent sequence $\{y_n\}$. On setting x = v and y = x_{2n+1} in (3.2) one gets (for t > 0),

$M(Av, y_{2n+1}, t) = M(Av, Bx_{2n+1}, t)$

$\geq \varphi(\min\{M(u, y_{2n}, t), M(Av, u, t), M(y_{2n+1}, y_{2n}, t)\})$ and

$N(Av, y_{2n+1}, t) = N(Av, Bx_{2n+1}, t)$

$\leq \psi(\max\{N(u, y_{2n}, t), N(Av, u, t), N(y_{2n+1}, y_{2n}, t)\})$.

Letting limit as $n \to \infty$, we have

$M(Av, u, t) \geq \varphi(M(Av, u, t)) > M(Av, u, t)$ and

$N(Av, u, t) \leq \psi(N(Av, u, t)) < N(Av, u, t)$, imply Av = u = Sv, which shows that the pair (A, S) has a point of coincidence.

As $A(X) \subset T(X)$ and Av = u implies that $u \in T(X)$. Let $w \in T^{-1}u$, then Tw = u. Now using (3.2) again $M(y_{2n}, Bw, t) = M(Ax_{2n}, Bw, t)$

$\geq \varphi(\min\{M(y_{2n-1}, Tw, t), M(y_{2n-1}, y_{2n}, t), M(u, Bw, t)\})$

and

$N(y_{2n}, Bw, t) = N(Ax_{2n}, Bw, t)$

$\leq \psi(\max\{N(y_{2n-1}, Tw, t), N(y_{2n-1}, y_{2n}, t), N(u, Bw, t)\})$.

Taking limit as $n \to \infty$, we have

$M(u, Bw, t) \geq \varphi(M(u, Bw, t)) > M(u, Bw, t)$ and

$N(u, Bw, t) \leq \psi(N(u, Bw, t)) < N(u, Bw, t)$, imply u = Bw. Thus we have shown u = Av = Sv = Bw = Tw, which amounts to say that both pairs have point of coincidence. If one assumes T (X) to be complete, then an analogous argument establishes this claim.

The remaining two cases pertain essentially to the previous cases. Indeed if A(X) is complete, then $u \in A(X) \subset T(X)$ and if B(X) is complete, then $u \in B(X) \subset S(X)$. Thus (i) and (ii) are completely established.

Since the pairs (A, S) and (B, T) are coincidentally commuting at v and w respectively, then Au = A(Sv) = S(Av) = Su and Bu = B(Tw) = T (Bw) = Tu.

If Au ≠ u, then for t > 0

$M(Au, u, t) = M(Au, Bw, t)$

$\geq \varphi(\min\{M(Su, Tw, t), M(Su, Au, t), M(Bw, Tw, t)\}$

$= \varphi(\min\{M(Au, u, t), 1, 1\} = \varphi(M(Au, u, t)) > M(Au, u, t)$ and

$N(Au, u, t) = N(Au, Bw, t)$

$\leq \psi(\max\{N(Su, Tw, t), N(Su, Au, t), N(Bw, Tw, t)\}$

$= \psi(\max\{N(Au, u, t), 0, 0\} = \psi(N(Au, u, t)) < N(Au, u, t)$, imply Au = u. Similarly, one can show that Bu = u. Thus u is a common fixed point of A, B, S and T. The uniqueness of a common fixed point follows easily.

Remark 3.2. If $\min\{M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t)\} = M(Sx, Ty, t)$,

$\max\{N(Sx, Ty, t), N(Ax, Sx, t), N(By, Ty, t)\} = N(Sx, Ty, t)$ by setting A = B and S = T , one obtains a substantially improved version of

Turkoglu,Alaca ,Cho and Yildiz (Theorem 2 ,[24]) as our result is proved under tight commutativity condition without any continuity requirement.

Next, we prove a common fixed point theorem for weakly compatible maps along with E.A. property.

Theorem 3.2. Let A, B, T and S be self mappings of a intuitionistic fuzzy metric space (X, M, N, *, ◊) satisfying (3.1) and the following:

(3.7) pairs (A, S) and (B, T) are weakly compatible,

(3.8) pairs (A, S) or (B, T) satisfies E.A. property,

(3.9)

$M(Ax, By, t) \geq \varphi(\min\{M(Sx, Ty, t), M(Sx, By, t), M(Ty, By, t)\})$ and

$N(Ax, By, t) \leq \psi(\max\{N(Sx, Ty, t), N(Sx, By, t), N(Ty, By, t)\})$ for all $x, y \in X$.

If any one of A(X), B(X), S(X) and T(X) is a complete subspace of X, then A, B,S and T have a unique common fixed point.

Proof. Suppose that (B, T) satisfies the E.A property,then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty}Bx_n = \lim_{n\to\infty}Tx_n = p$ for some $p \in X$.

Since $B(X) \subset S(X)$ there exists a sequence $\{y_n\} \in X$ such that $\lim_{n\to\infty}Bx_n = \lim_{n\to\infty}Sy_n = p$.Now we shall show that $\lim_{n\to\infty}Ay_n = p$.

From (3.9) ,

$M(Ay_n, Bx_n, t) \geq \varphi(\min \{M(Sy_n, Tx_n, t), M(Sy_n, Bx_n, t), M(Tx_n, Bx_n, t)\})$ and

$N(Ay_n, Bx_n, t) \leq \psi(\max \{N(Sy_n, Tx_n, t), N(Sy_n, Bx_n, t), N(Tx_n, Bx_n, t)\})$

Proceeding limit as $n\to\infty$, one obtain $\lim_{n\to\infty}M(Ay_n, Bx_n, t) \geq \varphi(\min \{1, 1, 1\}) \geq 1$ and $\lim_{n\to\infty}N(Ay_n, Bx_n, t) \leq \psi(\max \{0, 0, 0\}) \leq G(0) = 0$. Therefore, $\lim_{n\to\infty}Ay_n = \lim_{n\to\infty}Bx_n = p$. Suppose that S(X) is a complete subspace of X. Then p = Su for some $u \in X$. Subsequently, we have $\lim_{n\to\infty}Ay_n = \lim_{n\to\infty}Bx_n = \lim_{n\to\infty}Tx_n = \lim_{n\to\infty}Sy_n = p = Su$. Now, we shall show that Au = Su. From (3.9), we have

$M(Au, Bx_n, t) \geq \varphi(\min \{M(Su, Tx_n, t), M(Su, Bx_n, t), M(Tx_n, Bx_n, t)\})$ and

$N(Au, Bx_n, t) \leq \psi(\max \{N(Su, Tx_n, t), N(Su, Bx_n, t), N(Tx_n, Bx_n, t)\})$.

Letting limit as $n\to\infty$, we get

$$\lim_{n\to\infty}M(Au, Bx_n, t) \geq \varphi(\min \{M(Su, Su, t), M(Su, Su, t), M(Su, Su, t)\} \geq \varphi(\min \{1, 1, 1\}) \geq F(1) = 1 \text{ and}$$

$$\lim_{n\to\infty}N(Au, Bx_n, t) \leq \psi(\max \{N(Su, Su, t), N(Su, Su, t), N(Su, Su, t)\} \leq \psi(\max \{0, 0, 0\}) \leq G(0) = 0.$$

Therefore, $\lim_{n\to\infty}M(Au, Bx_n, t) = 1$ and $\lim_{n\to\infty}N(Au, Bx_n, t) = 0$,which implies Au = Su.

Since A and S are weak compatible which implies ASu = SAu and then AAu = ASu = SAu = SSu.Since $A(X) \subset T(X)$, so there exists $v \in X$ such that Au = Tv.

Now, we claim that Tv = Bv. Suppose that Tv ≠ Bv. Then (3.9) implies that

$$M(Au, Bv, t) \geq \varphi(\min \{M(Su, Tv, t),M(Su,Bv, t),M(Tv,Bv, t)\}) \geq \varphi(\min \{M(Au, Au, t),M(Au,Bv, t),M(Au,Bv, t)\}) \text{ and}$$

$$N(Au, Bv, t) \leq \psi(\max \{N(Su, Tv, t), N(Su, Bv, t), N(Tv,Bv, t)\}) \leq \psi(\max \{N(Au, Au, t), N(Au, Bv, t), N(Au, Bv, t)\}),$$

implies Au = Bv. Hence Tv = Bv.Thus we have Au = Su = Tv = Bv.

The weak compatibility of B and T implies that BTv = TBv = TTv = BBv. In conclusion, we shall show that Au is the common fixed point of A, B, T and S. Suppose that AAu ≠ Au. Then using (3.9), one obtain

$$\begin{aligned} M(Au, AAu, t) &= M(AAu, Bv, t) \\ &\geq \varphi(\min\{M(SAu, Tv, t), M(SAu, Bv, t), M(Tv, Bv, t)\}) \\ &\geq \varphi(\min\{M(AAu, Au, t), M(AAu, Au, t), 1\}) \text{ and} \end{aligned}$$

$$\begin{aligned} N(Au, AAu, t) &= M(AAu, Bv, t) \\ &\leq \psi(\max\{N(SAu, Tv, t), N(SAu, Bv, t), N(Tv, Bv, t)\}) \\ &\leq \psi(\max\{N(AAu, Au, t), N(AAu, Au, t), 0\}) \text{ ,imply } AAu = Au. \end{aligned}$$

Therefore Au = AAu = SAu is the common fixed point of A and S.
Similarly we prove that Bv is the common fixed point of B and T. Since Au = Bv, Au is the common fixed point of A, B, T and S. The proof is similar when T(X) is assumed to be a complete subspace of X. The cases in which A(X) or B(X) is a complete subspace of X are similar to the cases in which T(X) or S(X), respectively is complete subspace of X, since $A(X) \subset T(X)$ and $B(X) \subset S(X)$. Uniqueness follows easily.

4. Fixed Point Results for different variants of R-weakly commuting mappings

Alace et al. [3] proved the following theorem:

Theorem 4.1. Let A, B, S and T be self maps of a complete intuitionistic fuzzy metric spaces (X, M, N, *, ◊) with continuous t-norm * and continuous t-conorm ◊ defined by $a*a \geq a$ and $(1-a) \lozenge (1-a) \leq (1-a)$ for all $a \in [0, 1]$ satisfying the following conditions:

(a) $A(X) \subset T(X)$, $B(X) \subset S(X)$
(b) S and T are continuous
(c) The pairs (A, S) and (B, T) are compatible
(d) There exists a number $k \in (0,1)$ such that

$$\begin{aligned} M(Ax, By, kt) \geq\ & M(Sx, Ty, t) * M(Ax, Sx, t) * M(By, Ty, t) \\ & * M(By, Sx, 2t) * M(Ax, Ty, t) \text{ and} \end{aligned}$$

$$\begin{aligned} N(Ax, By, kt) \leq\ & N(Sx, Ty, t) \lozenge N(Ax, Sx, t) \lozenge N(By, Ty, t) \\ & \lozenge N(By, Sx, 2t) \lozenge N(Ax, Ty, t), \text{ for all } x, y \in X \text{ and } t > 0. \end{aligned}$$

Then A, B, S and T have a unique common fixed point in X.

Now we prove our main result which improves Theorem 4.1.

Theorem 4.2. Let A, B, S and T be self maps of an intuitionistic fuzzy metric spaces (X,M,N,*, ◊) with continuous t-norm * and continuous t-conorm ◊ defined by $a*a \geq a$ and $(1-a) \lozenge (1-a) \leq (1-a)$ for all $a \in [0, 1]$ satisfying the followings:

(4.1) $A(X) \subset T(X)$, $B(X) \subset S(X)$

(4.2) There exists a number $k \in (0,1)$ such that

$$M(Ax, By, kt) \geq M(Sx, Ty, t) * M(Ax, Sx, t) * M(By, Ty, t) * M(By, Sx, 2t) * M(Ax, Ty, t) \text{ and}$$

$$N(Ax, By, kt) \leq N(Sx, Ty, t) \Diamond N(Ax, Sx, t) \Diamond N(By, Ty, t) \Diamond N(By, Sx, 2t) \Diamond N(Ax, Ty, t), \text{ for all } x, y \in X \text{ and } t > 0.$$

If one of A(X), T(X), B(X) and S(X) is a complete subspace of X, then the pairs (A, S) and (B, T) have a coincidence point. Moreover, if pairs (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point in X.

Proof: By (4.1), since $A(X) \subset T(X)$, for any $x_0 \in X$, there exists a point $x_1 \in X$ such that $Ax_0 = Tx_1$. Since $B(X) \subset S(X)$, for this point x_1, we can choose a point $x_2 \in X$ such that $Bx_1 = Sx_2$.

Inductively, we can find a sequence $\{y_n\}$ in X as follows:

$$y_{2n+1} = Ax_{2n} = Tx_{2n+1} \quad \text{and} \quad y_{2n+2} = Bx_{2n+1} = Sx_{2n+2} \quad \text{for } n = 0, 1, 2, \ldots .$$

By Theorem 17 of Alaca et al. [3], we have $\{y_n\}$ is Cauchy sequence in X. Now suppose that S(X) is a complete subspace of X, then the subsequence $y_{2n+1}= Sx_{2n+2}$ must get a limit in S(X). Call it to be v and $u \in S^{-1}v$. As $\{y_n\}$ is a Cauchy sequence containing a convergent subsequence $\{y_{2n+1}\}$, therefore the sequence $\{y_n\}$ also converges implying thereby the convergence of $y = x_{2n+1}$ in (4.2) one gets for $t > 0$.

Now using (v) and (xi) of Definition 2.3 and (4.2), we have

$$\begin{aligned} M(Au, y_{2n+1}, (k+1)t) &= M(Au, Bx_{2n+1}, (k+1)t) \geq M(Au, Bx_{2n+1}, kt) * M(Bx_{2n+1}, z, t) \\ &\geq M(Su, Tx_{2n+1}, t) * M(Au, Su, t) * M(Bx_{2n+1}, Tx_{2n+1}, t) \\ &\quad * M(Bx_{2n+1}, Su, 2t) * M(Au, Tx_{2n+1}, t) * M(Bx_{2n+1}, z, t) , \end{aligned}$$

and

$$\begin{aligned} N(Au, y_{2n+1}, (k+1)t) &= N(Au, Bx_{2n+1}, (k+1)t) \leq N(Au, Bx_{2n+1}, kt) \Diamond N(Bx_{2n+1}, z, t) \\ &\leq N(Su, Tx_{2n+1}, t) \Diamond N(Au, Su, t) \Diamond N(Bx_{2n+1}, Tx_{2n+1}, t) \\ &\quad \Diamond N(Bx_{2n+1}, Su, 2t) \Diamond N(Au, Tx_{2n+1}, t) \Diamond N(Bx_{2n+1}, z, t). \end{aligned}$$

Taking limit as $n\to\infty$, we obtain Au = v = Su, which shows that pair (A, S) has a coincidence point.

Since $A(X) \subset T(X)$ implies that $v \in T(X)$. Let $w \in T^{-1}v$, then Tw = v. Now, again using (4.2) one obtains

$$\begin{aligned} M(y_{2n}, Bw, kt) &= M(Ax_{2n}, Bw, kt) \\ &\geq M(Sx_{2n}, Tw, t) * M(Ax_{2n}, Sx_{2n}, t) * M(Bw, Tw, t) \\ &\quad * M(Bw, Sx_{2n}, 2t) * M(Ax_{2n}, Tw, t) \text{ and} \end{aligned}$$

$$\begin{aligned} N(y_{2n}, Bw, kt) &= N(Ax_{2n}, Bw, kt) \\ &\leq N(Sx_{2n}, Tw, t) \Diamond N(Ax_{2n}, Sx_{2n}, t) \Diamond N(Bw, Tw, t) \\ &\quad \Diamond N(Bw, Sx_{2n}, 2t) \Diamond N(Ax_{2n}, Tw, t) . \end{aligned}$$

Proceeding limit as $n\to\infty$, we have v = Bw = Tw. Thus Au = Su = Bw = Tw = v, which amounts to say that both pairs have a coincidence point.
If one assumes T(X) to be complete, then an analogous argument establishes this claim.The remaining two cases pertain essentially to the previous cases. Indeed if A(X) is complete, then $v \in A(X) \subset T(X)$ and if B(X) is complete, then $v \in B(X) \subset S(X)$. Thus both pairs have coincidence point.

Since the pairs (A, S) and (B, T) are weakly compatible at u and w respectively, then Av = A(Su) = S(Au) = Sv and Bv = B(Tw) = T(Bw) = Tv. (4.3)
If $Av \neq v$, then from (4.2), we get

$$M(Av, v, kt) = M(Av, Bw, kt) \geq M(Sv, Tw, t) * M(Av, Sv, t) * M(Bw, Tw, t) * M(Bw, Sv, 2t) * M(Av, Tw, t) = M(Sv, v, t) * 1 * 1 * M(Av, v, t) * M(v, Sv, 2t),$$

and

$$N(Av, v, kt) = N(Av, Bw, kt) \leq N(Sv, Tw, t) \lozenge N(Av, Sv, t) \lozenge N(Bw, Tw, t) \lozenge N(Bw, Sv, 2t) \lozenge N(Av, Tw, t),$$

which implies Av = v. Hence Av = Sv = v. Similarly, pair of maps B and T are weakly compatible, we can show Bv = Tv = v. Thus Av = Bv = Sv = Tv = v. Hence v is a common fixed point of A, B, S and T. The uniqueness of the common fixed point follows easily from (4.2). This completes the proof of the theorem.

Remark.4.1. Kumar and Vats [14] proved similar results for complete intuitionistic fuzzy metric spaces.

Theorem 4.3. Theorem 4.2 (Theorem 3.1) remains true if a weakly compatible mappings property is replaced by any one (retaining the rest of the hypotheses) of the following:

(i) R-weakly commuting property(ii) R-weakly commuting of type (A_f) property
(iii) R-weakly commuting of type (A_g) property(iv) R-weakly commuting of type (P) property,(v) weakly commuting property.

Proof: Since all the conditions of Theorem 4.2 (Theorem 3.1) are satisfied, then the existence of coincidence points for both the pairs is insured. Let x be an arbitrary point of coincidence for the pair (A, S), then using R-weak commutativity one gets

$$M(ASx, SAx, t) \geq M(Ax, Sx, t/R) = 1 \text{ , and}$$

$$N(ASx, SAx, t) \leq N(Ax, Sx, t/R) = 0,$$

which amounts to say that ASx = SAx. Thus the pair (A, S) is weakly compatible. Similarly (B, T) commutes at all of its coincidence points. Now applying Theorem 4.2 (Theorem 3.1) one concludes that A, B, S and T have a unique common fixed point.

In case (A, S) is an R-weakly commuting of type (A_f), one obtain

$$M(ASx, S^2x, t) \geq M(Ax, Sx, t/R) = 1, \text{ and}$$

$N(ASx, S^2x, t) \leq N(Ax, Sx, t/R) = 0$,which amounts to say that $ASx = S^2x$.

Now $M(ASx, SAx, t) \geq M(ASx, S^2x, \frac{t}{2}) * M(S^2x, SAx, \frac{t}{2}) = 1 * 1 = 1$, and

$N(ASx,SAx, t) \leq N(ASx, S^2x, \frac{t}{2}) \lozenge N(S^2x, SAx, \frac{t}{2}) = 0 \lozenge 0 = 0$, yielding thereby $ASx = SAx$. Similarly, if pair is R-weakly commuting mappings of type (A_g) or type (P) or weakly commuting and then pair (A, S) also commutes at their points of coincidence. Similarly, one can show that the pair (B, T) is also weakly compatible. Now in view of Theorem 4.2 (Theorem 3.1) , in all four cases A, B, S and T have a unique common fixed point. This completes the proof.

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