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Introducing Preorder to Hilbert C^*-Modules

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Abstract

We comment on the triangle (in)equality for a C^*-valued norm defined on a Hilbert C^*-module V. A C^*-valued norm is used to define a preorder on V. In this preorder, one can interpolate the convex combination of two elements "between" any two elements of V that satisfy certain condition.

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1 Preliminaries and introduction

Let A be a C^*-algebra i. e. a Banach linear space with an involution $*$ and a norm $\|\cdot\|$ with the C^*-property $\|a^*a\| = \|a\|^2, a \in A$. A (right) Hilbert C^*-module V over a C^*-algebra A is a right A-module V with an A-valued inner product $(\cdot,\cdot) : V \times V \to A$ with the following properties:

1. $(x, \alpha y + z) = \alpha\,(x, y) + (x, z)$, for $x, y, z \in V, \alpha \in \mathbf{C}$,
2. $(x, ya) = (x, y)\,a$, for $x, y \in V, a \in A$,
3. $(x, y)^* = (y, x)$, for $x, y \in V$,
4. $(x, x) \geq 0$ and $(x, x) = 0$ if and only if $x = 0$, for $x \in V$,

5. V is complete in the norm $\|x\| = \| (x,x) \|^{1/2}$, $x \in V$.

Every C^*-algebra A becomes a Hilbert C^*-module over itself with the inner product $(a,b) = a^*b$, $a, b \in A$. Furthermore, a Hilbert C^*-module norm on A coincides with the C^*-norm on A.
Throughout, V denotes a Hilbert C^*-module over a C^*-algebra A.

On V there is the C^*-valued "norm" $|\cdot|$ defined by

$$|x| = (x,x)^{\frac{1}{2}}, \ x \in V.$$

For every $x \in V$, $|x|$ is positive (a positive square root of $(x,x) \in A$) and we have $|\alpha x| = |\alpha||x|$, $\alpha \in \mathbf{C}$. The list of norm properties of the "norm" $|\cdot|$ stops here.

The triangle inequality $|x+y| \leq |x|+|y|$ need not hold on V ([5], p. 4). R. Harte gave example of this fact in case of C^*-algebras in [3]. We begin Section 2 by presenting another example with essentially different proof. After that, we comment on conditions for the triangle inequality to hold.

By [2], the triangle equality $|x + y| = |x| + |y|$ is characterized by the property $(x,y) = |x||y|$. We use this property extensively in Section 3. There we use the norm $|\cdot|$ to define a preorder $\preceq$ on V by setting: $x \preceq y \Leftrightarrow |x| \leq |y|$, $x, y \in V$. In Theorem 4 we prove that for $x, y \in V$ such that $(x,y) = |x||y|$, $x \preceq y$ is equivalent to $x \preceq \alpha x + (1-\alpha)y \preceq y$, for a real number α, $0 \leq \alpha \leq 1$. In particular, for $x, y \in V$ such that $x \preceq y$ and $(x,y) = |x||y|$ we can interpolate the convex combination of x and y "between" x and y .

2 On the triangle (in)equality

Let V be a Hilbert C^*-module over a C^*-algebra A. The triangle inequality $|x + y| \leq |x| + |y|$, $x, y \in V$ is in fact an order relation on A. Recall that for selfadjoint $a, b \in A$ we have $a \leq b$ if and only if $b - a \geq 0$ (i.e. $b - a$ is selfadjoint with positive real spectrum). We provide an example that the triangle inequality fails to be true on a Hilbert C^*-module V.

Example 1 *Denote by $M_2(\mathbf{C})$ the space of 2×2 matrices with complex entries. With matrix multiplication and involution given by $(a_{ij})^* = (a_{ji}^*)$, $M_2(\mathbf{C})$ is a C^*-algebra, hence a Hilbert C^*-module as indicated in the introduction. Let:*

$$x = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

We have

$$|x| = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix}, |y| = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{2} \end{pmatrix}.$$

Further,

$$x + y = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = (x+y)^*.$$

The above matrix is positive, hence

$$|x+y| = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Let $\mathbf{b} = |x| + |y| - |x+y|$, *i. e.*

$$\mathbf{b} = \begin{pmatrix} \sqrt{2}-1 & -1 \\ -1 & \sqrt{2}-1 \end{pmatrix}.$$

In general (see [8]) a selfadjoint matrix $\mathbf{a} \in M_n(A)$ *(where a* C^**-algebra* A *is taken to be faithfully represented on a Hilbert space* $\mathbf{H}$*, and therefore* $M_n(A)$ *is considered as a* C^**-algebra of operators on* $\mathbf{H}^n = \mathbf{H} \oplus \ldots \oplus \mathbf{H}$*) is positive if and only if* $(\mathbf{a}\xi, \xi) \geq 0$ *for all vectors* $\xi \in \mathbf{H}^n$*. If we take a vector* $\xi = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ *to test matrix* $\mathbf{b}$ *for positivity, we get*

$$(\mathbf{b}\xi, \xi) = (\mathbf{b}\xi)^*\xi = \begin{pmatrix} \sqrt{2}-2 & \sqrt{2}-2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2\sqrt{2} - 4 < 0.$$

■

Under what conditions does the triangle inequality hold on a Hilbert C^*-module? Consider first a C^*-algebra case. It was proved in [1] that for a C^*-algebra A with unit e, for every a, b in A and arbitrary $\epsilon \geq 0$ there are unitaries $u, v \in A$ such that

$$|a+b| \leq u|a|u^* + v|b|v^* + \varepsilon e.$$

In order to get the triangle inequality on A, it suficces for $|a|$ and $|b|$ to fall into $Z(A)$, the center of A. In the case of Hilbert C^*-modules we have the similar result.

Theorem 1 *(Theorem 2 in [4]) Let* V *be a Hilbert* C^**-module over a* C^**-algebra* A*. For* $x, y \in V$ *such that* $|x|, |y| \in Z(A)$*, we have*

$$|x+y| \leq |x| + |y|.$$

The condition on $x, y \in V$ for which the triangle equality $|x+y| = |x|+|y|$ holds is already known.

Theorem 2 *(Theorem 2.3 of [2]) Let V be a Hilbert C^*-module over a C^*-algebra A and let $x, y \in V$. Then $|x+y| = |x|+|y|$ if and only if $(x, y) = |x||y|$.*

The next Theorem asserts that the condition $(x, y) = |x||y|$, together with $|x| \leq |y|$ is both necessary and sufficient for $|y-x| = |y| - |x|$ to hold.

Theorem 3 *Let V be a Hilbert C^*-module and $x, y \in V$. Then ($|x| \leq |y|$ and $(x, y) = |x||y|$) if and only if $|y| = |x| + |y-x|$.*

Proof: Let $x, y \in V$ be such that $|x| \leq |y|$ and $(x, y) = |x||y|$. Notice that

$$\begin{aligned} |y-x|^2 &= (y-x, y-x) = (y, y) - (y, x) - (x, y) + (x, x) = \\ &= |y|^2 - |y||x| - |x||y| + |x|^2 = \\ &= |y|(|y| - |x|) - |x|(|y| - |x|) = (|y| - |x|)^2 \end{aligned}$$

and the claim follows.
Now suppose that $|y| = |x| + |y-x|$. Then obviously $|x| \leq |y|$. We can write the supposition in the equivalent form $|x+y-x| = |x|+|y-x|$. This equality is by Theorem 2 equivalent to $(x, y-x) = |x||y-x|$. Now

$$(x, y-x) = |x||y-x| = |x|(|y| - |x|) = |x||y| - |x|^2,$$

hence $(x, y) = |x||y|$ as claimed. ■

The following fact will be used in the next section.

Proposition 1 *Let V be a Hilbert C^*-module over a C^*-algebra A and let $x, y \in V$ be such that $(x, y) = |x||y|$. Let α be a real number, $0 \leq \alpha \leq 1$. Then*

$$|\alpha x + (1-\alpha)y| = \alpha|x| + (1-\alpha)|y|.$$

Proof: We have

$$\begin{aligned} |\alpha x + (1-\alpha)y|^2 &= (\alpha x + (1-\alpha)y, \alpha x + (1-\alpha)y) = \\ &= \alpha^2|x|^2 + \alpha(1-\alpha)|x||y| + \alpha(1-\alpha)|y||x| + (1-\alpha)^2|y|^2 = \\ &= \alpha|x|(\alpha|x| + (1-\alpha)|y|) + (1-\alpha)|y|(\alpha|x| + (1-\alpha)|y|) = \\ &= (\alpha|x| + (1-\alpha)|y|)^2. \end{aligned}$$

The set of all positive elements of A is a cone, so the claim follows by taking the square root. ■

3 Interpolation of elements in a preorder on V

Definition 1 *Let V be a Hilbert C^*-module over a C^*-algebra A and $x, y \in V$. We define*

$$x \preceq y \overset{def}{\Longleftrightarrow} |x| \leq |y|.$$

The relation $\preceq$ is reflexive ($x \preceq x$) and transitive ($x \preceq y$ and $y \preceq z \Rightarrow x \preceq z$), but not antisymmetric ($x \preceq y$ and $y \preceq x \not\Rightarrow x = y$), and therefore it is a preorder on V.

Remark 1

1. If we consider a C^*-algebra A as a Hilbert C^*-module with the inner product $(a, b) = a^*b$ and take two positive $a, b \in A$, then obviously $a \preceq b$ if and only if $a \leq b$.

2. If V is a Hilbert C^* module over a commutative C^*-algebra A, then $x \preceq y$ if and only if $(x, x) \leq (y, y)$. Namely, we know from [6] that $|x| \leq |y|$ implies $(x, x) \leq (y, y)$. (The opposite implication is allways true on A.)

3. Let V be a Hilbert C^*-module over a C^*-algebra A. Due to Theorem 2, for $x, y \in V$ such that $(x, y) = |x||y|$ we have $x \preceq x + y$ and $y \preceq x + y$.

The following Lemma recalls equivalences for the order on a C^*-algebra A.

Lemma 1 *Let A be a C^*-algebra, α a real number and $a, b \in A$ selfadjoint. The relation $a \leq b$ is equivalent to every of the following order relations on A:*

$$\begin{aligned} a \leq \alpha a + (1-\alpha)b \leq b \quad &for \quad 0 \leq \alpha \leq 1, \\ b \leq \alpha a + (1-\alpha)b \quad &for \quad \alpha \leq 0, \\ \alpha a + (1-\alpha)b \leq a \quad &for \quad \alpha \geq 1. \end{aligned}$$

We have the next generalization of the first relation to a preorder on V.

Theorem 4 *Let V be a Hilbert C^*-module and $x, y \in V$ such that $(x, y) = |x||y|$. For a real number α, $0 \leq \alpha \leq 1$ we have*

$$x \preceq y \Leftrightarrow x \preceq \alpha x + (1-\alpha)y \preceq y.$$

Proof: By Proposition 1, the condition $(x, y) = |x||y|$ implies $|\alpha x+(1-\alpha)y| = \alpha|x| + (1-\alpha)|y|$, for $0 \leq \alpha \leq 1$. The claims now follow after noticing that

$$x \preceq y \Leftrightarrow |\alpha x + (1-\alpha)y| - |x| = (1-\alpha)(|y| - |x|) \geq 0$$

and

$$x \preceq y \Leftrightarrow |y| - |\alpha x + (1-\alpha)y| = \alpha(|y| - |x|) \geq 0.$$

■

We can continue with interpolation of elements in a preorder on V. First notice the following.

Lemma 2 *Let V be a Hilbert C^*-module and $x, y \in V$ such that $(x, y) = |x||y|$. For a real number α, $0 \leq \alpha \leq 1$ we have:*

$$\begin{aligned} (x, \alpha x + (1-\alpha)y) &= |x||\alpha x + (1-\alpha)y|, \\ (\alpha x + (1-\alpha)y, y) &= |\alpha x + (1-\alpha)y||y|. \end{aligned}$$

Proof: Indeed,

$$\begin{aligned} (x, \alpha x + (1-\alpha)y) &= \alpha|x|^2 + (1-\alpha)|x||y| = |x|(\alpha|x| + (1-\alpha)|y| = \\ &= |x||\alpha x + (1-\alpha)y|. \end{aligned}$$

Similarly, $(\alpha x + (1-\alpha)y, y) = |\alpha x + (1-\alpha)y||y|$. ■

The relation $x \preceq y$ implies $x \preceq \alpha x + (1-\alpha)y \preceq y$ for $0 \leq \alpha \leq 1$. With the same reasoning, from the relations $x \preceq \alpha x + (1-\alpha)y$ and $\alpha x + (1-\alpha)y \preceq y$ we get the next interesting result.

Theorem 5 *Let V be a Hilbert C^*-module and $x, y \in V$ such that $(x, y) = |x||y|$. If $x \preceq y$, then for real numbers α, β, $0 \leq \alpha, \beta \leq 1$ we have*

$$(\alpha + \beta - \alpha\beta)x + [1 - (\alpha + \beta - \alpha\beta)]y \preceq \alpha x + (1-\alpha)y \preceq \alpha\beta x + (1-\alpha\beta)y.$$

Proof: For $0 \leq \alpha \leq 1$, by Theorem 4 we have $x \preceq \alpha x + (1-\alpha)y \preceq y$. Further, by Lemma 2 we have $(x, \alpha x + (1-\alpha)y) = |x||\alpha x + (1-\alpha)y|$ and $(\alpha x + (1-\alpha)y, y) = |\alpha x+(1-\alpha)y||y|$. Now, for $0 \leq \beta \leq 1$, again by Theorem 4 we have:

$$x \preceq \beta x + (1-\beta)[\alpha x + (1-\alpha)y] \preceq \alpha x + (1-\alpha)y,$$

$$\alpha x + (1-\alpha)y \preceq \beta(\alpha x + (1-\alpha)y) + (1-\beta)y \preceq y.$$

In particular,

$$(\alpha + \beta - \alpha\beta)x + [1 - (\alpha + \beta - \alpha\beta)]y \preceq \alpha x + (1-\alpha)y \preceq \alpha\beta x + (1-\alpha\beta)y,$$

as claimed. ■

At the end, let us mention that it seems promising to consider this preorder in the setting of Finsler modules introduced in [7].

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