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Common Fixed Point Theorems for Fuzzy Mappings in Quasi-Metric Spaces

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Abstract

In this paper we prove common fixed point theorems for a class of fuzzy mappings in Smyth-complete quasi-metric spaces. Well-known theorems are special case of our results.

Keywords: Fuzzy mapping; Fixed point; Quasi-metric space; Smyth-complete; Left K-complete.

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1. Introduction

Heilpern [5] first introduced the concept of fuzzy mappings and proved a fixed point theorem for fuzzy mappings which is the generalization of the fixed point theorem for multivalued mappings of Nadler [8]. Recently, Gregori and Romaguera [4] showed the importance of the Smyth completeness regarding many areas of research in theoretical computer science and proved some fixed point theorems for fuzzy mappings in Smyth-complete and left K-complete quasi-metric spaces, respectively. Also Telci and Fisher [13] obtained a fixed point theorem for these mappings.

In this paper, we consider a generalized contractive type condition involving fuzzy mappings in Smyth-complete quasi-metric spaces and we establish a common fixed point theorem which extends many theorems obtained by many authors.

2. Basic notions and preliminary results

In the following, the letter Γ denotes the set of positive integers. If A is a subset of a topological space (X,τ), we will denote by $cl_\tau A$ the closure of A in (X,τ).

A quasi-metric on a nonempty set X is a nonnegative real valued function d on $X \times X$, such that, for all $x, y, z \in X$:

(a) $d(x,y) = d(y,x) = 0 \Leftrightarrow x = y$, and (b) $d(x,y) \le d(x,z) + d(z,y)$.

A pair (X,d) is called a quasi-metric space, if d is a quasi-metric on X.

Each quasi-metric d on X induces a T_0 topology $\tau(d)$ on X, which has a base the family of all d-balls $\{B_d(x,r) : x \in X, r > 0\}$, where $B_d(x,r) = \{y \in X : d(x,y) < r\}$.

If d is a quasi-metric on X, then the function d^{-1} defined on $X \times X$ by $d^{-1}(x,y) = d(y,x)$ is also a quasi-metric on X. By $d \wedge d^{-1}$ we denote $\min\{d, d^{-1}\}$ and also we denote d^S the metric on X by $d^S(x,y) = \max\{d(x,y), d(y,x)\}$, for all $x, y \in X$.

A sequence $(x_n)_{n\in\Gamma}$ in a quasi-metric space (X,d) is called left K-Cauchy [10], if for each $\varepsilon > 0$ there is a $n_\varepsilon \in \Gamma$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \in \Gamma$ such that $m \ge n \ge n_\varepsilon$.

The quasi-metric space (X,d) is said to be left K-complete [10] if each left K-Cauchy sequence in (X,d) converges (with respect to the topology $\tau(d)$). A quasi metric space (X,d) is said to be Smyth-complete [7] if each left K-Cauchy sequence in (X,d) converges in the metric space (X,d^S). Clearly, every Smyth-complete quasi-metric space is left K-complete. In general the converse implication does not hold.

Let (X,d) be a quasi-metric space and let $\mathscr{K}_0^S(X)$ be the collection of all nonempty compact subset of the metric space (X,d^S). Then the Hansdorff distance H_d on $\mathscr{K}_0^S(X)$ is defined by

$$H_d(A,B) = \max\{\sup d(a,B) : a \in A, \sup d(A,b) : b \in B\}$$

whenever $A, B \in \mathscr{K}_0^S(X)$.

A fuzzy set on X is an element of I^X where $I = [0,1]$. If A is a fuzzy set in X, then the number $A(x)$ is called the grade of membership of x in A. The α-level set of A, denoted by A_α, is defined by

$A_\alpha = \{x \in X : A(x) \geq \alpha\}$ for each $\alpha \in (0,1]$, and $A_0 = \overline{\{x \in X : A(x) > 0\}}$ where the closure is taken in (X, d^S).

Definition 2.1[4] Let (X,d) be a quasi-metric space. A fuzzy set A in the quasi-metric space (X,d) will be called an approximate quantity. The family $\mathscr{A}(X)$ of all fuzzy sets on (X,d) is defined by

$$\mathscr{A}(X) = \{A \in I^X : A_\alpha \text{ is } d^S\text{-compact for each } \alpha \in [0,1] \text{ and } \sup\{A(x) : x \in X\} = 1\}.$$

Definition 2.2[5] Let (X,d) be a quasi-metric space and let $A, B \in \mathscr{A}(X)$ and $\alpha \in [0,1]$. Then we define,

$$p_\alpha(A,B) = \inf\{d(x,y) : x \in A_\alpha, y \in B_\alpha\} = d(A_\alpha, B_\alpha)$$
$$D_\alpha(A,B) = H_d(A_\alpha, B_\alpha)$$
$$p(A,B) = \sup\{p_\alpha(A,B) : \alpha \in [0,1]\}$$
$$D(A,B) = \sup\{D_\alpha(A,B) : \alpha \in [0,1]\}.$$

Definition 2.3[4] A fuzzy mapping on a quasi-metric space (X,d) is a function F defined on X, which satisfies the following two conditions:

(1) $F(x) \in \mathscr{A}(X)$ for all $x \in X$
(2) If $a, z \in X$ such that $(F(z))(a) = 1$ and $p(a, F(a)) = 0$, then $(F(a))(a) = 1$.

We need the following lemmas for our main result which was given [4].

Lemma 2.4[4] Let (X,d) be a quasi-metric space. Then, for each $A \in \mathscr{A}(X)$ there exists $p \in X$ such that $A(p) = 1$.

Lemma 2.5[4] Let (X,d) be a quasi-metric space and let $A, B \in \mathscr{A}(X)$ and $x \in A_1$. There exists $y \in B_1$ such that $d(x,y) \leq D_1(A,B)$.

Lemma 2.6[4] Let (X,d) be a quasi-metric space and let $A,B\in \mathcal{W}(X)$.
Then $p(A,B)=p_1(A,B)$

Lemma 2.7[4] Let (X,d) be a quasi-metric space and let $A\in \mathcal{W}(X)$ and $y\in A_1$.Then $p(x,A)\le d(x,y)$ for each $x\in X$.

Lemma 2.8[4] Let (X,d) be a quasi-metric space and let $A\in \mathcal{W}(X)$. Then $p_\alpha(x,A)\le d(x,y)+p_\alpha(y,A)$ for each $x,y\in X$ and $\alpha\in[0,1]$.

Lemma 2.9[4] Let (X,d) be a quasi-metric space and let $A\in \mathcal{W}(X)$ and $x\in A_1$. Then $p_\alpha(x,B)\le D_\alpha(A,B)$ for each $B\in \mathcal{W}(X)$ and each $\alpha\in[0,1]$.

Lemma 2.10[4] Let (X,d) be a quasi-metric space and let $A\in \mathcal{W}(X)$. If $p(x,A)=0$, then there is $y\in cl_{\tau(d^{-1})}\{x\}$ such that $A(y)=1$.

Definition 2.11[4] We say that a fuzzy mapping F on a quasi-metric space (X,d) has a fixed point if there exists $a\in X$ such that $(F(a))(a)=1$.

3. Common fixed point theorem for fuzzy mappings.

We consider the set G of all continuous functions $g:[0,\infty)^5\to[0,\infty)$ with the following properties:

(i) g is non-decreasing in the 2nd, 3nd, 4nd, 5nd variable,

(ii) If $u,v\in[0,\infty)$ are such that $u\le g(v,v,u,u+v,0)$ or $u\le g(v,u,v,0,u+v)$ then $u\le hv$, where $0<h<1$ is a given constant,

Theorem 3.1 Let (X,d) be a Smyth-complete quasi-metric space and let F_1,F_2 be fuzzy mappings on X into $\mathcal{W}(X)$. If there exists a $g\in G$ such that for all $x,y\in X$

$$D(F_1(x),F_2(y))\le g(d(x,y),p(x,F_1(x)),p(y,F_2(y)),p(x,F_2(y),p(y,F_1(x))) \quad (3.1)$$

then F_1,F_2 have common fixed point.

Proof. Let x_0 be an arbitrary point in X. By lemma 2.4, there exists $x_1\in X$ such that $(F_1(x_0))(x_1)=1$. By lemmas 2.4 and 2.5, there exists $x_2\in X$ such that $(F_2(x_1))(x_2)=1$ and $d(x_1,x_2)\le D_1(F_1(x_0),F_2(x_1))$.
Then we obtain

$$d(x_1,x_2) \le D_1(F_1(x_0),F_2(x_1)) \le D(F_1(x_0),F_2(x_1))$$
$$\le g(d(x_0,x_1),p(x_0,F_1(x_0)),p(x_1,F_2(x_1)),p(x_0,F_2(x_1),p(x_1,F_1(x_0)))$$

By lemma 2.7 $p(x_0,F_1(x_0)) \le d(x_0,x_1)$, $p(x_1,F_2(x_1)) \le d(x_1,x_2)$, $p(x_0,F_2(x_1)) \le d(x_0,x_2) \le d(x_0,x_1)+d(x_1,x_2)$, and $p(x_1,F_1(x_0))=0$.

Thus we have

$$d(x_1,x_2) \le g(d(x_0,x_1),d(x_0,x_1),d(x_1,x_2),d(x_0,x_1)+d(x_1,x_2),0)$$

By (ii) condition we have

$$d(x_1,x_2) \le hd(x_0,x_1),\ \ 0<h<1.$$

Again

$$d(x_2,x_3) \le hd(x_1,x_2) \le h^2 d(x_0,x_1).$$

By induction, we produce a sequence $(x_n)_{n\in\Gamma}$ in X such that for $k \ge 0$,

$$(F_1(x_{2k}))(x_{2k+1})=1, \quad (F_2(x_{2k+1}))(x_{2k+2})=1$$

and

$$d(x_n,x_{n+1}) \le h^n d(x_0,x_1).$$

Furthemore, for $m>n$,

$$\begin{aligned} d(x_n,x_m) &\le d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+...+d(x_{m-1},x_m) \\ &\le (h^n+h^{n+1}+...+h^{m-1})d(x_0,x_1) \\ &\le \frac{h^n}{1-h}d(x_0,x_1) \end{aligned}$$

It follows that $(x_n)_{n\in\Gamma}$ is a left K-Cauchy sequence in the Smyth-complete quasi-metric space (X,d) and so there exists a $z\in X$ such that $d^S(z,x_n)\to 0$. Now by lemma 2.8, we have $p_1(z,F_2(z) \le d(z,x_{2n+1})+p_1(x_{2n+1},F_2(z))$ for all $n\in N$.

So by lemmas 2.6, 2.9 and inequality (3) we have

$$p(z,F_2(z)\le d(z,x_{2n+1})+D(F_1(x_{2n}),F_2(z))$$
$$\le d(z,x_{2n+1})+g(d(x_{2n},z),p(x_{2n},F_1(x_{2n})),p(z,F_2(z)),$$
$$p(x_{2n},F_2(z)),p(z,F_1(x_{2n})))$$

By lemmas 2.7, 2.8, we have

$$p(x_{2n},F_1(x_{2n})\le d(x_{2n},x_{2n+1}),\qquad p(z,F_1(x_{2n}))\le d(z,x_{2n+1}),$$
$$p(x_{2n},F_2(z)\le d(x_{2n},z)+p(z,F_2(z))$$

It follows that

$$p(z,F_2(z)\le d(z,x_{2n+1})+g(d(x_{2n},z),d(x_{2n},x_{2n+1}),p(z,F_2(z)),$$
$$d(x_{2n},z)+p(z,F_2(z)),d(z,x_{2n+1}))$$

As $n\to\infty$, we have

$$p(z,F_2(z))\le g(0,0,p(z,F_2(z)),p(z,F_2(z)),0).$$

By (ii) condition we have $p(z,F_2(z)=0$. Similarly, we have $p(z,F_1(z)=0$. So by lemma 2.10 there exists $z^*\in cl_{\tau(d^{-1})}\{z\}$ such that $(F_2(z))(z^*)=1$. Since $z^*\in cl_{\tau(d^{-1})}\{z\}$ we have $d(z,z^*)=0$.

We will prove now that z^* is fixed point of F_2.

By $d(x_n,z^*)\le d(x_n,z)+d(z,z^*)$ we have $d(x_n,z^*)\to 0$ and $x_n\to z^*$ as $n\to\infty$.

By Lemmas 2.8 and 2.9 we have

$$p(z^*,F_2(z^*))\le d(z^*,x_{2n+1})+p(x_{2n+1},F_2(z^*))$$
$$\le d(z^*,x_{2n+1})+D(F_1(x_{2n}),F_2(z^*)) \tag{3.2}$$

Using the inequality (3.1), we have

$$D(F_1(x_{2n}),F_2(z^*))\le g(d(x_{2n},z^*),p(x_{2n},F_1(x_{2n})),p(z^*,F_2(z^*)),$$
$$p(x_{2n},F_2(z^*)),p(z^*,F_1(x_{2n}))) \tag{3.3}$$

By Lemmas 2.7 and 2.9 we have

$$p(x_{2n}, F_1(x_{2n})) \le d(x_{2n}, x_{2n+1}),$$
$$p(x_{2n}, F_2(z^*)) \le d(x_{2n}, z^*) + p(z^*, F_2(z^*)), \quad p(z^*, F_1(x_{2n})) \le d(z^*, x_{2n+1})$$

So

$$\begin{aligned} p(z^*, F_2(z^*)) &\le d(z^*, x_{2n+1}) + D(F_1(x_{2n}), F_2(z^*)) \\ &\le d(z^*, x_{2n+1}) + g(d(x_{2n}, z^*), d(x_{2n}, x_{2n+1}), p(z^*, F_2(z^*)), \\ &\quad d(x_{2n}, z^*) + p(z^*, F_2(z^*)), d(z^*, x_{2n+1})) \end{aligned} \tag{3.4}$$

As $n \to \infty$ we have

$$p(z^*, F_2(z^*)) \le g(0, 0, p(z^*, F_2(z^*)), p(z^*, F_2(z^*)), 0)$$

So, by (ii) condition

$$p(z^*, F_2(z^*)) \le hp(z^*, F_2(z^*)) \text{ with } 0 < h < 1,$$

and so

$$p(z^*, F_2(z^*)) = 0 \tag{3.5}$$

By Definition 2.3 we see that $(F_2(z^*), z^*) = 1$ and so z^* is fixed point of F_2.
Similarly, by Lemma 2.10 there exists $z_1^* \in cl_{\tau(d^{-1})}\{z\}$ such that $(F_1(z))(z_1^*) = 1$ and z_1^* is fixed point of F_1.
By the triangle inequality $d(z^*, z_1^*) \le d(z^*, x_n) + d(x_n, z_1^*)$ and $d(z_1^*, z^*) \le d(z_1^*, x_n) + d(x_n, z^*)$.
So as $n \to \infty, d(z^*, z_1^*) = d(z_1^*, z^*) = 0$ and F_1, F_2 have common fixed point.

If in theorem 3.1 $F_1 = F_2 = F$, we get the following fixed point theorem.

Theorem 3.2 Let (X, d) be a Smyth-complete quasi-metric space and let F be fuzzy mapping on X into $\mathscr{I}(X)$. If there exists a $g \in G$ such that for all $x, y \in X$

$$D(F(x), F(y)) \le g(d(x, y), p(x, F(x)), p(y, F(y)), p(x, F(y), p(y, F(x)))$$

then F have a fixed point.

Corollary 3.3 [4;Theorem 1] Let (X, d) be a Smyth-complete quasi-metric space and let F be fuzzy mapping from X into $\mathscr{I}(X)$. If there exists a constant $h,\ 0 \le h < 1$, such that for each $x, y \in X$

$$D(F(x),F(y)) \leq h\max\{d(x,y), p(x,F(x)), p(y,F(y)), [p(x,F(y))+p(y,F(x))]/2\}$$

Then, F has a fixed point.

Proof. We consider the function $g:[0,\infty)^5 \to [0,\infty)$ defined by

$$g(x_1,x_2,x_3,x_4,x_5) = h\max\{x_1,x_2,x_3,[x_4+x_5]/2\}$$

Since $g \in G$ we can apply Theorem 3.2 and obtain Corollary 3.3.

Corollary 3.4 [13;Theorem 3.4] Let (X,d) be a Smyth-complete quasi-metric space and let F be fuzzy mapping from X into $\mathscr{W}(X)$. If there exists a constant h, $0 \leq h < 1$, such that for each $x, y \in X$

$$D(F(x),F(y)) \leq h\max\{d(x,y), p(x,F(x)), p(y,F(y)), [p(x,F(y))+p(y,F(x))]/2, r[p(x,F(y))\cdot p(y,F(x))]^{1/2}\}$$

Where $rh < 2$,then F has a fixed point.

Proof. We consider the function $g:[0,\infty)^5 \to [0,\infty)$ defined by

$$g(x_1,x_2,x_3,x_4,x_5) = h\max\{x_1,x_2,x_3,[x_4+x_5]/2, r[x_4\cdot x_5]^{1/2}\}$$

Since $g \in G$ we can apply Theorem 3.2 and obtain Corollary 3.4.

Remark 3.5 Theorems 3.1 and 3.2 generalize the results obtained in the [4, 9, 13, et al] in the settings of Smyth-complete quasi-metric spaces.

Note that Smyth-completeness cannot be relaxed to left K-completeness in Theorem 3.1 (see [4, Example 5]) hence the result of the Theorem 3.1 do not remain valid in the settings of left K-complete quasi-metric spaces.

4. Conclusions

In this paper we proved common fixed point theorems for a pair of the classes G of fuzzy mappings in the settings of Smyth-complete quasi-metric spaces. Well known results are special case of our results. The Smyth completeness cannot be relaxed to left K-completeness, therefore the issue of finding a more general class of fuzzy mappings such as the conclusions of our theorem stand, remains open.

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