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New Approaches for Generalized Continuous Functions

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Abstract

For a topological space (X, τ) and non open set B, the topology $\tau(B)$ generated by τ and B is finer than τ. For each $B \notin \tau$, new class of open sets arise, these classes are applied in introducing new forms of generalized closed sets and new forms of generalized continuous functions. Properties of these classes are investigated, examples and counter examples are given and a comparison between new types and these similar classes are obtained.

Keywords: Topological space, generalized closed set, B-generalized closed set, continuous function, generalized continuous function

1. Introduction

The study and research about near open and near closed sets have specific importance, it helps in the modifications of topological spaces via adding new concepts and facts or constructing new classes. In 1970, Levine[9] introduced g-closed sets in topological spaces as a generalization of closed sets. Arya and Nour in 1990 [3] defined and studied the notion of gs-closed sets. Maki et al. in 1994 [12] introduced the concept of αg-closed sets. In 1995 Dontchev [6] introduced gsp-closed sets. The notion of gp-closed sets was introduced and investigated by Noiri, Maki and Umehara in 1998 [17] and in 1996 [13]. Levine[10], Mashhour et al.[14, 15], Abd El-Monsef et al.[1], Balachandran et al.[4], Devi et al.[5], Dontchev[6] and Gnanambal [7] introduced and investigated semi-continuity, pre-continuity, α-continuity, β-continuity, g-continuity, gs-continuity, gsp-continuity, αg-continuity and gpr-continuity respectively which are weaker than continuity in topological spaces.

2. Preliminaries

A topological space [8] is a pair (X, τ) consisting of a set X and family τ of subsets of X satisfying the following conditions:
(i) $\varnothing\in\tau$ and $X\in\tau$.
(ii) τ is closed under arbitrary union.
(iii) τ is closed under finite intersection.

Throughout this paper (X, τ) denotes a topological space, the elements of X are called points of the space, the subsets of X belonging to τ are called open sets in the space, the complement of the subsets of X belonging to τ are called closed sets in the space. The family of all open sets of (X, τ) is denoted by τ and the family of all closed sets of (X, τ) is denoted by C(X). For a subset A of a space (X, τ), Cl(A), Int(A) denote the closure of A and is given by $Cl(A)=\cap\{F\subseteq X:A\subseteq F$ and F is closed set in $\tau\}$. Evidently, Cl(A) is the smallest closed subset of X which contains A, the interior of A and is given by $Int(A) = \cup\{G \subseteq X : G \subseteq A$ and $G\in\tau\}$. Evidently, Int(A) is the largest open subset of X which contained in A, A^c or X\A denote the complement of A in X.

Let us recall the following definitions, which are useful in the sequel.

Definition 2.1. A subset A of a space (X, τ) is called:

(i) semi-open [10] (briefly s-open) if $A \subseteq Cl (Int((A))$,

(ii) semi-preopen briefly (sp-open) [2] (=β-open [1]) if $A \subseteq Cl (Int(Cl(A)))$,

(iii) pre-open [14] (briefly p-open) if $A \subseteq Int(Cl(A))$,
(iv) α-open [16] if $A \subseteq Int(Cl(Int(A)))$.

The complement of a s-open (resp. β-open, p-open and α-open) set is called s-closed (resp. β-closed, p-closed and α-closed) set. The family of all s-open (resp. p-open, α-open and β-open) sets of (X, τ) is denoted by SO(X) (resp. PO(X), τ_α and βO(X)). The family of all s-closed (resp. p-closed, α-closed and β-closed) sets of (X, τ) is denoted by SC(X) (resp. PC(X), αC(X), and βC(X)).

The semi-closure (resp. α-closure, pre-closure, semi-pre-closure) of a subset A of (X,τ), denoted by sCl(A) (resp. αCl(A), pCl(A), spCl(A)) and defined to be the intersection of all semi-closed (resp. α-closed, p-closed, sp-closed) sets containing A.

Definition 2.2. A subset A of a space (X, τ) is said to be:

(i) generalized closed [9] (briefly, g-closed) if $Cl(A)\subseteq U$ whenever $A\subseteq U$ and U is open in (X, τ),

(ii) generalized semi-closed [3] (briefly, gs-closed) if sCl(A)$\subseteq$U whenever A$\subseteq$U and U is open in (X, τ),

(iii) generalized semi-preclosed [6] (briefly, gsp-closed) if spCl(A)$\subseteq$U whenever A$\subseteq$U and U is open in (X, τ),

(iv) α-generalized closed [12] (briefly, αg-closed) if αCl(A)$\subseteq$U whenever A$\subseteq$U and U is open in (X, τ),

(v) generalized preclosed [17] (briefly, gp-closed) if pCl(A)$\subseteq$U whenever A$\subseteq$U and U is open in (X, τ).

Definition 2.3. A function f:(X, τ)$\rightarrow$(Y, σ) is called:
(i) a continuous function if f^{1}(V) is a closed set in (X, τ) for every closed set V in (Y, σ),
(ii) a semi-continuous function[10] if f^{1}(V) is a semi-closed set in (X, τ) for every closed set V in (Y, σ),
(iii) a pre-continuous function[14] if f^{1}(V) is a pre-closed set in (X, τ) for every closed set V in (Y, σ),
(iv) an α-continuous function[15] if f^{1}(V) is an α-closed set in (X, τ) for every closed set V in (Y,σ).
(v) a β-continuous function[1] if f^{1}(V) is a β-closed set in (X, τ) for every closed set V in (Y,σ).

Definition 2.4. A function f:(X, τ)$\rightarrow$(Y, σ) is called:
(i) a g-continuous function[4] if f^{1}(V) is a g-closed set in (X, τ) for every closed set V in (Y, σ),
(ii) a gs-continuous function[5] if f^{1}(V) is a gs-closed set in (X, τ) for every closed set V in (Y, σ),
(iii) an αg-continuous function[7] if f^{1}(V) is an αg-closed set in (X, τ) for every closed set V in (Y, σ),
(iv) a gpr-continuous function[7] if f^{1}(V) is a gpr-closed set in (X, τ) for every closed set V in (Y, σ),
(v) a gsp-continuous function [6] if f^{1}(V) is a gsp-closed set in (X, τ) for every closed set V in (Y, σ).

Definition 2.5. Levine [11], 1963 defined $\tau(B) = \{O \cup (Ó \cap B) : O, Ó \in \tau \}$ and called it simple expansion of τ by B, where B$\notin\tau$.

3. New forms of continuity by suitable choice of B

Proposition 3.1. Let f:(X, τ)→(Y, σ) be a continuous function and B be a non open subset in (Y, σ) if f^{-1}(B) is an open subset of (X, τ) then f:(X, τ)→(Y, σ(B)) is a continuous function.

Proof. Let f:(X, τ)→(Y, σ) be a continuous function and B be a non open subset in (Y, σ) and G be an open set in (Y, σ(B))**,** then G= O ∪ (Ó ∩ B) where O and Ó are open sets in (Y, σ). f^{-1}(G)= f^{-1}(O ∪ (Ó ∩ B))= f^{-1}(O) ∪(f^{-1}(Ó) ∩ f^{-1}(B)), since f is a continuous function and by a assumption f^{-1}(B) is an open subset of (X, τ), then f^{-1}(G) is an open subset of (X, τ), it implies f:(X, τ)→(Y, σ(B)) is a continuous function.

Note 3.2 Let f:(X, τ)→(Y, σ) be a continuous function and B be a non open subset in (X, τ), then f:(X, τ(B))→(Y, σ) is a continuous function.

Proof. It follows from every open set in (X, τ) is an open set in (X, τ(B)).

Remark 3.3. Let f:(X, τ)→(Y, σ) be a continuous function and B_1 be a non open subset in (X, τ) and B_2 be a non open subset in (Y, σ), if f^{-1}(B_2) is an open set in (X, τ(B_1)) then f:(X, τ(B_1))→(Y, σ(B_2)) is a continuous function.

Corollary 3. 4. Let f:(X, τ)→(X, τ) be a continuous function and B be a non open subset in (X, τ), if f^{-1}(B) is an open set in (X, τ(B)) then f:(X, τ(B))→ (X, τ(B)) is a continuous function.

Definition 3. 5. Let f:(X, τ)→(Y, σ) and B be a non open subset of (X, τ), then f is called a B-continuous function if f^{-1}(V) is an open set in (X, τ(B)) for every open set V in (Y, σ).

Proposition 3. 6. Every continuous function is a B-continuous function.

Proof. It follows from every open set in (X, τ) is an open set in (X, τ(B)).

Remark 3. 7. The converse of Proposition 3.5 not always true as shown by the following example.

Example 3. 8. Let X = {a, b, c}, τ = {X, ∅, {a, b}} and B = {a, c}, then τ(B)={X, ∅, {a}, {a, b}, {a, c}}, let Y={a, b, c}, σ={Y, ∅, {a}} and *f*(a)={a}, *f*(b)={c}, *f*(c)={b}, then *f* is a B-continuous function but not continuous function.

Proposition 3. 9. Let *f*:(X, τ)→(Y, σ) be an α-continuous function and B be a non open subset in (Y, σ) if f^{-1}(B) is an α-open subset of (X, τ) then *f*:(X, τ)→(Y, σ(B)) is an α-continuous function.

Proof. Let *f*:(X, τ)→(Y, σ) be an α-continuous function and B be a non open subset in (Y, σ) and G be an open set in (Y, σ(B))**,** then G= O ∪ (Ó ∩ B) where O and Ó are open sets in (Y, σ). f^{-1}(G)= f^{-1}(O ∪ (Ó ∩ B))= f^{-1}(O) ∪(f^{-1}(Ó) ∩ f^{-1}(B)), since *f* is an α-continuous function and by a assumption f^{-1}(B) is an α-open subset of (X, τ), then f^{-1}(G) is an α-open subset of (X, τ)[since every open set is an α-open set and τ_α is closed under forming arbitrary unions and finite intersection], it implies *f*:(X, τ)→(Y, σ(B)) is an α-continuous function.

Proposition 3. 10. Let *f*:(X, τ)→(Y, σ) be a semi-continuous function and B be a non open subset in (Y, σ) if f^{-1}(B) is an open subset of (X, τ) then *f*:(X, τ)→(Y, σ(B)) is a semi-continuous function.

Proof. Let *f*:(X, τ)→(Y, σ) be a semi-continuous function and B be a non open subset in (Y, σ) and G be an open set in (Y, σ(B))**,** then G= O ∪ (Ó ∩ B) where O and Ó are open sets in (Y, σ). f^{-1}(G)= f^{-1}(O ∪ (Ó ∩ B))= f^{-1}(O) ∪(f^{-1}(Ó) ∩ f^{-1}(B)), since *f* is a semi-continuous function and by a assumption f^{-1}(B) is an open subset of (X, τ), then f^{-1}(G) is a semi-open subset of (X, τ)[since every open set is a semi-open set and the intersection of a semi-open set and an open set is a semi-open set. And SO(X) is closed under forming arbitrary unions], it implies *f*:(X, τ)→(Y, σ(B)) is a semi-continuous function.

Proposition 3. 11. Let f:(X, τ)→(Y, σ) be a pre-continuous function and B be a non open subset in (Y, σ) if f^{-1}(B) is an open subset of (X, τ) then f:(X, τ)→(Y, σ(B)) is a pre-continuous function.

Proof. Let f:(X, τ)→(Y, σ) be a pre-continuous function and B be a non open subset in (Y, σ) and G be an open set in (Y, σ(B))**,** then G= O ∪ (Ó ∩ B) where O and Ó are open sets in (Y, σ). f^{-1}(G)= f^{-1}(O ∪ (Ó ∩ B))= f^{-1}(O) ∪(f^{-1}(Ó) ∩ f^{-1}(B)), since f is a pre-continuous function and by a assumption f^{-1}(B) is an open subset of (X, τ), then f^{-1}(G) is a pre-open subset of (X, τ)[since every open set is a pre-open set and the intersection of a pre-open set and an open set is a pre-open set, and PO(X) is closed under forming arbitrary unions], it implies f:(X, τ)→(Y, σ(B)) is a pre-continuous function.

Proposition 3. 12. Let f:(X, τ)→(Y, σ) be a β-continuous function and B be a non open subset in (Y, σ) if f^{-1}(B) is an open subset of (X, τ) then f:(X, τ)→(Y, σ(B)) is a β-continuous function.

Proof. Let f:(X, τ)→(Y, σ) be a β-continuous function and B be a non open subset in (Y, σ) and G be an open set in (Y, σ(B))**,** then G= O ∪ (Ó ∩ B) where O and Ó are open sets in (Y, σ). f^{-1}(G)= f^{-1}(O ∪ (Ó ∩ B))= f^{-1}(O) ∪(f^{-1}(Ó) ∩ f^{-1}(B)), since f is a β-continuous function and by a assumption f^{-1}(B) is an open subset of (X, τ), then f^{-1}(G) is a β-open subset of (X, τ)[since every open set is a β-open set and the intersection of a β-open set and an open set is a β-open set, and βO(X) is closed under forming arbitrary unions], it implies f:(X, τ)→(Y, σ(B)) is a β-continuous function.

4. New forms of generalized continuity by suitable choice of B

Definition 4. 1. A subset A of a space (X, τ) is said to be B-generalized closed set (briefly, Bg-closed) if BCl(A) ⊆ U whenever A ⊆ U and U is open in (X, τ).

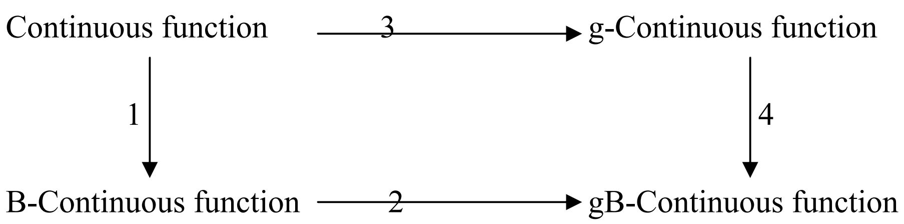
Where BCl(A) is given by BCl(A) = ∩{S ⊆ X : A ⊆ S and S is a closed set in τ(B)}.

A subset of X belonging to τ(B) is denoted by B-open set, the complement of B-open set is denoted by B-closed set. The family of all B-open sets is denoted by BO(X) and the family of all B-closed sets is denoted by BC(X).

Definition 4. 2. A function f:(X, τ)→(Y, σ) is called a B-continuous function if f^{-1}(V) is a B-closed set in (X, τ) for every closed set V in (Y, σ).

Definition 4. 3. A function f:(X, τ)→(Y, σ) is called a gB-continuous function if f^{-1}(V) is a Bg-closed set in (X, τ) for every closed set V in (Y, σ).

Proposition 4. 4. For a subset of a space (X, τ) and A function f:(X, τ)→(Y, σ) from the definition stated above, we have the following diagram of implications:



Proof.

(1) Since f is a continuous function then f^{-1}(V) is a closed set in (X, τ) for every closed set V in (Y, σ) but τ ⊆ τ(B) ⇒ every closed set in (X, τ) is a closed set in (X, τ(B))⇒f^{-1}(V) is a B-closed set in (X, τ) for every closed set V in (Y, σ)⇒ f is a B-continuous function.

(2) Since f is a B-continuous function then f^{-1}(V) is a B-closed set in (X, τ) for every closed set V in (Y, σ), but every B-closed set in (X, τ) is a Bg-closed set in (X, τ) ⇒ f^{-1}(V) is a Bg-closed set in (X, τ) for every closed set V in (Y, σ)⇒ f is a gB-continuous function.

(3) Since f is a continuous function then f^{-1}(V) is a closed set in (X, τ) for every closed set V in (Y, σ) but every closed set (X, τ) is a g-closed set in (X, τ) this implies f^{-1}(V) is a g-closed set in (X, τ) for every closed set V in (Y, σ)⇒ f is a g-continuous function.

(4) Since *f* is a g-continuous function then $f^{-1}(V)$ is a g-closed set in (X, τ) for every closed set V in (Y, σ) but τ ⊆ τ(B) ⇒ every closed set in (X, τ) is a closed set in (X, τ(B)) (i. e., every closed set is a B-closed set) ⇒ every g-closed set is a Bg-closed set ⇒ $f^{-1}(V)$ is a Bg-closed set in (X, τ) for every closed set V in (Y, σ)⇒ *f* is a gB-continuous function.

Remark 4. 5. None of the implications in the above proposition is reversible as shown by the following examples.

Example 4. 6. Let X = Y = {a, b, c}, τ = {X, ∅, {a, b}}, σ= {Y, ∅, {a, b}} and B= {a, c}, then τ(B) = {X, ∅, {a}, {a, b}, {a, c}}, if *f*:(X, τ)→(Y, σ) defined by:

(1) *f*(a)=a, *f*(b)=b and *f*(c)=c, then *f* is a B-continuous function but not a continuous function.

(2) *f*(a)=*f*(c)=b and *f*(b)=a, then *f* is a gB-continuous function but not a B-continuous function.

(3) *f*(a)=*f*(c)=b and *f*(b)=a, then *f* is a g-continuous function but not a continuous function.

(4) *f*(a)=c, *f*(b)=b and *f*(c)=a, then *f* is a gB-continuous function but not a g-continuous function.

Proposition 4. 7. Let (X, τ) be a topological space, B any non open subset of X and let *f*:(X, τ)→(Y, σ) be a gB-continuous function if B is:

(i) α-open set in (X, τ) then *f* is an α-generalized continuous function,

(ii) semi-open set in (X, τ) then *f* is a semi-generalized continuous function,

(iii) pre-open set in (X, τ) then *f* is a pre-generalized continuous function,

(iv) spr-open set in (X, τ) then *f* is a spr-generalized continuous function.

Proof.

(i) Let A be a Bg-closed set ⇒ BCl(A)⊆U whenever A⊆U and U is open in (X, τ), but since B is an α-open set then $\tau(B) \subseteq \tau_\alpha$ (i.e., αCl(A)⊆BCl(A)) ⇒ αCl(A)⊆U whenever A⊆U and U is open in (X, τ) ⇒ A is an αg-closed set, that implies every Bg-closed set is an αg-closed set, but since *f* is a gB-continuous function then f^{1}(V) is a Bg-closed set in (X, τ) for every closed set V in (Y ,σ) ⇒ f^{1}(V) is an αg-closed set in (X, τ) for every closed set V in (Y, σ) this implies *f* is an αg-continuous function.

(ii) Let A be a Bg-closed set ⇒ BCl(A)⊆U whenever A⊆U and U is open in (X, τ), but since B is a semi-open set then τ(B) ⊆ SO(X)**.** (i.e., sCl(A)⊆BCl(A)), we get sCl(A)⊆U whenever A⊆U and U is open in (X, τ) ⇒ A is a gs-closed set that implies every Bg-closed set is an gs-closed set, but since *f* is a gB-continuous function then f^{1}(V) is a Bg-closed set in (X, τ) for every closed set V in (Y,σ) ⇒ f^{1}(V) is a gs-closed set in (X, τ) for every closed set V in (Y, σ) this implies *f* is an a gs-continuous function.

(iii) Let A be a Bg-closed set ⇒ BCl(A)⊆U whenever A⊆U and U is open in (X, τ), but since B is a pre-open set then τ(B) ⊆ PO(X)**.** (i.e., pCl(A)⊆BCl(A)) ⇒ pCl(A)⊆U whenever A⊆U and U is open in (X, τ) ⇒ A is a gp-closed set that implies every Bg-closed set is an gp-closed set, but since *f* is a gB-continuous function then f^{1}(V) is a Bg-closed set in (X, τ) for every closed set V in (Y, σ) ⇒ f^{1}(V) is a gp-closed set in (X,τ) for every closed set V in (Y,σ) this implies *f* is a gp-continuous function.

(iv) Let A be a Bg-closed set ⇒ BCl(A)⊆U whenever A⊆U and U is open in (X, τ), but since B is a semi-pre-open set then τ(B) ⊆ βO(X)**.** (i.e., spCl(A)⊆BCl(A)) this implies spCl(A)⊆U whenever A⊆U and U is open in (X, τ) ⇒ A is a gsp-closed set that implies every Bg-closed set is an gsp-closed set, but since *f* is a gB-continuous function then $f^{-1}(V)$ is a Bg-closed set in (X, τ) for every closed set V in (Y,σ) this means $f^{-1}(V)$ is a gsp-closed set in (X, τ) for every closed set V in (Y, σ) this implies *f* is a gsp-continuous function.

5. Conclusions

Granular computing is a recent branch in the field of computer science that uses topological structure as granulation models suggested approach for generalized closed sets give new methods for generating the classes of subsets whose lower and upper approximations are contained in elementary sets which in turn help in the process of decision making under both quantities and qualitative information.

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