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Ideal Amenability of Triangular Banach Algebras

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Abstract

A Banach algebra A is called ideally amenable if $H^1(A, I^*) = 0$ for each closed ideal I of A. Let X be an A-B-module, we show that the triangular Banach algebra

$$T = \{\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} : \ a \in A, x \in X, b \in B\}$$

asociated to X is ideally amenable if and only if A and B are ideally amenable.

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1 Introduction

Let A be a Banach algebra and let X be a Banach A-module, Then X^* is a Banach A-module if for each $a \in A$ and $x \in X$ and $f \in X^*$ we define af and fa belong to X^* as follows:

$$af(x) = f(xa) \quad , \quad fa(x) = f(ax).$$

If X is a Banach A-module, then a derivation from A into X is a continuouse linear operator D such that

$$D(ab) = D(a)b + aD(b) \quad ; \quad (a, b \in A).$$

The linear space of all bounded derivation from A into X is denoted by $Z^1(A, X)$. For example if $x \in X$ and we define $\delta_x : A \to X$ by $\delta_x(a) =$

$ax - xa$, Then δ_x is a continuouse derivation. Derivations of this form are called inner derivations and we denoted by $N^1(A, X)$ the linear space of inner derivations from A into X. A Banach algebra A is called amenable if every derivation from A into every Dual A-module X^* is inner i.e. $H^1(A, X^*) := Z^1(A, X^*)/N^1(A, X^*) = \{0\}$, $H^1(A, X^*)$ is called the first cohomology group from A with cofficient in X^*. This concept was first defined by B. E. Johnson in [5], who showed that the group algebra $L^1(G)$ is amenable if and only if G is an amenable group. Bade , Curtis and Dales [6] later defined the notion of weak amenability for a commutative Banach algebra. A Banach algebra A is called weak amenable if every derivation from A into A^* is inner. A Banach algebra A is called ideally amenable if $H^1(A, I^*) = \{0\}$ for every closed (two-sided) ideal I of A. Eshaghi-Gordji and Yazdanpanah have introduced the concept of ideal amenability of Banach algebras in [4]. In [3] authors studied the n-ideal amenability of Banach algebras. Now let X be an $A - B$-module. B. E. Forrest and L. W. Marcoux in [1] showed that the triangular Banach algebra $T = \{\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} : \quad a \in A, x \in X, b \in B\}$ is weak amenable if and only if A and B are weak amenable. We prove the same theorem for ideal amenability of T.

2 Triangular Banach algebras

Definition 2.1. *Let A and B be Banach algebras. Let X be a Banach $A - B$-module. Put*

$$T = \{\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} : \quad a \in A, x \in X, b \in B\}$$

Consider T with the usual operations associated with 2×2 matrices, Then T becomes a complex algebra. We can define a norm on T by

$$\| \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \| = \|a\| + \|x\| + \|b\|.$$

It is obvious that T is infact a Banach algebra. T is called the triangular Banach algebra associated to $A - B$-module X. Morever if A and B are unital with the unit elements 1_A and 1_B, respectively, and X is unital, that is, $1_A x 1_B = x$ for each $x \in X$. Then T is unital.

Throughout this paper we assume that A and B are unital Banach algebra, X is a unital Banach $A - B$-module and T is it's associated triangular Banach algebra.

Now it is easy to show that I is a closed ideal in T, if and only if there exist closed ideals I_1 in A, closed ideal I_2 in B and a closed $A - B$-submodule Y of

X such that $I = \begin{pmatrix} I_1 & Y \\ 0 & I_2 \end{pmatrix}$ and that $I_1 X \cup X I_2 \subset Y$. Each element of I^* can be present as the form $\begin{pmatrix} f & h \\ 0 & g \end{pmatrix}$ where $f \in I_1^*, h \in Y^*, g \in I_2^*$ and

$$\begin{pmatrix} f & h \\ 0 & g \end{pmatrix} [\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}] = f(a) + h(x) + g(b) \quad (a \in I_1, x \in Y, b \in I_2).$$

Now suppose that $t = \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \in T$ and $i = \begin{pmatrix} f & h \\ 0 & g \end{pmatrix} \in I^*$. T is an I^*-bimodule which the actions given by

$$t \cdot i = \begin{pmatrix} af + xh & bh \\ 0 & bg \end{pmatrix},$$

$$i \cdot t = \begin{pmatrix} fa & ha \\ 0 & hx + gb \end{pmatrix}.$$

Lemma 2.2. *Let $I = \begin{pmatrix} I_1 & Y \\ 0 & I_2 \end{pmatrix}$ be a closed ideal of T and $\delta_A : A \longrightarrow I_1^*$ be a continuous derivation. Then*

$$\begin{array}{rcl} D_{\delta_A} : T & \longrightarrow & I^* \\ \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} & \longmapsto & \begin{pmatrix} \delta_A(a) & 0 \\ 0 & 0 \end{pmatrix} \end{array}$$

is also a continuous derivation. Morever δ_A is inner if and only if D_{δ_A} is inner. Similarliy, if $\delta_B : B \longrightarrow I_2^*$ is a continuous derivation. Then

$$\begin{array}{rcl} D_{\delta_B} : T & \longrightarrow & I^* \\ \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} & \longmapsto & \begin{pmatrix} 0 & 0 \\ 0 & \delta_B(b) \end{pmatrix} \end{array}$$

is a continuous derivation and it is inner precisely when δ_B is inner.

Proof. By simple calculations one can show that D_{δ_A} and D_{δ_B} are derivations. Suppose δ_A is inner. Then there exists $f \in I_1^*$ such that $\delta_A(a) = af - fa$. Consider $t = \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} \in I^*$. Then

$$\begin{array}{rcl} D_t \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} & := & \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \cdot \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \\ & = & \begin{pmatrix} af - fa & 0 \\ 0 & 0 \end{pmatrix} \\ & = & \begin{pmatrix} \delta_A(a) & 0 \\ 0 & 0 \end{pmatrix} \\ & = & D_{\delta_A} \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \end{array}$$

So $D_{\delta_A} = D_t$ is inner. Conversely, suppose that D_{δ_A} is inner. Then there exists $i = \begin{pmatrix} f & h \\ 0 & g \end{pmatrix} \in I^*$ such that for each $a \in A$

$$\begin{aligned} D_{\delta_A}\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} f & h \\ 0 & g \end{pmatrix} - \begin{pmatrix} f & h \\ 0 & g \end{pmatrix} \cdot \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} af - fa & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

On the other hand

$$D_{\delta_A}\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \delta_A(a) & 0 \\ 0 & 0 \end{pmatrix}$$

So for any $a \in A$, we have $\delta_A(a) = af - fa$, and so δ_A is inner. Similarly D_{δ_B} is inner if and only if δ_B is inner. □

Lemma 2.3. *Let $D : T \to I^*$ be a continuous derivation. Then there exists continuous derivations $\delta_1 : A \to I_1^*$, $\delta_2 : B \to I_2^*$, and element $f_0 \in Y^*$ such that*

$$D\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} = \begin{pmatrix} \delta_1(a) - xf_0 & f_0a - bf_0 \\ 0 & \delta_2(b) + f_0x \end{pmatrix} \quad (a \in A, b \in B, x \in X).$$

Proof. We use some ideas of Proposition 2.1 of [1]. By simple computation one can verify that

$(i) D\begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & f_0 \\ 0 & 0 \end{pmatrix}$ for some $f_0 \in Y^*$.

$(ii) D\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \delta_1(a) & f_0a \\ 0 & 0 \end{pmatrix}$ for some continuous derivation $\delta_1 : A \to I_1^*$.

$(iii) D\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -xf_0 & 0 \\ 0 & f_0x \end{pmatrix}$

$(iv) D\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & -bf_0 \\ 0 & \delta_2(b) \end{pmatrix}$ for some continuous derivation $\delta_2 : B \to I_2^*$.

This complete the proof. □

Theorem 2.4. *Let $I = \begin{pmatrix} I_1 & Y \\ 0 & I_2 \end{pmatrix}$ be an closed ideal of triangular Banach algebra T. Then*

$$H^1(T, I^*) \simeq H^1(A, I_1^*) \oplus H^1(B, I_2^*)$$

Proof. Let D be continuous derivation from T to I^*. Let δ_1, δ_2 and f_0 be as in the above lemma. Define the map

$$\begin{aligned} R : Z^1(T, I^*) &\to H^1(A, I_1^*) \oplus H^1(B, I_2^*) \\ D &\mapsto (\delta_1 + N^1(A, I_1^*), \delta_2 + N^1(B, I_2^*)) \end{aligned}$$

Clearly R is linear.
Now for given $\delta_A \in Z^1(A, I_1^*)$ and $\delta_B \in Z^1(B, I_2^*)$, by lemma 2.2, D_{δ_A} and D_{δ_B} belong to $Z^1(T, I^*)$ and so that $D := D_{\delta_A} + D_{\delta_B} \in Z^1(T, I^*)$ and we have

$$R(D) = (D_{\delta_A} + N^1(A, I_1^*), D_{\delta_B} + N^1(B, I_2^*)).$$

This show that R is surjective.
If $D \in kerR$. Then $\delta_1 \in N^1(A, I_1^*)$ and $\delta_2 \in N^1(B, I_2^*)$. By Lemma 2.2 $D_{\delta_1}, D_{\delta_2} \in N^1(T, I^*)$, and hence $D_1 := D_{\delta_A} + D_{\delta_B} \in N^1(T, I^*)$. Thus $D = D_{\begin{pmatrix} 0 & f_0 \\ 0 & 0 \end{pmatrix}} + D_1 \in N^1(T, I^*)$. Also it is clear that if $D \in N^1(T, I^*)$, then $R(D) = 0$. This show that $kerR = N^1(T, I^*)$.
Therefore

$$H^1(A, I_1^*) \oplus H^1(B, I_2^*) = ranR \simeq \frac{Z^1(T, I^*)}{kerR} = H^1(T, I^*).$$

□

Corollary 2.5. *Triangular Banach algebra* $T = \begin{pmatrix} A & X \\ 0 & B \end{pmatrix}$ *is ideally amenable if and only if* A *and* B *are ideally amenable.*

In [4] authers have shown that each C^*-algebra is ideally amenable. Combine this with the above corollary we have

Corollary 2.6. *For each* C^**-algebra* A*, the Banach algebras* $\begin{pmatrix} A & A \\ 0 & A \end{pmatrix}$ *and* $\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ *are ideally amenable.*

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