Int. Journal of Math. Analysis, Vol. 4, 2010, no. 21, 1041 - 1047

Approximation of Functions Belonging to Lipschitz Class by Triangular Matrix Method of Fourier Series

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Abstract

In this paper, a new theorem on the degree of approximation of the function $f \in lip\alpha$ by lower triangular matrix summability method of Fourier series has been established.

Mathematics Subject Classification: 42B05, 42B08

Keywords: Lipschitz Class of functions, Fourier series, Degree of approximation, Matrix summability method

1. Introduction

The degree of approximation of functions f belonging to $Lip\alpha$ class by (C,1), (C, δ), $\delta > 0$, Nörlund means (N, p_n), Riesz means $(\tilde{N}, p_n)$ and generalized Nörlund means (N,p,q) has been determined by several investigators like Berstein [4], Alexits [1], Chandra [5], Sahney and Goel [7] , Alexits and Kralik [3], Alexits and Leindler[2]. The degree of approximation of functions $f \in \mathrm{lip}\,\alpha$ using lower triangular matrix summability has not been studied so far. Lower triangular matrix summability method includes Cesáro means, Nörlund means and generalized Nörlund means as particular cases. In this paper, the degree of approximation of a function $f \in \mathrm{Lip}\ \alpha$, $0 < \alpha \le 1$, by lower triangular matrix method its Fourier series has been determined.

2. Definitions and Notations

A function $f \in \mathrm{Lip}\alpha$ if $|f(x+t) - f(x)| = O(|t|^{\alpha})$ for $0 < \alpha \le 1$.
Let f be a periodic with 2π and integrable over $(-\pi,\pi)$ in the Lebesgue sense and $f \in \mathrm{Lip}\,\alpha$. Let its Fourier series be given by

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx). \tag{1}$$

The degree of approximation of a function $f : R \to R$ by a trigonometric polynomial T_n of order n is given by $\|T_n - f\|_{\infty} = \sup\{|T_n(x) - f(x)| : x \in R\}$ (Zygmund ,[10] p.114).

Let $T = (a_{n,k})$ be an infinite triangular matrix satisfying the Töeplitz[9] condition of regularity, i.e. $\sum_{k=0}^{n} a_{n,k} \to 1$, as $n \to \infty$, $a_{n,k} = 0$ for $k > n$ and $\sum_{k=0}^{n} |a_{n,k}| \le M$, a finite constant. Let $\sum_{m=0}^{\infty} u_n$ be an infinite series such that whose n^{th} partial sum $s_n = \sum_{k=0}^{n} u_k$. The sequence-to-sequence transformation $t_n = \sum_{k=0}^{n} a_{n,k}\ s_k$ defines the sequence $\{t_n\}$ of lower triangular matrix means of the sequence $\{s_n\}$ generated by the sequence of coefficients $(a_{n,k})$. The series $\sum_{n=0}^{\infty} u_n$ is said to be summable to the sum s by matrix method if $\lim_{n\to\infty} t_n$ exists and is equal to s (Zygmund (10) p. 74) and it is denoted by $t_n \to s(t)$, as $n \to \infty$.
We write $\phi(t) = f(x+t) + f(x-t) - 2f(x)$,

$$A_{n,\tau} = \sum_{k=n-\tau}^{n} a_{n,n-k}\,, \quad \tau = [1/t] = \text{the largest integer in}\,(1/t)\,.$$

3. Theorem

We prove following theorem:

Theorem. Let T= ($a_{n,k}$) be an infinite lower triangular matrix such that the elements ($a_{n,k}$) be non-negative, non-decreasing with $k \le n$ $A_{n,\tau} = \sum_{k=n-\tau}^{n} a_{n,n-k}$ and $A_{n,n} = 1 \ \forall \ n \ge 0$

Let $f \in L^1[0, 2\pi]$ be a 2π-periodic function belonging to $\text{Lip}\,\alpha$ $(0 < \alpha \le 1)$, then the degree of approximation of f by lower triangular matrix means t_n of its Fourier series (1) is given by

$$\|t_n - f\|_\infty = \begin{cases} O\left((n+1)^{-\alpha}\right), & 0 < \alpha < 1, \\ O\left(\log(n+1)\pi e/(n+1)\right), & \alpha = 1, \end{cases} \tag{2}$$

for n = 0,1,2,3…

4. Lemmas

For the proof of our theorem following lemmas are required.

Lemma 1. $M_n(t) = O(n+1)$, if $0 < t \le 1/(n+1)$.

Proof. For $0 < t \le 1/(n+1)$, $\sin(n+1)t \le (n+1)t$,

$$|M_n(t)| \le \left| \frac{1}{2} \sum_{k=0}^{n} a_{n,k} \frac{(2k+1)(t/2)}{t} \right|$$

$$\le \tfrac{(2n+1)}{4} \sum_{k=0}^{n} a_{n,k} = \tfrac{(2n+1)}{4} A_{n.n} = O(n+1).$$

Lemma 2. If ($a_{n,k}$) is non-negative, non-decreasing with $k \le n$, then,

$$\left| \sum_{k=0}^{n} a_{n,k} e^{ikt} \right| = O(A_{n,\tau}), \text{ uniformly for } 0 < t \le \pi.$$

Proof. Let $\tau = [1/t] \le n$, then

$$\left| \sum_{k=0}^{n} a_{n,k} e^{ikt} \right| = \left| \sum_{k=0}^{n-\tau-1} a_{n,k} e^{ikt} + \sum_{k=n-\tau}^{n} a_{n,k} e^{ikt} \right|$$

$$\leq \left|\sum_{k=0}^{n-\tau-1} a_{n,k}\, e^{ikt}\right| + \left|\sum_{k=n-\tau}^{n} a_{n,k}\, e^{ikt}\right|;$$

But by Abel`s lemma, $\left|\sum_{k=0}^{n-\tau-1} a_{n,k}\, e^{ikt}\right| \leq 2\,a_{n,n-\tau-1} \max_{0\leq p\leq n-\tau-1}\left|\frac{1-e^{i(p+1)t}}{1-e^{it}}\right|$

$$\leq 4\,a_{n,n-\tau-1}\left|\frac{e^{it/2}}{e^{-it/2}-e^{it/2}}\right|$$

$$\leq 2\,a_{n,n-\tau-1}\left|\frac{1}{\sin(t/2)}\right|$$

$$\leq \frac{2\pi\, a_{n,n-\tau-1}}{t}$$

and $A_{n,\tau} = \sum_{k=n-\tau}^{n} a_{n,k} = a_{n,n-\tau} + a_{n,n-\tau+1} + \ldots + a_{n,n}$

$$\geq a_{n,n-\tau-1} + a_{n,n-\tau-1} + \ldots + a_{n,n-\tau-1}$$

$$= (\tau+1)a_{n,n-\tau+1}$$

$$\geq \frac{a_{n,n-\tau-1}}{t},$$

therefore, $\left|\sum_{k=0}^{n-\tau-1} a_{n,k}\, e^{ikt}\right| \leq 2\pi A_{n,\tau}.$

Also

$$\left|\sum_{k=n-\tau}^{n} a_{n,k}\; e^{ikt}\right| \leq \sum_{k=n-\tau}^{n} a_{n,k}\;\left|e^{ikt}\right| = \sum_{k=n-\tau}^{n} a_{n,k} = A_{n,\tau}.$$

Thus $\left|\sum_{k=0}^{n} a_{n,k}\, e^{ikt}\right| \leq (2\pi+1)A_{n,\tau} = O(A_{n,\tau}).$

Lemma 3. $M_n(t) = O\left(\frac{A_{n,\tau}}{t}\right)$, if $1/(n+1) < t \leq \pi$.

Proof. For $1/(n+1) < t \leq \pi$, $\sin(t/2) \geq (t/\pi)$, we have

$$|M_n(t)| \leq \left|\frac{1}{2t}\,\mathrm{Im}\sum_{k=0}^{n} a_{n,k}\, e^{i(k+\frac{1}{2})t}\right|$$

$$\leq \frac{1}{2t}\left|\sum_{k=0}^{n} a_{n,k}\, e^{ikt}\right|\left|e^{it/2}\right|$$

$$= O\left(\frac{A_{n,\tau}}{t}\right), \text{ using lemma 2.}$$

5. Proof of the theorem

Following Titchmarsh [8], $s_k(x)$ of Fourier (1) is given by, we have

$$s_k(x) - f(x) = \frac{1}{2\pi}\int_0^{\pi} \phi(t)\frac{\sin(k+1/2)t}{\sin(t/2)}dt$$

then , $$\sum_{k=0}^{n} a_{n,k}\left(s_{n,k}(x) - f(x)\right) = \frac{1}{2\pi}\int_0^{\pi}\phi(t)\sum_{k=0}^{n} a_{n,k}\frac{\sin\left(k+1/2\right)t}{\sin(t/2)}dt$$

or $$t_n(x) - f(x) = \int_0^{\pi}\phi(t)\ M_n(t)\ dt$$

$$= \left[\int_0^{1/(n+1)}\phi(t)\ M_n(t)\,dt + \int_{1/(n+1)}^{\pi}\phi(t)\ M_n(t)\,dt\right]$$

$$= I_1 + I_2, \quad \text{say} \tag{3}$$

Using Lemma 1 and $f \in Lip\,\alpha \Rightarrow \phi \in Lip\,\alpha$ [[6] ; Lemma 5.27], we have

$$|I_1| = O(n+1)\int_0^{1/(n+1)} t^{\alpha}dt$$

$$= O\left((n+1)^{-\alpha}\right) \tag{4}$$

Next, using lemma 3, $$|I_2| = O\left[\int_{1/(n+1)}^{\pi} t^{\alpha-1}A_{n,\tau}\ dt\right]$$

$$= O\left[\int_{1/\pi}^{n+1}\left(\frac{A_{n,u}}{u^{\alpha+1}}\right)du\right]$$

$$= O\left(\frac{A_{n,\,n}}{n+1}\right)\left[\int_{1/\pi}^{n+1}\frac{du}{u^{\alpha}}\right] ,$$

$\left(\because \left(\frac{A_{n,u}}{u}\right) \text{ is monotonic}\right)$

$$= O\left(\frac{1}{n+1}\right)\begin{cases}\frac{1}{(1-\alpha)}\left\{(n+1)^{-\alpha+1} - (1/\pi)^{-\alpha+1}\right\}, & 0<\alpha<1\\ \log(n+1)\pi, & \alpha = 1\end{cases}$$

$$= \begin{cases} O\left((n+1)^{-\alpha}\right), & 0<\alpha<1\\ O(\log(n+1)\pi), & \alpha=1\end{cases} \tag{5}$$

Combining (3) to (5) and writing log e =1, we have

$$\left\| t_n - f \right\|_\infty = \sup_{0\le x\le 2\pi} \left| t_n(x) - f(x) \right| = \begin{cases} O\left((n+1)^{-\alpha}\right), & 0<\alpha<1, \\ O\left(\log(n+1)\pi e/(n+1)\right), & \alpha = 1. \end{cases}$$

6. Corollaries

Following corollaries can be derived from the main theorem.

Cor.1. If $a_{n,k} = 1/(n+1)$ then degree of approximations $f \in Lip\,\alpha$ by Cesáro means $\sigma_n = \frac{1}{n+1}\sum_{k=0}^{n} s_k$ of Fourier series (2) is given by

$$\left\| \sigma_n - f \right\|_\infty = \begin{cases} O\left((n+1)^{-\alpha}\right), & 0<\alpha<1, \\ O\left(\log(n+1)\pi e/(n+1)\right), & \alpha = 1. \end{cases}$$

Cor. 2. If $a_{n,k} = \frac{p_{n-k}}{P_n}$ then degree of approximations $f \in Lip\,\alpha$ by Nörlund means

$t_n^N = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}\, s_k$ is given by $\left\| t_n^N - f \right\|_\infty = \begin{cases} O\left((n+1)^{-\alpha}\right), & 0<\alpha<1, \\ O\left(\log(n+1)\pi e/(n+1)\right), & \alpha = 1. \end{cases}$

7. Remarks

Remark(1) . There are several estimations of the function $f \in Lip\,\alpha$ by (C,1), (C, δ), δ >0, (N, p_n),(N, p, q) methods of its Fourier series, for example, Berstein [4], Alexits [1], Chandra [5], Sahney and Goel [7] , Alexits and Kralik [3], Alexits and Leindler[2] but most of these results are not satisfies for n=0,1 or $\alpha = 1$. Therefore this deficiency has been motivated to investigate the generalization of these results by a most general lower triangular matrix considering cases (1) $0<\alpha<1$ (2) $\alpha = 1$ separately. Our estimation is sharper and better than all previously known estimations of this direction.

(2) By our theorem, $\left\| t_n - f \right\|_\infty \to \infty$ *as* $n \to \infty$ then this estimate is correct and the best estimation. [Zygmund [9], p. 115]

(3) Our corollaries provide simplified and corrected forms of new results.

References

[1] **Alexits, G**.: Über die Annäherung einer stetigen Funktion durch die Cesàroschen Mittel ihrer Fourierreihe,Math. Ann. , **100**(1928), 264-277.

[2] **Alexits, G. and Leindler, L**.: Über die Approximation im starken Sinne (German), Acta Math. Acad. Sci. Hunger , **16**(1965) , 27-32.

[3] **Alexits, G. and Kralic** : Über die Approximation mit starken de la valle-poussinchen mitteln, Acta Math. Acad. Sci. Hunger **16**(1965) , 43-49.

[4] **Bernstein S.,** Surl Order de la Meilleure approximation desfunctions continues pardes polynomes de degree, donne`s Memories Acad.Roy.Belyique (2),4(1912),1-104.

[5] **Chandra Prem**, On the degree of approximation of functions belonging to Lipschitz class, Nanta Math. **8**(1975) no. 1 ,88-91.

[6] **McFadden, Leonard:** Absolute Nörlund summability**,** *Duke Math. J.* **9** (1942), 168-207.

[7] **Sahney B. N. and Geol D.S.** ,On degree of approximation of contineous functions, Rachi Univ. Math. J. 4(1973) , 50-53.

[8] **Titchmarsh E.C**. ,The theory of functions, 2nd Edition, Oxford University Press (1939),402-403.

[9] **Töeplitz O**. ,Uberallgemeine linear Mittelbudungen, PMF, 22(1913), 113-119.

[10] **Zygmund A**. ,Trignometic series, second edition Volume 1, Cambridge University Press, Cambridge (1959),74, 114-115.

Received: September, 2009