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Operators on Lorentz-Karamata Spaces

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Abstract

In this paper, we study the composition operators and weighted composition operators on Lorentz–Karamata spaces $L_{p,q;b}(R,\mu)$, $1 < p \leq \infty$ and $1 \leq q \leq \infty$.

Keywords: Composition operators, Lorentz-Karamata spaces, measurable transformations, semigroups, weighted composition operators

1. INTRODUCTION

Let $R = (R, \Sigma, \mu)$ be a σ-finite complete measure space. By $L(\mu)$, we denote the linear space of all equivalence classes of Σ-measurable functions on R, where we identify any two functions that are equal μ-almost everywhere on R. Let $T: R \to R$ be a non-singular measurable transformation, that is, $T^{-1}(A) \in \Sigma$ for each $A \in \Sigma$ and $\mu o T^{-1}(A) = 0$ for each $A \in \Sigma$ whenever $\mu(A) = 0$. The Radon-Nikodym theorem ensures the existence of a non-negative locally integrable function f_T on R such that

$$\mu o T^{-1}(A) = \int_A f_T(t)\, d\mu(t) \qquad \text{for each } A \in \Sigma.$$

Any measurable non-singular transformation T induces a composition operator from $L(\mu)$ into itself defined by

$$C_T f(t) = f o T(t) = f(T(t)) ; \quad t \in R, \ f \in L(\mu).$$

Let $\pi : R \to \mathbf{C}$ be a complex-valued measurable function and we define a linear operator (called weighted composition operator) from $L(\mu)$ into itself defined by

$$W_{\pi,T} f(t) = \pi(t) f(T(t)); \ t \in R, \ f \in L(\mu).$$

If $\pi \equiv 1$, then $W_{\pi,T} \equiv C_T : f \to f o T$ is called a composition operator induced by T. If T is the identity map, then $W_{\pi,T} \equiv M_\pi : f \to \pi f$ is a multiplication operator induced by

π. The study of these operator on L^p-spaces, Orlicz spaces, Lorentz spaces and Orlicz-Lorentz spaces has been made in [1], [2], [3], [4] and [5] and references therein.

For $c \approx d$, we mean that c is bounded above by a multiple of d, the multiple being independent of any variables in c and d and vice versa.

Definition 1.1. A Lebesgue measurable function $b : [1,\infty) \to (0,\infty)$ is said to be slowly varying if given $\varepsilon > 0$, $t \to t^{\varepsilon} b(t)$ is equivalent to a non-decreasing function and $t \to t^{-\varepsilon} b(t)$ is equivalent to a non-increasing function on $[1, \infty)$

Given any slowly varying function b, we denote by γ_b the function defined on $(0,\infty)$ by

$$\gamma_b(t) = b(\max\{t, \frac{1}{t}\}), \quad t > 0.$$

The detailed study of Karamata theory, properties and examples of slowly varying functions can be found in [6] and [7]. We shall need the following properties of slowly varying functions for which we refer to [7, Lemma 3.1].

Lemma 1.2. Let b be any slowly varying function.

(i) Let r be any real number. Then b^r is a slowly varying function and $\gamma_b^r(t) = \gamma_{b^r}(t)$ for all $t > 0$.

(ii) Given positive numbers $\in$ and k, $\gamma_b(kt) \approx \gamma_b(t)$, i.e., there are positive constants c_ε and C_ε such that

$$c_\varepsilon \min\{k^{-\varepsilon}, k^{\varepsilon}\}\, \gamma_b(t) \le \gamma_b(kt) \le C_\varepsilon \max\{k^{-\varepsilon}, k^{\varepsilon}\}\, \gamma_b(t)$$

for all $t > 0$.

(iii) Let $\alpha > 0$. Then

$$\int_0^t s^{\alpha-1} \gamma_b(s)ds \approx t^{\alpha} \gamma_b(t) \text{ and } \int_0^t s^{-\alpha-1} \gamma_b(s)\, ds \approx t^{-\alpha} \gamma_b(t) \text{ for all } t > 0.$$

Let f be a complex-valued function defined on R. Then the distribution function $\mu_f(.)$ and the decreasing rearrangement $f^*(.)$ of f are given by

$$\mu_f(\lambda) = \mu(\{x \in R : |f(x)| > \lambda\}), \quad (\lambda \ge 0)$$

and

$$f^*(t) = \inf\{\lambda > 0 : \mu_f(\lambda) \le t\}, \quad (t \ge 0)$$

respectively.

We assume that $\inf \phi = \infty$. Also, the average (or maximal) function of f on $(0, \infty)$ is given by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds.$$

Definition 1.3. Let $p, q \in (0,\infty]$ and let b be a slowly varying function. The Lorentz-Karamata (LK) spaces $L_{p,q;b}(R,\mu)$ is defined to be the linear space of all measurable functions such that

$$\| f \,|L_{p,q;b}(R,\mu)\| := \| t^{\frac{1}{p}-\frac{1}{q}} \gamma_b(t) f^{**}(t) \mid L_q(0,\infty)\|$$

is finite. Here $\|.\ |L_q(0,\infty)\|$ stands for the usual L_q quasi-norm over the interval $(0,\infty)$.

Note that $\|\ .\ |L_{p,q;b}(R,\mu)\|$ is a norm if $q \geq 1$ and moreover, $(L_{p,q;b}(R,\mu), \|.\ |L_{p,q;b}(R,\mu)\|$ is a Banach space for $1 < p \leq \infty$, $1 \leq q \leq \infty$. For Lorentz-Karamata spaces, one can refer to [6] and [8].

The Lebesgue, Lorentz, Zygmund, Lorentz-Zygmund and Generalized Lorentz-Zygmund spaces obtained by making particular choices of the slowly varying functions.

2. COMPOSITION OPERATORS ON LK SPACES

In this section, we characterize the boundedness and the compactness of composition operator C_T on LK spaces.

Theorem 5.2.1. Let $T : R \to R$ be a non-singular measurable transformation. Then T induces a bounded composition operator on $L_{p,q;b}(R,\mu)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$ if and only if there exists a constant $M > 0$ such that

$$\mu o T^{-1}(A) \leq M\ \mu(A) \tag{5.2.1}$$

for each $A \in \sum$.

Proof. Suppose the condition (5.2.1) hold. Then for each $f \in L_{p,q;b}(R,\mu)$, we have

$$\mu_{C_T f}(\lambda) \leq M\ \ \mu_f(\lambda), \qquad (\lambda \geq 0).$$

Also, for each $t \geq 0$, we have

$$(C_T f)^*(t) \leq f^*(t/M)$$

which implies that

$$(C_T f)^{**}(t) \leq f^{**}(t/M).$$

So, for every $f \in L_{p,q;b}(R,\mu)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$, we have

$$\begin{aligned}
\|C_T f \mid L_{p,q;b}(R,\mu\| &= \| t^{\frac{1}{p}-\frac{1}{q}}\ \gamma_b(t)(C_T f)^{**}(t)|\ L_q(0,\infty)\| \\
&\leq \| t^{\frac{1}{p}-\frac{1}{q}}\ \gamma_b(t)\ f^{**}(t/M)|\ L_q(0,\infty\| \\
&= M^{1/p}\ \| t^{\frac{1}{p}-\frac{1}{q}}\ \gamma_b(Mt)\ f^{**}(t)|\ L_q(0,\infty)\| \\
&\approx M^{1/p}\ \| t^{\frac{1}{p}-\frac{1}{q}} \gamma_b(t)\ f^{**}(t)|\ L_q(0,\infty)\| \\
&= M^{1/p}\ \| f\ |L_{p,q;b}(R,\mu)\|.
\end{aligned}$$

Thus, we see that C_T is a bounded operator on $L_{p,q;b}(R,\mu)$, $1 < p \leq \infty$, and $1 \leq q \leq \infty$.

Conversely, suppose that C_T is bounded on $L_{p,q;b}(R,\mu)$. For $E \in \sum$, with $0 < \mu(E) < \infty$, we have $\chi_E \in L_{p,q;b}(R,\mu)$ as $\|\chi_E \mid L_{p,q;b}(R,\mu)\| \approx (\mu(E))^{1/p}\ \gamma_b(\mu(E)) < \infty$. By using the boundedness of C_T on $L_{p,q;b}(R,\mu)$, we have

$$\|C_T\ \chi_E \mid L_{p,q;b}(R,\mu)\| \leq K\ \|\chi_E \mid L_{p,q;b}(R,\mu)\|$$

for some $K > 0$.

Then there exists some positive constant c and C such that

$$c(\mu o T^{-1}(E)^{1/p}\ \ \gamma_b(\mu o T^{-1}(E)) \leq K\ C(\mu(E))^{1/p}\ \ \gamma_b(\mu(E))$$

which implies that

$$\mu o T^{-1}(E) \le M\, \mu(E)$$

where $M = \left(\frac{kC}{c}\frac{\gamma_b(\mu(E))}{\gamma_b(\mu o T^{-1}(E))}\right)^p > 0$. Thus, for each $E \in \sum$ and for some $M > 0$, we have

$$\mu o T^{-1}(E) \le M(\mu(E).$$

Hence the result.

Theorem 5.2.2. Let C_T be a bounded composition operator on $L_{p,q;b}(R,\mu)$, $1 < p \le \infty$, $1 \le q \le \infty$ induced by a non-singular measurable transformation $T : R \to R$ and let $\{A_n\}_{n\in \mathbf{N}}$ be the atoms of R with $\mu(A_n) = a_n$, for each n. Then C_T is compact if and only if μ is purely atomic and

$$b_n = \frac{\mu o T^{-1}(A_n)}{\mu(A_n)} \to 0.$$

Proof. Since $R = R_1 \cup R_2$, where $R_1 \cap R_2 = \phi$; μ/R_1 is non-atomic and μ/R_2 is purely atomic. So that $\mu = \mu/R_1 + \mu/R_2$. Since $\mu o T^{-1} << \mu$, by the Radon-Nikodym theorem, there exists a function g locally integrable on R_1 such that $\mu o T^{-1}(E) = \int_E g(t)\, d\mu(t)$ for $E \in \sum \cap R_1$. Let $E_o = \{x \in R_1 : g(t) > 0\}$. We will show that $\mu(E_o) = 0$. Assume for the contrary that $\mu(E_0) > 0$. Then there is some $\varepsilon > 0$ such that the set $E_1 = \{x \in E_o : g(x) \ge \varepsilon\}$ has positive measure. Consider a sequence $\{E_n\}_{n=1}^{\infty}$ of pairwise disjoint subset of $\sum \cap E_1$ with $0 < \mu(E_n) < 1/2^n$ for $n \in \mathbf{N}$ large enough such that $n_o \in \mathbf{N}$ and $n > n_o$. Take

$$f_n(x) = \frac{\chi_{E_n}(x)}{\|\chi_{E_n} \mid L_{p,q;b}(R,\mu)\|}\ ;\ x \in R,\ n > n_o.$$

Then $f_n(x) \to 0$ weakly, $\| f_n \mid L_{p,q,b}(R,\mu)\| = 1$ for $n > n_o$ and

$$(C_T f_n)(x) = \frac{\chi_{T^{-1}(E_n)}(x)}{\|\chi_{E_n} \mid L_{p,q;b}(R,\mu)\|},\ x \in R.$$

For $n > n_o$, we have

$$\begin{aligned}\|C_T f_n \mid L_{p,q;b}(R,\mu)\| &\approx \frac{\gamma_b(\mu o T^{-1}(E_n))}{\gamma_b(\mu(E_n))}\frac{(\mu o T^{-1}(E_n))^{1/p}}{(\mu(E_n))^{1/p}}\\ &\approx \left(\frac{\mu o T^{-1}(E_n)}{\mu(E_n)}\right)^{1/p}\\ &\ge \varepsilon^{1/p}.\end{aligned}$$

Consequently, $C_T f_n \nrightarrow 0$ which contradicts the compactness of C_T. Thus, $\mu(E_o) = 0$. So, we conclude that μ is purely atomic.

Next, we will show that $b_n \to 0$. Suppose this is not true. Then there exists some $\varepsilon > 0$ such that $b_n \ge \varepsilon$ for all $n \in \mathbf{N}$. Let $R = \bigcup_{n=1}^{\infty} R_n$, where each R_n is an atom. For each $n \in \mathbf{N}$, let

$$f_n = \frac{\chi_{R_n}}{\| \chi_{T^{-1}(R_n)} \mid L_{p,q;b}(R,\mu) \|}$$

Then for each $n \in \mathbf{N}$, we have

$$\| f_n \mid L_{p,q;b}(R,\mu)\| \approx \left(\frac{\mu(R_n)}{\mu o T^{-1}(R_n)} \right)^{1/p}$$

$$= \frac{1}{(b_n)^{1/p}}$$

$$\leq 1/\varepsilon^{1/p}$$

and

$$\|C_T f_n \mid L_{p,q;b}(R,\mu)\| = 1.$$

Also, for $n \neq m$, we have

$$\begin{aligned} 1 &= ½ \|C_T f_n + C_T f_n \mid L_{p,q;b}(R,\mu)\| \\ &= ½ \|C_T f_n - C_T f_m + C_T f_m + C_T f_n \mid L_{p,q;b}(R,\mu)\| \\ &\leq ½(\|C_T f_n - C_T f_m \mid L_{p,q;b}(R,\mu)\| + \|C_T f_m + C_T f_n \mid L_{p,q;b} (R,\mu)\|) \\ &= \|C_T f_n - C_T f_m \mid L_{p,q;b} (R,\mu)\| \end{aligned}$$

which is a contradiction. Hence $b_n \to 0$.

Conversely, suppose the condition hold. Note that f and $\sum f(A_n) \chi_{A_n}$ are equal μ - almost everywhere. For each $N \in \mathbf{N}$, we define $C_T^{(N)}$ by

$$C_T^{(N)} f = \sum_{n \leq N} f(A_n) \chi_{T^{-1}(A_n)}$$

Then for each $\lambda > 0$, we have

$$\begin{aligned} \mu_{(C_T - C_T^{(N)}) f}(\lambda) &\leq \sum_{n>N, |f(A_n)|>\lambda} \mu(T^{-1}(A_n)) \\ &\leq (\sup_{n>N} b_n) \sum_{|f(A_n)|>\lambda} \mu(A_n). \\ &\leq (\sup_{n>N} b_n)\, \mu_f(\lambda). \end{aligned}$$

Therefore, we can see that

$$\|C_T - C_T^{(N)} \mid L_{p,q;b}(R,\mu)\| \leq (\sup_{n>N} b_n)^{1/p} \to 0$$

as $N \to \infty$. Since C_T is the limit of finite rank operators $C_T^{(N)}$, it is compact. Hence the result.

3. WEIGHTED COMPOSITION OPERATORS ON LK SPACES

In this section, we investigate the boundedness of weighted composition operators on LK spaces.

Theorem 3.1. Let $(R, \sum, \mu)$ be a σ-finite measure space and $\pi : R \to \mathbf{C}$ be a measurable function. Let $T : R \to R$ be any non-singular measurable transformation

such that the Radon–Nikodym derivative $f_T = \dfrac{d\mu \circ T^{-1}}{d\mu}$ is in $L^\infty(R,\mu)$ and $T(E_r) \subseteq E_r$ for each $r > 0$, where $E_r = \{x : |\pi(x)| > r\}$. Then $W_{\pi,T}$ is bounded on $L_{p,q;b}(R,\mu)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$ if and only if $\pi \in L^\infty(R,\mu)$.

Proof: Proof is on the similar lines as in [10].

Assume that for each $r > 0$, E_r is invariant under T. Let $E_m = \{x \in R : |\pi(x)| > m\}$ be a set having finite positive measure for every $m \in \mathbf{N}$. Clearly $\chi_{E_m} \in L_{p,q;b}(R,\mu)$. For every finite $n \in \mathbf{N}$, we find that

$$\begin{aligned}
\{x \in R : |\chi_{E_m}(x)| > \lambda\} &\subseteq \{x \in R : |\chi_{T^{-n}(E_m)}(x)| > \lambda\} \\
&\subseteq \{x \in R : |\pi o T^{n-1} \chi_{T^{-n}(E_m)}(x)| > m\lambda\} \\
&\;\;\vdots \\
&\subseteq \{x \in R : |\pi(x)\, \pi o T(x) \ldots \\
&\qquad \pi o T^{n-1}(x)\ \chi_{T^{-n}(E_m)}(x)| > m^n\lambda\}.
\end{aligned}$$

That is,

$$\mu_{\chi_{E_m}}(\lambda) \leq \mu_{W^n_{\pi,T}\chi_{E_m}}(m^n\lambda) \text{ for } \lambda > 0.$$

Therefore, for each $t \geq 0$,

$$(W^n_{\pi,T}(\chi_{E_m}))^* \geq m^n \chi^*_{E_m}(t)$$

which implies that

$$(W^n_{\pi,T}(\chi_{E_m}))^{**} \geq m^n \chi^{**}_{E_m}(t).$$

Now, we have the following corollaries.

Corollary 3.2. Under the assumption of Theorem 3.1, $W^n_{\pi,T}$ is a bounded operator on $L_{p,q;b}(R,\mu)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$ for every finite $n \in \mathbf{N}$ if and only if $\pi \in L^\infty(R,\mu)$.

Corollary 3.3. Let $T : R \to R$ be a non-singular measurable transformation such that $f_T = \dfrac{d\mu o T^{-1}}{d\mu} \leq 1$ almost everywhere and let $\pi : R \to \mathbf{C}$ be a essentially bounded

measurable function such that $\|\pi\|_\infty \leq 1$. Then for every $n \in \mathbf{N}$, $W^n_{\pi,T}$ is a contractive operator.

4. EXAMPLE

Let W be the following collection of weights:

$$\{\pi \in L^\infty(R,\mu) : 0 < \pi(t) \leq \pi_0(ts) \text{ for each } t \in R \text{ and for some } s \in R\},$$

where $\pi_0 \in L^\infty(R,\mu)$. Suppose that W is closed under multiplication. Next for each $\pi \in W$ and $n \in \mathbf{N}$. We define the operator $W_{\pi,T^n} : L_{p,q;b}(R,\mu) \to L_{p,q;b}(R,\mu)$ by

$$(W_{\pi,T^n} f)(t) = \pi(t) f o T^n(t); \; f \in L_{p,q;b}(R,\mu), \; t \in R$$

where T is a non-singular measurable transformation such that $T^{-1}(A) \subset A$ for every $A \in \sum$. Let $S = \{W_{\pi,T^n} : \pi \in W \text{ and } n \in \mathbf{N}\}$. Then S is a multiplicative semigroup.

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