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Hypersurface of Finsler Space with Exponential Change of (α, β) Metric

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Abstract

The purpose of the present paper is to find the hypersurface of a Finsler space with exponential change of (α, β) metric $L = \alpha e^{\beta/\alpha} + \beta$ given by $b(x)$ = constant. We shall find the conditions under which the hypersurface be a hyperplane of the first or second kinds have been obtained. This hypersurface is not a hyperplane of third kind.

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1 Introduction

Let $F^n = (M^n, L)$ be an n -dimensional Finsler space, where M^n is an n-dimensional differentiable manifold and L (x, y) is the fundamental function. The concept of an (α, β) metric was introduced in 1972 by Matsumoto [4]. A Finsler space F^n is called an (α, β) metric if L is positively homogeneous function of α and β of degree one, where $\alpha^2 = a_{ij}(x) y^i y^j$ and $\beta = b_i(x) y^i$ is one form of M^n. As well-known examples are Randers metric L = α + β [5] Kropina metric $L = \frac{\alpha^2}{\beta}$ [1]. In 1989 M. Matsumoto while studying the slope of mountain introduced an (α, β) metric given by $L = \frac{\alpha^2}{\alpha-\beta}$, which has been called Matsumoto space [5].

2 Fundamental quantities of Finsler space with exponential change of (α, β) metric

The Finsler space with exponential change of (α, β) metric is given by

(2.1) $L(\alpha, \beta) = \alpha e^{\beta/\alpha} + \beta$, where $\alpha^2 = a_{ij}(x) y^i y^j$ and $\beta = b_i(x) y^i$

The derivatives of (2.1) with respect to α and β are given by

(2.2) (a) $L_\alpha = \frac{(\alpha-\beta)}{\alpha} e^{\beta/\alpha}$, (b) $L_\beta = e^{\beta/\alpha} + 1$.

(2.3) (a) $L_{\alpha\alpha} = \frac{\beta^2}{\alpha^3} e^{\beta/\alpha}$, (b) $L_{\alpha\beta} = -\frac{\beta}{\alpha^2} e^{\beta/\alpha}$, (c) $L_{\beta\beta} = \frac{1}{\alpha} e^{\beta/\alpha}$

Where,

$L_\alpha = \frac{\partial L}{\partial \alpha}$, $L_\beta = \frac{\partial L}{\partial \beta}$, $L_{\alpha\alpha} = \frac{\partial^2 L}{\partial \alpha^2}$, $L_{\alpha\beta} = \frac{\partial^2 L}{\partial \alpha\, \partial \beta}$ and $L_{\beta\beta} = \frac{\partial^2 L}{\partial \beta^2}$.

The normalized element of support $l_i = \frac{\partial L}{\partial y^i}$ is given by [8].

(2.4) $l_i = \frac{1}{\alpha} L_\alpha Y_i + L_\beta b_i$ where $Y_i = a_{ij} y^j$.

The angular metric tensor $h_{ij} = L \frac{\partial^2 L}{\partial y^i \partial y^j}$ is given by [8].

(2.5) $h_{ij} = p a_{ij} + q_0 b_i b_j + q_1 (b_i Y_j + b_j Y_i) + q_2 Y_i Y_j$, where

(2.6) $p = L L_\alpha \alpha^{-1} = \frac{e^{\beta/\alpha}}{\alpha^2} (\alpha - \beta)(\alpha e^{\beta/\alpha} + \beta)$.

(2.7) $q_0 = L L_{\beta\beta} = \frac{e^{\beta/\alpha}}{\alpha} (\alpha e^{\beta/\alpha} + \beta)$.

(2.8) $q_1 = L L_{\alpha\beta} \alpha^{-1} = \frac{e^{\beta/\alpha}}{\alpha^3} \beta (\alpha e^{\beta/\alpha} + \beta)$,

(2.9) $q_2 = L \alpha^{-2} (L_{\alpha\alpha} - L_\alpha \alpha^{-1}) = \frac{e^{\beta/\alpha}}{\alpha^5} (\beta^2 - \alpha^2 + \alpha\beta)(\alpha e^{\beta/\alpha} + \beta)$.

The fundamental metric tensor

$g_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j}$ is given by [8] .

(2.10) $g_{ij} = p a_{ij} + p_0 b_i b_j + p_1 (b_i Y_j + b_j Y_i) + p_2 Y_i Y_j$

(2.11) $p_0 = q_0 + L^2{}_\beta = \frac{e^{\beta/\alpha}}{\alpha}(\alpha e^{\beta/\alpha} + \beta) + (e^{\beta/\alpha} + 1)^2$

(2.12) $p_1 = q_1 + p L_\beta L^{-1} = \frac{e^{\beta/\alpha}}{\alpha^3}[\alpha(\alpha - \beta)(e^{\beta/\alpha} + 1) - \beta(\alpha e^{\beta/\alpha} + \beta)]$

(2.13) $p_2 = q_2 + p^2 L^{-2} = \frac{e^{\beta/\alpha}}{\alpha^5}[(\beta^2 - \alpha^2 + \alpha\beta)(\alpha e^{\beta/\alpha} + \beta) + \alpha(\alpha-\beta)^2 e^{\beta/\alpha}]$

Moreover, the reciprocal tensor g^{ij} of g_{ij} is given by

(2.14) $g^{ij} = p^{-1} a^{ij} - s_0 b^i b^j - s_1(b^i y^j + b^j y^i) - s_2 y^i y^j$,where

(2.15) (a) $b^i = a^{ij} b_j$, $b^2 = a_{ij} b^i b^j$,

(b) $s_0 = \frac{1}{\tau p}[p p_o + (p_o p_2 - p_1^2)\alpha^2]$,

(c) $s_1 = \frac{1}{\tau p}[p p_1 + (p_o p_2 - p_1^2)\beta]$,

(d) $s_2 = \frac{1}{\tau p}[p p_2 + (p_o p_2 - p_1^2) b^2]$,

(e) $\tau = p(p + p_o b^2 + p_1 \beta) + (p_o p_2 - p_1^2)(\alpha^2 b^2 - \beta^2)$

The hv - torsion tensor $C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}$ is given by [9] .

(2.16) $2p C_{ijk} = p_1(h_{ij} m_k + h_{jk} m_i + h_{ki} m_j) + \gamma_1 m_i m_j m_k$.

Where

(2.17) (a) $\gamma_1 = p\frac{\partial p_o}{\partial \beta} - 3 p_1 q_0$, (b) $m_i = b_i - \alpha^{-2}\beta Y_i$.

It is noted that the covariant vector m_i is a non-vanishing one and is orthogonal to the element of support y^i.

Let $\{ {i \atop jk} \}$ be the component of Christoffel's symbol of associated Riemannian space R^n and ∇_k denotes the covariant differentiation with respect to x^k relative to the Christoffel's symbol. We shall use the following tensors.

(2.18) (a) $2E_{ij} = b_{ij} + b_{ji}$, (b) $2F_{ij} = b_{ij} - b_{ji}$,

where , $b_{ij} = \nabla_j b_i$.

If we denote the Cartan`s connection CΓ as (Γ^{*i}_{jk} , Γ^{*i}_{ok} , C^i_{jk})

then the difference tensor $D^i_{jk} = \Gamma^{*i}_{jk} - \{ {i \atop jk} \}$ of exponential change of Finsler space with (α, β) metric $L = \alpha e^{\beta/\alpha} + \beta$ is given by [2] .

(2.19) $D^i_{jk} = B^i E_{jk} + F^i_k B_j + F^i_j B_k + B^i_j b_{ok} + B^i_k b_{oj} - b_{om} g^{im} B_{jk} - C^i_{jm} A^m_k - C^i_{km} A^m_j + C_{jkm} A^m_s g^{is} + \lambda^s (C^i_{jm} C^m_{sk} + C^i_{km} C^m_{sj} - C^i_{ms} C^m_{jk})$,

where

(2.20) (a) $B_k = p_o b_k + p_1 Y_k$

(b) $B^i = g^{ij} B_j$,

(c) $F^k_i = g^{kj} F_{ji}$,

(d) $B_{ij} = \frac{1}{2}\{ p_1 (a_{ij} - \alpha^{-2} Y_i Y_j) + \frac{\partial p_o}{\partial \beta} m_i m_j \}$,
(e) $B_i^k = g^{kj} B_{ji}$,
(f) $A_k^m = B_k^m E_{oo} + B^m E_{ko} + B_k F_o^m + B_o F_k^m$,
(g) $\lambda^m = B^m E_{oo} + 2 B_o F_o^m$, (h) $B_o = B_i y^i$.

Where 'o' denote the contraction with y^i except for the quantities p_o, q_0 and s_0.

3 Induced Cartan Connection

Let $F^{n-1} = (M^{n-1}, L(u, v))$ be a hypersurface of $F^n = (M^n, L(x, y))$ given by the equations $x^i = x^i(u^\alpha)$, where α = 1,2,3---------n-1
Suppose that the matrix of the projection factor $B_\alpha^i = \frac{\partial x^i}{\partial u^\alpha}$ is of rank n-1.
The element of support y^i of F^n is to be taken tangential to F^{n-1} i. e.

(3.1) $y^i = B_\alpha^i(u)\, v^\alpha$.

Thus v^α is the element of support of F^{n-1} at the point u^α. The metric tensor $g_{\alpha\beta}$ and hv – torsion tensor $C_{\alpha\beta\gamma}$ of F^{n-1} is given by

(3.2) (a) $g_{\alpha\beta} = g_{ij} B_\alpha^i B_\beta^j$, (b) $C_{\alpha\beta\gamma} = C_{ijk} B_\alpha^i B_\beta^j B_\gamma^k$.

At each point u^α of F^{n-1} a unit normal vector $N^i(u, v)$ is defined by

(3.3) (a) $g_{ij}(x(u), y(u,v)) B_\alpha^i N^j = 0$,
(b) $g_{ij}(x(u), y(u,v)) N^i N^j = 1$.

As for the angular metric tensor h_{ij} we have

(3.4) (a) $h_{\alpha\beta} = h_{ij} B_\alpha^i B_\beta^j$, (b) $h_{ij} B_\alpha^i N^j = 0$, (c) $h_{ij} N^i N^j = 1$.

If (B_i^α, N_i) denote the inverse of (B_α^i, N^i) then we have

(3.5) (a) $B_i^\alpha = g_{ij} g^{\alpha\beta} B_\beta^j$, (b) $B_\alpha^i B_i^\beta = \delta^\beta_\alpha$,
(c) $B_i^\alpha N^i = 0$, (d) $B_\alpha^i N_i = 0$,
(e) $N_i = g_{ij} N^j$, (f) $B_\alpha^i B_j^\alpha + N^i N_j = \delta^i_j$.

The induced connection ICΓ = ($\Gamma_{\beta\gamma}^{*\alpha}$, G_β^α, $C_{\beta\gamma}^\alpha$) of F^{n-1} induced from the Cartan's connection

CΓ = (Γ_{jk}^{*i}, Γ_{ok}^{*i}, C_{jk}^i) is given by [6].

(3.6) $\Gamma_{\beta\gamma}^{*\alpha} = B_i^\alpha (B_{\beta\gamma}^i + \Gamma_{jk}^{*i} B_\beta^j B_\gamma^k) + M_\beta^\alpha H_\gamma$.

(3.7) $G_\beta^\alpha = B_i^\alpha (B_{o\beta}^i + \Gamma_{oj}^{*i} B_\beta^j)$.

(3.8) $C_{\beta\gamma}^\alpha = B_i^\alpha C_{jk}^i B_\beta^j B_\gamma^k$.

Where

(3.9) (a) $M_{\beta\gamma} = N_i C_{jk}^i B_\beta^j B_\gamma^k$, (b) $M_\beta^\alpha = g^{\alpha\gamma} M_{\beta\gamma}$.

(3.10) $H_\beta = N_i\ (B^i_{o\beta} + \Gamma^{*i}_{oj}\ B^j_\beta\ )$.

(3.11) (a) $B^i_{\beta\gamma} = \frac{\partial B^i_\beta}{\partial u^\gamma}$, (b) $B^i_{o\beta} = B^i_{\gamma\beta}\ v^\gamma$.

The quantities $M_{\beta\gamma}$ and H_β are called second fundamental v- tensor and normal curvature vector respectively [6].

The second fundamental h- tensor $H_{\beta\gamma}$ is defined as [6].

(3.12) $H_{\beta\gamma} = N_i\ (B^i_{\beta\gamma} + \Gamma^{*i}_{jk}\ B^j_\beta\ B^k_\gamma) + M_\beta\ H_\gamma$,where

(3.13) $M_\beta = N_i\ C^i_{jk}\ B^j_\beta\ N^k$.

The relative h and v – covariant derivatives of projection factor B^i_α with respect to ICΓ are given by

(3.14) (a) $B^i_{\alpha|\beta} = H_{\alpha\beta}\ N^i$, (b) $B^i_\alpha|\ \beta = M_{\alpha\beta}\ N^i$.

The equation (3.12) shows that $H_{\beta\gamma}$ is generally not symmetric and

(3.15) $H_{\beta\gamma} - H_{\gamma\beta} = M_\beta\ H_\gamma - M_\gamma\ H_\beta$.

Furthermore (3.10), (3.12), (3.13) yield

(3.16) (a) $H_{o\gamma} = H_\gamma$, (b) $H_{\gamma o} = H_\gamma + M_\gamma\ H_o$.

We quote the following lemma which is due to Matsumoto [6].

Lemma 3.1 - The normal curvature $H_o = H_\gamma\ v^\gamma$ vanishes if and only if normal curvature vector H_β vanishes.

The hyperplanes of first, second and third kind are defined in [6] and we only quote the following.

Lemma3.2 - A hypersurface F^{n-1} is a hyperplane of the first kind iff $H_\alpha = 0$

Lemma3.3 -A hypersurface F^{n-1} is a hyperplane of the second kind iff $H_\alpha = 0$ and $H_{\alpha\beta} = 0$.

Lemma3.4 - A hypersurface F^{n-1} is a hyperplane of the third kind iff $H_\alpha = 0$ and $M_{\alpha\beta} = H_{\alpha\beta} = 0$.

4 Hypersurface F^{n-1}(c) of Finsler space with exponential change of (α, β) metric

Let us consider the Finsler space with exponential change of (α, β) metric $L = \alpha e^{\beta/\alpha} + \beta$ with a gradient $b_i(x) = \partial_i b$ for a scalar function b(x) =c (constant). From parametric equations $x^i = x^i(u^\alpha)$ of F^{n-1} (c) we get

$\frac{\partial}{\partial u^\alpha}$ { b(x(u)) } = 0, which implies that $b_i B^i_\alpha = 0$.

So that $b_i(x)$ are regarded as covariant components of a normal vector of F^{n-1} (c). Therefore, along the F^{n-1} (c) we have

(4.1) (a) $b_i B^i_\alpha = 0$, (b) $b_i y^i = 0$.

In general, the induced metric L (u, v) from the F^{n-1} (c) is given by

$$L(u,v) = \frac{a_{ij}\, B^i_\alpha\, B^j_\beta\, v^\alpha\, v^\beta}{\sqrt{a_{ij} B^i_\alpha\, B^j_\beta\, v^\alpha\, v^\beta - b_i\, B^i_\alpha\, v^\alpha}} .$$

Therefore, the induced metric of F^{n-1} (c) becomes

(4.2) (a) $L(u, v) = \sqrt{a_{\alpha\beta}(u) v^\alpha v^\beta}$, (b) $a_{\alpha\beta} = a_{ij}\ B^i_\alpha\ B^j_\beta$,

which is the Riemannian metric. At the point of F^{n-1} (c) from (2.6), (2.7), (2.8) and (2.9) we have

(4.3) $p=1, q_0 = 1,\ q_1 = 0, q_2 = -\frac{1}{\alpha^2}$.

The quantities in the equations (2.11), (2.12) and (2.13) reduces to

(4.4) $p_o = 5,\quad p_1 = \frac{2}{\alpha},\quad p_2 = 0$

Whereas (2.15) reduces to

(4.5) (a) $\tau = (1+b^2)$, (b) $s_0 = \frac{1}{(1+b^2)}$,(c) $s_1 = \frac{2}{\alpha(1+b^2)}$,(d) $s_2 = -\frac{4b^2}{\alpha^2(1+b^2)}$.

Therefore from (2.14) we get

(4.6) $g^{ij} = a^{ij} - \frac{1}{(1+b^2)} b^i b^j - \frac{2}{\alpha(1+b^2)} b^i y^j + b^j y^i) + \frac{4b^2}{\alpha^2(1+b^2)} y^i y^j$.

Thus along F^{n-1} (c), (4.1) and (4.6) gives

(4.7) $g^{ij}\ b_i b_j = \frac{b^2}{(1+b^2)}$. Therefore, we get,

(4.8) $b_i(x(u)) = \sqrt{\frac{b^2}{(1+b^2)}}\ N_i$,

where $b^2 = a^{ij}\ b_i b_j$ and b is the length of the vector b^i

Again from (4.6) and (4.8) we have

(4.9) $b^i = a^{ij}\ b_j = \sqrt{b^2(1+b^2)}\ N^i + \frac{2b^2}{\alpha^2} y^i$.

Hence we have the following.

Theorem 4.1 - Let F^n be an exponential change of Finsler space with (α, β) metric $L = \alpha e^{\beta/\alpha} + \beta$ with gradient $b_i(x) = \partial_i b(x)$ and let F^{n-1} (c) be a hypersurface of F^n which is given by b(x) =c (constant). Suppose the Riemannian metric $a_{ij}(x)\, dx^i\, dx^j$ be positive definite and b_i be non-zero field. Then the induced metric on F^{n-1} (c) is a Riemannian metric given by (4.2) and the relations (4.8) and (4.9) hold.

Along F^{n-1} (c) the angular metric tensor and metric tensor are given by

(4.10) $h_{ij} = a_{ij} + b_i b_j - \frac{1}{\alpha^2} Y_i\ Y_j$, and

(4.11) $g_{ij} = a_{ij} + 5\, b_i b_j + \frac{2}{\alpha}(b_i Y_j + b_j Y_i)$.

If $h^{(a)}_{\alpha\beta}$ denote the angular metric tensor of the Riemannian $a_{ij}(x)$ then using (4.1) in (4.10) we have along F^{n-1} (c)

(4.12) $h_{\alpha\beta} = h^{(a)}_{\alpha\beta}$. From (2.11) we have

(4.13) $\frac{\partial p_o}{\partial\beta} = \frac{5}{\alpha}$ (along F^{n-1} (c)) . Therefore (2.17) gives

(4.14) (a) $\gamma_1 = -\frac{1}{\alpha}$, (b) $m_i = b_i$.

Therefore hv - torsion tensor becomes

(4.15) $C_{ijk} = \frac{1}{\alpha}(h_{ij}\, b_k + h_{jk}\, b_i + h_{ki}\, b_j) - \frac{1}{2\alpha} b_i\, b_j\, b_k$.

Hence from (3.9), (4.1), (4.8), and (4.15) we have

(4.16) $M_{\alpha\beta} = \frac{1}{\alpha}\sqrt{\frac{b^2}{(1+b^2)}}\; h_{\alpha\beta}$.

Also from (3.4), (3.13), (4.1) and (4.15) we have

(4.17) $M_\alpha = 0$. On using (4.17) in (3.15) we have

(4.18) $H_{\alpha\beta} = H_{\beta\alpha}$.

Theorem 4.2 - The second fundamental v- tensor $M_{\alpha\beta}$ of F^{n-1} (c) is given by (4.16) and the second fundamental h- tensor $H_{\alpha\beta}$ is symmetric.

Next from (4.1) we get

$b_{i|\beta}\, B^i_\alpha + b_i\, B^i_{\alpha|\beta} = 0$.

Therefore from (3.14) and the fact that

$b_{i|\beta} = b_{i|j}\, B^j_\beta + b_i|_j\, N^j\, H_\beta$, we get

(4.19) $b_{i|j}\, B^i_\alpha\, B^j_\beta + b_i|_j\, B^i_\alpha\, N^j\, H_\beta + b_i\, H_{\alpha\beta}\, N^i = 0$.

Since $b_i|_j = -\, b_h\, C^h_{ij}$, from (3.13), (4.8) and (4.17) we get

(4.20) $b_{i|j}\, B^i_\alpha\, B^j_\beta + \sqrt{\frac{b^2}{(1+b^2)}}\; H_{\alpha\beta} = 0$.

Since $b_{i|j}$ is symmetric tensor Contracting (4.20) with respect to v^β and using (3.16) we get

(4.21) $b_{i|j}\, B^i_\alpha\, y^j + \sqrt{\frac{b^2}{(1+b^2)}}\; H_\alpha = 0$.

Further contracting (4.21) with respect to v^α we get

(4.22) $b_{i|j}\, y^i\, y^j + \sqrt{\frac{b^2}{(1+b^2)}}\; H_o = 0$.

In view of Lemma (3.1) and (3.2) the hypersurface F^{n-1} (c) is a hyperplane of the first kind if and only if $H_o = 0$. Thus from (4.22) it follows that F^{n-1} (c) is a hyperplane of the first kind if and only if $b_{i|j}\, y^i y^j = 0$. This $b_{i|j}$ being the covariant derivative with respect to CΓ of F^n , it may depend on y^i . On the other hand $\nabla_j b_i = b_{ij}$ is the covariant derivative with respect to the Riemannian tensor $\{ {}^{\,i}_{jk} \}$ constructed from $a_{ij}(x)$, therefore b_{ij} does not depend on y^i . We shall consider the difference $b_{i|j} - b_{ij}$ in the following . The difference tensor

$D^i_{jk} = \Gamma^{*i}_{jk} - \{ {}^{\,i}_{jk} \}$ is given by (2.19) .

Since b_i is a gradient vector from (2.18) we have

(4.23) $E_{ij} = b_{ij}$, $F_{ij} = 0$, $F^i_j = 0$.

Thus (2.19) reduces to

(4.24) $D^i_{jk} = B^i\ b_{jk} + B^i_j\ b_{ok} + B^i_k\ b_{oj} - b_{om}\ g^{im}\ B_{jk} - C^i_{jm}\ A^m_k - C^i_{km}\ A^m_j + C_{jkm}\ A^m_s\ g^{is} + \lambda^s\ (C^i_{jm}\ C^m_{sk} + C^i_{km} C^m_{sj} - C^i_{ms} C^m_{kj}\ )$.

But in view of (4.3), (4.4) and (4.6) the expression (2.20) reduces to

(4.26)

(a) $B_i = 5\ b_i + \frac{2}{\alpha} Y_i$,

(b) $B^i = g^{ij}\ B_j = \frac{1}{(1+b^2\ )} b^i + \frac{2}{\alpha(1+b^2\ )} y^i$,

(c) $B_{ij} = \frac{1}{\alpha}\ (\ a_{ij} - \frac{1}{\alpha^2} Y_i\, Y_j\ ) + \frac{5}{2\alpha}\ b^i\, b_j$,

(d) $B^i_j = \frac{1}{\alpha}(\delta^i_j - \frac{1}{\alpha^2}\ y^i\ Y_j\ ) + \frac{3}{2\,\alpha(1+b^2\ )} b^i b_j - \frac{(\,2+5\,b^2\ )}{\alpha^2(1+b^2\,)} y^i\ b_j$,

(e) $A^m_k = B^m_k\ b_{oo} + B^m\ b_{ko}$,

(f) $\lambda^m = B^m\ b_{oo}$.

By virtue of (4.1) we have $B^i_o = o$, $B_{io} = o$ which gives $A^m_o = B^m\ b_{oo}$ .We have therefore

(4.26) $D^i_{jo} = B^i\, b_{jo} + B^i_j\ b_{oo} - B^m\ C^i_{jm}\ b_{oo}$.

Again contracting (4.26) with y^j, we get

(4.27) $D^i_{oo} = B^i\ b_{oo} = [\ \frac{b^i}{(1+b^2\ )} + \frac{2}{\alpha(1+b^2\ )} y^i\ ]\ b_{oo}$,

Thus paying attention to (4.1) along the F^{n-1} (c) we finally get

(4.28) $b_i\ D^i_{jo} = \frac{b^2}{(1+b^2\ )} b_{jo} + \frac{(2\ +5\,b^2\ )}{2\,\alpha(1+\ b^2\,)} b_j\, b_{oo} - \frac{1}{(1+b^2\ )} b_i\, b^r C^i_{jr}\ b_{oo}$.

On contracting (4.28) by y^j we have

(4.29) $b_i\ D^i_{oo} = \frac{b^2}{(1+b^2\ )} b_{oo}$.

From (3.13), (4.8), (4.9) and (4.17) we have

$b_i\ b^m\ C^i_{jm}\ B^j_\alpha = b^2 M_\alpha = 0$. Therefore the relation

$b_{i|j} = b_{ij} - b_r\ D^r_{ij}$ and the equations (4.28) and (4.29) give

(4.30) $b_{i|j}\ y^i y^j = b_{oo} - b_r\ D^r_{oo} = \frac{1}{(1+b^2\ )}\ b_{oo}$.

Consequently (4.21) and (4.22) may be written as

(4.31) $\sqrt{\frac{b^2}{(1+b^2\ )}}\ H_\alpha + \frac{1}{(1+b^2\ )}\ b_{io} B^i_\alpha = 0$.

(4.32) $\sqrt{\frac{b^2}{(1+b^2\ )}}\ H_o + \frac{1}{(1+b^2\ )}\ b_{oo} = 0$.

Thus the condition $H_o = 0$ is equivalent to b_{oo}, where b_{ij} does not depend on y^i Since y^i is to satisfy (4.1), the condition is written as $b_{ij}\ y^i y^j = (b_i\ y^i)$ $(C_j\, y^j)$ for some $C_j(x)$, so that we have

(4.33) $2 b_{ij} = b_i C_j + b_j C_i$. From (4.1) and (4.33) it follows that

(4.34) (a) $b_{ij} = 0$, (b) $b_{ij} B^i_\alpha B^j_\beta = 0$, (c) $b_{ij} B^i_\alpha y^j = 0$.

Hence (4.32) gives $H_o = 0$.

Again from (4.24), (4.25) and (4.33) we have

(4.35) (a) $b_{io} b^i = \frac{1}{2} c_o b^2$, (b) $\lambda^r = 0$ (c) $A^i_j B^j_\beta = 0$, (d) $B_{ij} B^i_\alpha B^j_\beta = \frac{1}{\alpha} h_{\alpha\beta}$.

Thus from (4.23) we have

(4.36) $b_r D^r_{ij} B^i_\alpha B^j_\beta = - \frac{1}{2\alpha}(g^{rs} b_r b_s) C_o h_{\alpha\beta} + b_r C_{ijm} A^m_s g^{rs} B^i_\alpha B^j_\beta$.

Also from (4.6) we find

(4.37) $b_r b_s g^{rs} = \frac{b^2}{(1+ b^2)}$. With the help of (3.9), (4.9), (4.16), (4.24)and (4.25) we get (4.38) $b_r C_{ijm} A^m_s g^{rs} B^i_\alpha B^j_\beta = \frac{C_o b^2}{2\alpha(1+ b^2)} (b_r b_s g^{rs}) h_{\alpha\beta}$.

Substituting (4.37), and (4.38) in (4.36), we get

(4.39) $b_r D^r_{ij} B^i_\alpha B^j_\beta = - \frac{C_o b^2}{2\alpha (1+b^2)^2} h_{\alpha\beta}$ .Therefore (4.20) reduces to

(4.40) $\sqrt{\frac{b^2}{(1+ b^2)}} H_{\alpha\beta} + \frac{C_o b^2}{2\alpha (1+b^2)^2} h_{\alpha\beta} = 0$.

Hence the hypersurface F^{n-1} (c) is umbilic.

Theorem 4.3- The necessary and sufficient condition for F^{n-1} (c) to be hyperplane of first kind is (4.23) and in this case the second fundamental tensor of F^{n-1} (c) is proportional to its angular metric tensor.

In view of Lemma (3.3) F^{n-1} (c) is a hyperplane of second kind iff $H_\alpha = 0$ and $H_{\alpha\beta} = 0$. Thus from (4.40) we get $C_o = C_i (x) y^i = 0$.

Therefore, there exist a function Ψ(x) such that

(4.41) $C_i (x) = \Psi(x) b_i (x)$. Thus (4.33) gives

(4.42) $b_{ij} = \Psi b_i b_j$.

Theorem 4.4 – The necessary and sufficient condition for F^{n-1} (c) to be hyperplane of second kind is (4.42).

Finally (4.16), (4.17) and lemma (3.4) shows that F^{n-1} (c) does not become a hyperplane of the third kind.

Theorem 4.5 – The hypersurface F^{n-1} (c) is not a hyperplane of the third kind.

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