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Explicit Examples of Bounded Sequences with Countably Infinite Subsequential Limits and Limit Points

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Abstract

We construct explicit examples of bounded sequences $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ with prescribed behaviors for their accumulation properties. Specifically, we present one sequence whose set of subsequential limits $S = \{\lim\limits_{k\to\infty} a_{n_k} \mid \{a_{n_k}\}$ is a convergent subsequence of $\{a_n\}\}$ has cardinality $\aleph_0$, the smallest infinite cardinality. We also construct a different example where the set of limit points $T = \{x \in \mathbb{R} \mid x$ is a limit point of $\{a_n\}\}$ has cardinality $\aleph_0$ as well.

These examples illustrate that not only can S and T differ in structure, but that both sets can be countably infinite—a possibility not often emphasized in introductory analysis. This work contributes to a deeper understanding of the diversity of limiting behavior in sequences and highlights the subtle distinctions between subsequential limits and limit points.

Keywords: bounded sequence; subsequential limits; limit points; cardinality; countable infinity

1 Introduction

Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence in $\mathbb{R}$, and define the set of its subsequential limits as

$$S = \left\{ \lim_{k\to\infty} a_{n_k} \,\middle|\, a_{n_k} \text{ is a convergent subsequence of } a_n \right\}.$$

Similarly, define the set of its limit points as

$$T = \{x \,|\, x \text{ is a limit point of } \{a_n\}_{n=1}^{\infty}\}.$$

For basic definitions such as sequence, boundedness, convergence, subsequence, and limit point, we refer the reader to [2]. For foundational set theory definitions, see [1].

It is known that both S and T are bounded subsets of $\mathbb{R}$ and that $S \supseteq T$ by the definition of limit points. Furthermore, by the Bolzano-Weierstrass Theorem, $T \neq \varnothing$ if $|\{a_n\}| = \infty$, where $|A|$ denotes the cardinality of the set A. See Theorem 2.3.8 on page 78 of [2]. It is possible for S to properly contain T. For example, when $a_n = a$ is a constant sequence, then $S = \{a\}$ while $T = \varnothing$.

The purpose of this article is to investigate the cardinalities of the sets S and T for various sequences $\{a_n\}_{n=1}^{\infty}$. Specifically, we aim to construct explicit examples where both S and T have countably infinite cardinality.

Interestingly, while the question can be readily posed to AI systems, several prominent large language models—including ChatGPT, Deepseek, and Grok—consistently fail to provide correct answers. In most cases, their responses do not yield a sequence with a countably infinite set of limit points, instead producing constructions similar to those in Proposition 1.

It is worth noting that a countable set may indeed have an uncountable set of limit points, as exemplified by the rational numbers in the real line. Consequently, a rigorous analysis is essential to ensure correctness. However, none of the LLMs tested were capable of delivering a rigorous argument comparable to the proof in Proposition 2.

As subsets of $\mathbb{R}$, both S and T have cardinalities at most $\mathfrak{c} = 2^{\aleph_0}$. This upper bound is sharp: for instance, if $\{a_n\}$ is the enumeration of all rational numbers in $(0, 1)$, then $S = T = [0, 1]$, and thus both sets have cardinality $\mathfrak{c}$.

It is also easy to construct examples where $|S|$ and $|T|$ are finite. For any integer $k \geq 1$, let $\{a_{n,m}\}_{n=1}^{\infty}$ be a sequence in $(m, m + 1)$ converging to m, for each $1 \leq m \leq k$. Then, for the sequence

$$\{a_{1,1}, a_{1,2}, \ldots, a_{1,k},\ a_{2,1}, a_{2,2}, \ldots, a_{2,k},\ \ldots,\ a_{n,1}, a_{n,2}, \ldots, a_{n,k}, \ldots\},$$

we have $S = T = \{1, 2, \ldots, k\}$ and $|S| = |T| = k$.

So far, we have seen examples where both $|S|$ and $|T|$ can take any finite value or the uncountable cardinality $\mathfrak{c}$. But what about the countably infinite case $\aleph_0$?

2 Main results

In this article, we construct two explicit sequences: $\{x_n\}$ for which $|S| = \aleph_0$, and $\{y_n\}$ for which $|T| = \aleph_0$.

For an example involving an unbounded sequence, see [3].

Starting with the sequence $\{\dfrac{1}{n}\}_{n=1}^{\infty}$ as our base, we construct a new sequence $\{x_n\}_{n=1}^{\infty}$ such that its set of subsequential limits is exactly

$$S = \{0\} \cup \left\{\frac{1}{n}\right\}_{n=1}^{\infty},$$

ensuring that S is countably infinite. The construction of $\{x_n\}$ is designed so that each value $\frac{1}{n}$ appears infinitely often in a controlled fashion while also allowing accumulation at 0.

To achieve this, we define $\{x_n\}$ iteratively, as illustrated in Figure 1:

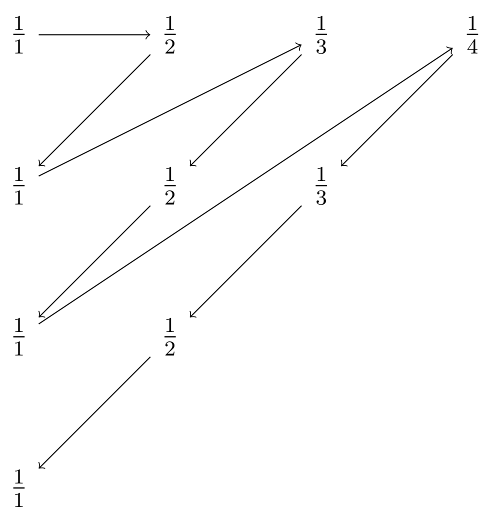

Figure 1: A sequence with countably infinite subsequential limits

The essential idea behind the construction is that the countable product of countable sets is still countable. The sequence $\{x_n\}$ is generated by creating infinitely many initial segments of $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$, then enumerating all elements diagonally—an idea commonly used in texts. See, for example, the first picture on page 76 of [1] or the illustration on page 19 of [2].

Proposition 1. *Let $\{x_n\}$ be the sequence defined as:*

$$\left(\frac{1}{1}\right),\ \left(\frac{1}{2},\frac{1}{1}\right),\ \left(\frac{1}{3},\frac{1}{2},\frac{1}{1}\right),\ \left(\frac{1}{4},\frac{1}{3},\frac{1}{2},\frac{1}{1}\right),\ \ldots$$

as illustrated in Figure 1. Then the set of subsequential limits of $\{x_n\}$ is countably infinite.

Proof. We aim to show that

$$S = \{0\} \cup \left\{\frac{1}{n}\right\}_{n=1}^{\infty}.$$

From Figure 1, it is clear that each term $\frac{1}{n}$ appears infinitely often in the sequence. In fact, each vertical column in the figure corresponds to a constant subsequence converging to $\frac{1}{n}$. Similarly, every horizontal row reproduces the sequence $\left\{\frac{1}{n}\right\}$, whose limit is 0. Hence $\{0\} \cup \left\{\frac{1}{n}\right\}_{n=1}^{\infty} \subseteq S$.

To prove the reverse inclusion, suppose by contradiction that there exists $x \in S - (\{0\} \cup \left\{\frac{1}{n}\right\}_{n=1}^{\infty})$. Then some subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x, and $x \in (0,1)$ since each $x_n \in (0,1]$.

Because $(0,1) = \displaystyle\bigcup_{k=1}^{\infty}\left[\frac{1}{k+1}, \frac{1}{k}\right)$, there exists k such that $x \in \left[\frac{1}{k+1}, \frac{1}{k}\right)$. By the property of x, we know $\frac{1}{k+1} < x < \frac{1}{k}$. This implies

$$\delta := \min\left\{x - \frac{1}{k+1}, \frac{1}{k} - x\right\} > 0.$$

However, since $\{x_{n_k}\}$ converges to x, for any $\epsilon > 0$ there exists N such that for all $k > N$, $|x_{n_k} - x| < \epsilon$. But each x_{n_k} equals $\frac{1}{n}$ for some n, so the set $\left\{\frac{1}{n}\right\}$ would have to contain points within ϵ of x.

Yet, by the choice of δ, no term in $\left\{\frac{1}{n}\right\}$ lies within δ of x, which contradicts the convergence of $\{x_{n_k}\}$. Therefore, no such x exists, and we conclude $S = \{0\} \cup \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$, which is countably infinite. $\square$

The above example shows that $|S| = \aleph_0$ is possible. However, in that example, we have $T = \{0\}$, so $|T| = 1$. Indeed, suppose $x \in T - \{0\} \subseteq S - \{0\} = \{\frac{1}{n}\}_{n=1}^{\infty}$. Then $x = \frac{1}{n}$ for some $n \geq 1$. Define

$$\delta = \begin{cases} \min\left\{\frac{1}{n} - \frac{1}{n+1}, \frac{1}{n-1} - \frac{1}{n}\right\} = \frac{1}{n(n+1)} & \text{if } n \geq 2, \\ \frac{1}{2} & \text{if } n = 1. \end{cases}$$

Then, there is no other term of the form $\frac{1}{k}$ within the δ-neighborhood of $\frac{1}{n}$ except for $\frac{1}{n}$ itself. Therefore, x is not a limit point of the sequence $\{x_n\}_{n=1}^{\infty}$.

To construct a sequence such that $|T| = \aleph_0$, we will apply a similar idea to that in Figure 1, but with an appropriate shift to ensure accumulation at infinitely many distinct points.

Proposition 2. *Let $\{y_n\}$ be the sequence defined as*

$$\left\{(\frac{1}{1}), (\frac{1}{2}, \frac{1}{1} + \frac{1}{1}), (\frac{1}{3}, \frac{1}{2} + \frac{1}{1}, \frac{1}{1} + \frac{1}{2}), (\frac{1}{4}, \frac{1}{3} + \frac{1}{1}, \frac{1}{2} + \frac{1}{2}, \frac{1}{1} + \frac{1}{3}), \dots\right\},$$

as illustrated in Figure 2. Then the set of limit points of $\{y_n\}$ is countably infinite.

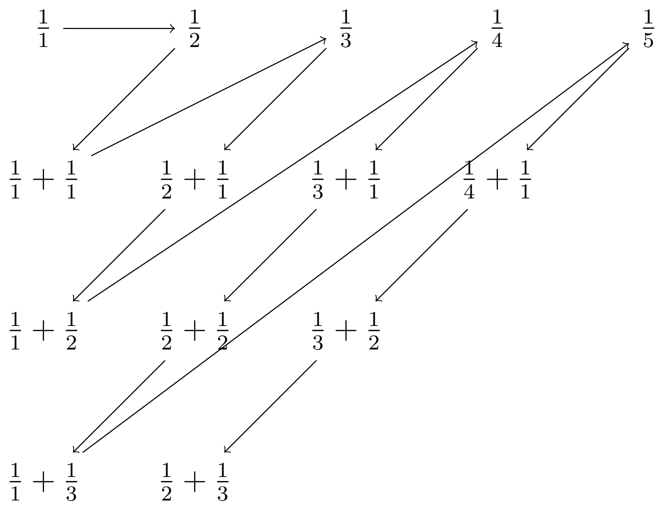

Figure 2: A sequence with countably infinite limit points

Proof. We aim to show that

$$T = \{0\} \cup \left\{\frac{1}{n}\right\}_{n=1}^{\infty}.$$

From Figure 2, it is clear that $\{\frac{1}{n}\}_{n=1}^{\infty} \subseteq T$. Indeed, for each $n \geq 1$, the n-th vertical column (or $(n+1)$-th horizontal row) in the figure corresponds to the sequence $\left\{\frac{1}{n} + \frac{1}{k}\right\}_{k=1}^{\infty}$, which converges to $\frac{1}{n}$.

Similarly, $0 \in T$ since the first horizontal row is the sequence $\left\{\frac{1}{k}\right\}_{k=1}^{\infty}$, which converges to 0.

Now suppose, for contradiction, that there exists $y \in T - \left(\{0\} \cup \left\{\frac{1}{n}\right\}_{n=1}^{\infty}\right)$, so that there is a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ converging to y. Since $\{y_n\} \subseteq (0, 2]$, we have $y \in (0, 2]$, and in particular $y \in (0, 2] - \{1\}$.

We first claim that $y \leq 1$. Suppose instead that $y > 1$. Let $a_{i,j}$ denote the (i, j)-entry in the infinite table of Figure 2. Observe that $a_{i,j}$ is decreasing in both rows and columns. So,

$$a_{i,j} \leq a_{3,3} = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}, \quad \forall i, j \geq 3.$$

Hence, any subsequence converging to $y > 1$ can only use finitely many entries from $a_{i,j}$ with $i, j \geq 3$. Therefore, it must have infinitely many terms in the first two rows. But the values in those rows converge to 0 or 1, so they cannot converge to $y > 1$.

So $y \in (0, 1)$. Since $(0, 1) = \displaystyle\bigcup_{k=1}^{\infty} \left[\frac{1}{k+1}, \frac{1}{k}\right)$, assume $y \in \left[\frac{1}{k+1}, \frac{1}{k}\right)$ for some

$k \geq 1$. By the property of y, we have $y > \frac{1}{k+1}$. Note that

$$a_{i,j} \leq a_{2k+4,2k+5} = \frac{1}{2k+4} + \frac{1}{2k+4} = \frac{1}{k+2}, \quad \forall i, j \geq 2k+5.$$

So, a subsequence converging to $y > \frac{1}{k+1}$ cannot contain infinitely many terms from the entries $a_{i,j}$ with $i, j \geq 2k+4$, as those values are all bounded above by $\frac{1}{k+2}$. Therefore, it must draw infinitely many terms from the first $2k+4$ columns. By the Pigeonhole Principle, at least one of these columns must contain infinitely many terms of the subsequence. However, each i-th column converges to $\frac{1}{i}$, for $1 \leq i \leq 2k+4$, which contradicts the assumption that $y \notin \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$.

Therefore, no such y exists, and we conclude $T = \{0\} \cup \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$. □

3 Discussions and Conclusion

In this article, we construct two bounded sequences:

- One with a subsequential limits set S that is countably infinite while its set of limit points T is finite.
- Another with T countably infinite.

These results demonstrate that all cardinalities—finite, countably infinite, and continuum—are possible for S and T under bounded sequences.

The constructions are specific to sequences in $\mathbb{R}$ and depend on careful enumeration and convergence design. Generalizing to broader topological spaces or higher-dimensional settings is not addressed here. We also do not explore the statistical or functional properties of such sequences beyond their limit sets.

References

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