International Journal of Contemporary Mathematical Sciences
Vol. 20, 2025, no. 1, 29 - 37
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ijcms.2025.91981

Hypersurfaces of Z-Shen Square Metric

S. K. Tiwari

Professor and Head
Department of Mathematics
K.S. Saket P.G. College Ayodhya (India) – 224123

Brijesh Kumar Maurya

Research Scholar
K.S. Saket P.G.College Ayodhya (India) – 224123

C. P. Maurya

Adarsh Inter College Saltauwa Gopalpur Basti
(India) – 272190

Abstract

We have considered the Z-Shen Square metric L given by $L^* = \frac{(L+\beta)^2}{L}$. We have obtained three types of hypersurfaces and hyperplanes of first, second and third kind invariant under certain condition.

Mathematics Subject Classification: 53B40, 53B60

Keywords: Z-Shen Square metric, hypersurfaces, hyperplane of first, second and third kind

1. Introduction

Let (M^n, L) be an n – dimensional Finsler Space on a differentiable Manifold M^n, equipped with the fundamental function L(x ,y) . In 1984 C . Shibta [4] introduced the transformation of Finsler metric

(1.1) $L^*(x,y) = f(L,\beta)$,

where $\beta(x,y) = b_i(x)y^i$ is a one forn on M^n and f is a positively homogeneous function of degree one in L and β . This change of Finsler metric is called a β-change.

A particular β-change of Finsler metric is a Z-Shen square metric is defined as [7]

(1.2) $L^* = \frac{(L+\beta)^2}{L}$.

If L(x , y) reduces to the metric function of Riemannian space then $L^*(x,y)$ reduces to the metric function of the space generated by Z-Shen square metric . Due to this reason this transformation (1.2) has been called the Z-Shen square metric .

On the other hand , in 1985 , M.Matsumoto investigated the theory of Finslerian hypersurface [6] . He was defined three types of hypersurfaces that were called a hyperplane of the first , second and third kind . In the year 2005 , Prasad and Tripathi [2] studied the Finslerian hypersurfaces and Kropina change of Finsler metric and obtained different results in this paper . Again in the year 2005 , Prasad , Chaubey and Patel [3] studied the Finsler hypersurfaces and Matsumoto change of a Finsler metric and obtained different results .

2. Induced Finsler connection of the hypersurface

The hypersurface F^{n-1} of the Finsler Space F^n may be represented parametrically by the equations $x^i = x^i(u^\alpha)$, $i = 1,2 \ldots\ldots, n$; $\alpha = 1,2, \ldots\ldots, n-1$; where u^α are Gaussian coordinates on F^{n-1} . We suppose that the matrix consisting of projection factors $B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}$ is of rank (n-1) . Then B^i_α may be regarded as (n-1) linearly independent vectors tangent to F^{n-1} at a point u^α. The supporting element y^i at a point u^α of F^{n-1} is assumed to be tangential to F^{n-1} so that

(2.1) $y^i = B^i_\alpha v^\alpha$.

Thus v^α is the supporting element of F^{n-1} at a point u^α. The metric tensor $g_{\alpha\beta}$ of F^{n-1} is given by

$$g_{\alpha\beta} = g_{ij} B^i_\alpha B^j_\beta \quad ,$$

where g_{ij} are components of the fundamental metric tensor of F^n . The components $N^i(x, y)$ of unit normal vector at a point u^α of F^{n-1} are given by

(2.2) (a) $g_{ij} B^i_\alpha N^j = 0$,

(b) $g_{ij} N^i N^j = 1$.

If (B^α_i, N_i) be the inverse matrix of (B^i_α, N^i) , then we have

(2.3) (a) $B^\alpha_i B^i_\beta = \delta^\alpha_\beta$,

(b) $B^i_\alpha N_i = 0$,

(c) $B^\alpha_i N^i = 0$,

(d) $N^i N_i = 1$,

(e) $B^i_\alpha B^\alpha_j + N^i N_j = \delta^i_j$.

The induced Finsler connection $\mathrm{IF}\Gamma = \left(\mathrm{F}^\alpha_{\beta\gamma}, N^\alpha_\beta, C^\alpha_{\beta\gamma}\right)$ on F^{n-1} , the second fundamental h-tensor $H_{\alpha\beta}$ and the normal curvature tensor H_α are respectively given by [1] :

(2.4) (a) $H_{\alpha\beta} = N_i\left(B^i_{\alpha\beta} + F^i_{jk} B^j_\alpha B^k_\beta\right) + M_\alpha H_\beta$,

(b) $H_\beta = N_i \left(B^i_{0\beta} + N^i_j B^j_\beta\right)$,

(c) $M_\alpha = C_{ijk} B^i_\alpha N^j N^k$,

where

$$B^i_{\alpha\beta} = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} \ , \quad B^i_{0\beta} = B^i_{\alpha\beta} v^\alpha \quad .$$

Contracting $H_{\alpha\beta}$ by v^α , we immediately get

$$H_{0\beta} = H_{\alpha\beta}\, v^\alpha = H_\beta \quad .$$

Furthermore the second fundamental v −tensor $M_{\alpha\beta}$ is given by [5]

(2.5) $M_{\alpha\beta} = C_{ijk} B^i_\alpha B^j_\beta N^k$.

3. Fundamental quantities of (M^n, L^*)

To find the relation between the fundamental quantities of (M^n, L) and (M^n, L^*) , we use the following results :

(3.1) $\dot{\partial}_i\beta = b_i$, $\dot{\partial}_i L = l_i$, $\dot{\partial}_j l_i = \frac{1}{L} h_{ij}$,

where $\dot{\partial}_i$ stands for $\frac{\partial}{\partial y^i}$ and h_{ij} are components of angular metric tensor of (M^n, L) given by

$$h_{ij} = g_{ij} - l_i l_j = L\dot{\partial}_i\dot{\partial}_j L \ .$$

The successive differentiation of (1.2) with respect to y^i and y^j gives :

(3.2) $l_i^* = \frac{(L+\beta)}{L^2}[(L-\beta)l_i + 2Lb_i]$.

(3.3) $h_{ij}^* = \frac{(L+\beta)^2}{L^4}\left[(L^2-\beta^2)h_{ij} + 2\beta^2 l_i l_j + 2L^2 b_i b_j - 2L\beta\left(l_i b_j + l_j b_i\right)\right]$.

The quantities corresponding to (M^n, L^*) will be denoted by putting star on the top of that quantity .

From (3.2) and (3.3) we get the following relation between the metric tensors of (M^n, L) and (M^n, L^*) :

(3.4) $g_{ij}^* = \frac{(L+\beta)^2}{L^4}\left[(L^2-\beta^2)g_{ij} + 2\beta(2\beta-L)l_i l_j + 6L^2 b_i b_j - 2L(2\beta - L)\left(l_i b_j + l_j b_i\right)\right]$.

Differentiating (3.4) with respect to y^k and using (3.1) , we get :

(3.5) $C_{ijk}^* = \frac{(L+\beta)^2(L^2-\beta^2)}{L^4}C_{ijk} - \frac{(L+\beta)^2(2\beta-L)}{L^4}\left(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j\right) + \frac{6(L+\beta)}{L^4}m_i m_j m_k$,

where $m_i = b_i - \frac{\beta}{L}l_i$.

It is to be noted that

(3.6) $m_i l^i = 0$, $m_i b^i = b^2 - \frac{\beta^2}{L^2}$, $h_{ij}l^j = 0$, $h_{ij}m^j = h_{ij}b^j = m_i$,

where

$$m^i = g^{ij}m_j = b^i - \frac{\beta}{L}l^i.$$

4. Hypersurface of (M^n, L^*)

Suppose a Finslerian hypersurface $F^{n-1} = \left(M^{n-1}, \bar{L}(u,v)\right)$ of the $F^n = (M^n, L(x,y))$ and another Finslerian hypersurface

$$F^{*n-1} = (M^{n-1}, \bar{L}^*(u,v)) \text{ of the } F^{*n} = (M^n, L^*(x,y))$$

given by Z-Shen square metric . Let $N^i(x,y)$ be the unit normal vector at each point of F^{n-1} and (B_i^α, N_i) be the inverse matrix of $\left(B_\alpha^i, N^i\right)$. The functions B_α^i may be considered components of (n-1) linearly independent tangent vectors of F^{n-1} and they are invariant under Z-Shen square metric .

Thus we shall show that a unit normal vector $N^{*i}(u,v)$ of F^{*n-1} is uniquely determined by

(4.1) (a) $g_{ij}^* B_\alpha^i N^{*j} = 0$,

(b) $g_{ij}^* N^{*i} N^{*j} = 1$.

On contracting (3.4) by $N^i N^j$ and using (2.1) (a) and $l_i N^i = 0$, we get :

$$g_{ij}^* N^i N^j = \frac{(L+\beta)^2}{L^4}\left[(L^2-\beta^2) + 6L^2\left(b_i N^i\right)^2\right].$$

Thus , we write

$$g_{ij}^*\left(\pm\frac{L^2 N^i}{(L+\beta)\sqrt{(L^2-\beta^2)+6L^2\left(b_i N^i\right)^2}}\right)\left(\pm\frac{L^2 N^j}{(L+\beta)\sqrt{(L^2-\beta^2)+6L^2\left(b_i N^i\right)^2}}\right) = 1 \ .$$

Now , we can put

$$\text{(4.2)} \quad N^{*i} = \frac{L^2 N^i}{(L+\beta)\sqrt{(L^2-\beta^2)+6L^2\left(b_i N^i\right)^2}} \ ,$$

where we have chosen the positive sign in order to fix an orientation .

From (3.4) , (4.1) (a) and (4.2) , we get

$$\left[6L^2 b_i B_\alpha^i - 2L(2\beta - L) l_i B_\alpha^i\right] \frac{(L+\beta) b_j N^j}{L^2\sqrt{(L^2-\beta^2)+6L^2\left(b_i N^i\right)^2}} = 0 \ .$$

If

$$6L^2 b_i B_\alpha^i - 2L(2\beta - L) l_i B_\alpha^i = 0 \ ,$$

then contracting it by v^α , we get L = 0 , which is a contradiction that L>0 .

Hence $b_i N^i = 0$.

Therefore (4.2) is rewritten as

(4.3) $$N^{*i} = \frac{L^2 N^i}{(L+\beta)\sqrt{(L^2-\beta^2)}} .$$

Thus we have from (4.3) :

Proposition(4.1) :

For a field of linear frame $\left(B_1^i, B_2^i, \dots \dots, B_{n-1}^i, N^i\right)$ of F^n there exists a field of linear frame

$\left(B_1^i, B_2^i, \dots \dots, B_{n-1}^i, N^{*i} = \frac{L^2 N^i}{(L+\beta)\sqrt{(L^2-\beta^2)}}\right)$ such that (4.1) is satisfied along F^{*n-1} and then b_i is tangential to both the hypersurface F^{n-1} and F^{*n-1} .

The quantities $B_i^{*\alpha}$ are uniquely defined along F^{n-1} by

$$B_i^{*\alpha} = g^{*\alpha\beta} g_{ij}^* B_\beta^j \quad ,$$

where ($g^{*\alpha\beta}$) is the inverse matrix of ($g_{\alpha\beta}^*$) .

Let ($B_i^{*\alpha}, N_i^*$) be the inverse of (B_α^i, N^{*i}) , then we have

$$B_\alpha^i B_i^{*\beta} = \delta_\alpha^\beta \quad ,$$

$$B_\alpha^i N_i^* = 0 \, ,$$

$$N^{*i} N_i^* = 1 \, ,$$

and

$$B_\alpha^i B_j^{*\alpha} + N^{*i} N_j^* = \delta_j^i \quad .$$

We also get

$N_i^* = g_{ij}^* N^{*j}$, which in view of (3.4) , (2.2) and (4.3) gives :

(4.4) $$N_i^* = \frac{(L+\beta)\sqrt{(L^2-\beta^2)}}{L^2} N_i \quad .$$

We denote the Cartan's connection of F^n and F^{*n} by $\left(F_{jk}^i, N_j^i, C_{jk}^i\right)$ and $\left(F_{jk}^{*i}, N_j^{*i}, C_{jk}^{*i}\right)$ respectively and put

$D^i_{jk} = F^{*i}_{jk} - F^i_{jk}$, which will be called the difference tensor. We chose that the vector field b_i in F^n such that

(4.5) $D^i_{jk} = A_{jk} b^i - B_{jk} l^i$,

where A_{jk} and B_{jk} are components of a symmetric covariant tensor of second order .

Now $N_i b^i = 0$ and $N_i l^i = 0$, from (4.5) , we get

(4.6) $N_i D^i_{jk} = 0$, $N_i F^{*i}_{jk} = N_i F^i_{jk}$ and $N_i\, D^i_{ok} = 0$.

Thus from (2.4)(b) and (4.3) , we get

(4.7) $H^*_\alpha = \frac{(L+\beta)\sqrt{(L^2-\beta^2)}}{L^2}\, H_\alpha$.

If each path of a hypersurface F^{n-1} with respect to the induced connection is also a path of enveloping space F^n , then F^{n-1} is called a hyperplane of the first kind [1] . A hyperplane of the first kind is characterized by $H_\alpha = 0$.

Hence we have from (4.7)

Theorem (4.1) :

If $b_i(x)$ be a vector field in F^n satisfying (4.5) , then a hypersurface F^{n-1} is a hyperplane of the first kind iff the hypersurface F^{*n-1} is a hyperplane of the first kind .

Again contracting (3.5) by $B^i_\alpha N^{*k} N^{*j}$ and paying attention to (4.4) , $m_i N^i = 0$, $h_{jk} N^j N^k = 1$, *and* $h_{ij} B^i_\alpha N^j = 0$, we get

(4.8) $M^*_\alpha = M_\alpha - \frac{(2\beta - L)}{(L^2-\beta^2)} m_i B^i_\alpha$.

From (2.4)(a) , (4.4) and (4.8) , we have

(4.9) $H^*_{\alpha\beta} = \frac{(L+\beta)\sqrt{(L^2-\beta^2)}}{L^2}\left(H_{\alpha\beta} - \frac{(2\beta-L)}{(L^2-\beta^2)} m_i B^i_\alpha H_\beta\right)$.

If each h-path of a hypersurface F^{n-1} with respect to the induced connection is also h-path of the enveloping space F^n, then F^{n-1} is called a hyperplane of the second kind [1] . A hyperplane of the second kind is characterized by $H_{\alpha\beta} = 0$.

Also $H_{\alpha\beta} = 0$ implies that $H_\alpha = 0$,

Thus we have from (4.7) and (4.9) :

Theorem (4.2) :

If $b_i(x)$ be a vector field in F^n satisfying (4.5) , then a hypersurface F^{n-1} is a hyperplane of the second kind iff the hypersurface F^{*n-1} is a hyperplane of the second kind .

Lastly contracting (3.5) by $B^i_\alpha B^j_\beta N^{*k}$ and using (4.3) , we have

(4.10) $M^*_{\alpha\beta} = \frac{(L+\beta)\sqrt{(L^2-\beta^2)}}{L^2} M_{\alpha\beta}$.

If the unit normal vector of F^{n-1} is parallel along each curve of F^{n-1} , then F^{n-1} is called a hyperplane of the third kind [1] . A hyperplane of third kind is characterized by $H_{\alpha\beta} = 0$, $M_{\alpha\beta} = 0$.

Thus we have from (4.7) , (4.9) and (4.10) :

Theorem (4.3) :

If $b_i(x)$ be a vector field in F^n satisfying (4.5) , then a hypersurface F^{n-1} is a hyperplane of the third kind iff the hypersurface F^{*n-1} is a hyperplane of the third kind .

References

[1] A. Rapesak, Eine neve characterisierung Finslershen Taume Konstanter Krummung and projectiveben Raume, *Acta Math. Acad. Sci. Hungar*, 8 (1957), 1-18. https://doi.org/10.1007/bf02025229

[2] B.N.Prasad and B.K.Tripathi, Finslerian hypersurfaces and Kropina chane of Finsler metric, Journal of the Tensor Society of India, **23** (2005), 49 – 58.

[3] B.N.Prasad, G.C.Chaubey and G.S.Patel, Finslerian hypersurfaces and Matsumoto change of Finsler metric, *Varahmihir Journal of Mathematical Sciences*, **5** (2005), np. 2, 337-342.
[4] C.Shibata, On invariant tensors of β – change of Finsler metrics, *J. Math. Kyoto Univ*, **24** (1984), 163-188. https://doi.org/10.1215/kjm/1250521391

[5] C.Shibata, U.P.Singh and A.K.Singh, On induced and intrinsic theories of hypersurface of Kropina space, *Hokkaido Univ. of Education (II A)*, **34** (1983), 1-11.

[6] M. Matsumoto, The induced and intrinsic connection of a hypersurface and Finslerian projective geometry, *J. Math. Kyoto Univ.*, **25** (1985), 107 -144. https://doi.org/10.1215/kjm/1250521161

[7] Z.Shen, G.Civi Yildirim, On a class of projectively flat metrics with constant flag curvature, *Can. J. Math.*, **60** (2008), no. 2, 443-456. https://doi.org/10.4153/cjm-2008-021-1

Received: March 15, 2025; Published: March 28, 2025