International Journal of Contemporary Mathematical Sciences
Vol. 20, 2025, no. 1, 119 - 123
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ijcms.2025.92010

On the Nonexistence of Hermitian Circulant Butson Hadamard Matrices

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Abstract

We give another proof of our previous results stating the one-to-one correspondence between circulant Hadamard matrices and Hermitian circulant complex Hadamard matrices and the nonexistence of Hermitian circulant q-Butson Hadamard matrices of order $n > 4$. In addition, we prove the nonexistence of skew-Hermitian circulant q-Butson Hadamard matrices of order $n > 4$.

Mathematics Subject Classification: 05B20, 15B57, 15B34

Keywords: Hermitian circlant complex Hadamard matrix, skew-Hermitian circlant complex Hadamard matrix, Butson matrix

1 Introduction

A Hadamard matrix H_n of order n is an $n \times n$ matrix with entries ± 1 such that $H_n {H_n}^t = nI_n$, where ${H_n}^t$ is the transpose of H_n and I_n is the identity matrix of order n. Moreover, if H_n is a circulant matrix, then H_n is called a circulant Hadamard matrix. A generalization of a Hadamard matrix is a complex Hadamard matrix, which is defined as an $n \times n$ matrix K_n with complex entries of absolute 1 such that $K_n {K_n}^* = nI_n$, where ${K_n}^*$ is the conjugate transpose of K_n. In particular, a complex Hadamard matrix whose entries are q-th roots of unity is called a q-Butson Hadamard matrix [2].

In 1963, Ryser [6] conjectured that there is no circulant Hadamard matrix of order $n > 4$. Brualdi [1] proved Ryser's conjecture provided that circulant

Hadamard matrices are symmetric. This result was generalized by Craigen and Kharaghani [3] to Hermitian circulant 4-Butson Hadamard matrices and skew-Hermitian circulant 4-Butson Hadamard matrices. In [5] we established the one-to-one correspondence between circulant Hadamard matrices and Hermitian circulant complex Hadamard matrices and generalized the Craigen and Kharaghani's result to Hermitian circulant q-Butson Hadamard matrices.

In this paper we give a more systematic proof of the above one-to-one correspondence and the nonexistence of Hermitian circulant q-Butson Hadamard matrices. In addition, we prove the nonexistence of skew-Hermitian circulant q-Butson Hadamard matrices.

2 The one-to-one correspondence

We denote by $\mathrm{circ}(a_0, \dots, a_{n-1})$ the circulant matrix whose first row is $(a_0 \cdots a_{n-1})$, by $\mathrm{diag}(b_0, \dots, b_{n-1})$ the diagonal matrix whose $j+1$-th diagonal element is b_j, and by F_n the $n \times n$ Fourier matrix whose $(j+1, k+1)$-th entry is $n^{-1/2}{\zeta_n}^{jk}$, where $\zeta_n = e^{2\pi i/n}$. It is well-known that $A_n = \mathrm{circ}(a_0, \dots, a_{n-1})$ can be expressed as $A_n = F_n \mathrm{diag}(b_0, \dots, b_{n-1}){F_n}^*$, where $b_j = \sum_{k=0}^{n-1} a_k {\zeta_n}^{jk}$ for $0 \le j \le n-1$.

Let Γ_n be the set of circulant complex matrices of order n. For $C_n = n^{1/2} F_n \mathrm{diag}(d_0, \dots, d_{n-1}){F_n}^* \in \Gamma_n$, we define transformation $T : \Gamma_n \to \Gamma_n$ by $T(C_n) = \mathrm{circ}(d_0, \dots, d_{n-1})$. Then T has the following properties.

Lemma 2.1. *Let* $C_n = \mathrm{circ}(c_0, \dots, c_{n-1}) = n^{1/2} F_n \mathrm{diag}(d_0, \dots, d_{n-1}){F_n}^*$. *Then*

(i) $T({C_n}^t) = T(C_n)^t$,

(ii) $T(\overline{C_n}) = T(C_n)^*$,

(iii) $T(T(C_n)) = {C_n}^t$,

(iv) $T(\alpha C_n) = \alpha C_n$ *for* $\alpha \in \mathbb{C}$.

Proof. We first note that ${C_n}^t = \mathrm{circ}(c_0, c_{n-1}, \dots, c_1)$. (i) and (ii) follow from

$$\sum_{k=0}^{n-1} c_{n-k} {\zeta_n}^{jk} = n^{1/2} d_{n-k}, \quad \sum_{k=0}^{n-1} \overline{c_k} {\zeta_n}^{jk} = \sum_{k=0}^{n-1} \overline{c_k {\zeta_n}^{(n-j)k}} = n^{1/2} \overline{d_{n-j}},$$

where $c_n = c_0$ and $d_n = d_0$. (iii) can be derived as follows:

$$\begin{aligned} T(T(C_n)) &= T(\mathrm{circ}(d_0, \ldots, d_{n-1})) \\ &= n^{-1/2}\mathrm{circ}\left(\sum_{j=0}^{n-1} d_j {\zeta_n}^{0j}, \ldots, \sum_{j=0}^{n-1} d_j {\zeta_n}^{(n-1)j}\right) \\ &= n^{-1}\mathrm{circ}\left(\sum_{j=0}^{n-1}\sum_{k=0}^{n-1} c_k {\zeta_n}^{jk+0j}, \ldots, \sum_{j=0}^{n-1}\sum_{k=0}^{n-1} c_k {\zeta_n}^{jk+(n-1)j}\right) \\ &= \mathrm{circ}(c_0, c_{n-1}, \ldots, c_1) = {C_n}^t. \end{aligned}$$

From the definition of T, (iv) is obvious. □

Lemma 2.2. *$T : \Gamma_n \to \Gamma_n$ is a bijection.*

Proof. Let C_n and C'_n be circulant complex matrices. By the definition, it is obvious that $C_n = C'_n$ if $T(C_n) = T(C'_n)$. Hence T is injective. The surjectivity of T follows from Lemma 2.1 (iii). □

Lemma 2.3. *If C_n is a circulant complex Hadamard matrix of order n, then $T(C_n)$ is a circulant complex Hadamard matrix of order n.*

Proof. Let $C_n = \mathrm{circ}(c_0, \ldots, c_{n-1}) = n^{1/2}F_n\mathrm{diag}(d_0, \ldots, d_{n-1}){F_n}^*$ be a circulant complex Hadamard matrix. Since

$$C_n{C_n}^* = nF_n\mathrm{diag}(d_0\overline{d_0}, \ldots, d_{n-1}\overline{d_{n-1}}){F_n}^* = nI_n,$$

it holds that $|d_j| = 1$ for $0 \leq j \leq n-1$. Moreover, since

$$\begin{aligned} T(C_n) &= \mathrm{circ}(d_0, \ldots, d_{n-1}) \\ &= F_n\mathrm{diag}\left(\sum_{j=0}^{n-1} d_j {\zeta_n}^{0j}, \ldots, \sum_{j=0}^{n-1} d_j {\zeta_n}^{(n-1)j}\right){F_n}^* \\ &= n^{-1/2}F_n\mathrm{diag}\left(\sum_{j=0}^{n-1}\sum_{k=0}^{n-1} c_k {\zeta_n}^{jk+0j}, \ldots, \sum_{j=0}^{n-1}\sum_{k=0}^{n-1} c_k {\zeta_n}^{jk+(n-1)j}\right){F_n}^* \\ &= n^{1/2}F_n\mathrm{diag}(c_0, c_{n-1}, \ldots, c_1){F_n}^*, \end{aligned}$$

we have

$$T(C_n)T(C_n)^* = nF_n\mathrm{diag}(c_0\overline{c_0}, c_{n-1}\overline{c_{n-1}}, \ldots, c_1\overline{c_1}){F_n}^* = nI_n.$$

Thus $T(C_n)$ is a circulant complex Hadamard matrix. □

The following theorem is a modification of our previous results [5, Lemmas 2.2 and 2.3].

Theorem 2.4. *If H_n is a circulant Hadamard matrix of order n, then $T(H_n)$ is a Hermitian circulant complex Hadamard matrix of order n.*

If K_n is a Hermitian circulant complex Hadamard matrix of order n, then $T(K_n)$ is a circulant Hadamard matrix of order n.

Proof. By Lemmas 2.1 and 2.3, if H_n is a circulant Hadamard matrix, then $T(H_n)$ is a circulant complex Hadamard matrix satisfying $T(H_n) = T(\overline{H_n}) = T(H_n)^*$. Hence $T(H_n)$ is a Hermitian circulant complex Hadamard matrix.

Conversely, if K_n is a Hermitian circulant complex Hadamard matrix, then $T(K_n)$ is a circulant complex Hadamard matrix satisfying $T(K_n) = T({K_n}^*) = \overline{T(K_n)}$. Hence $T(K_n)$ is a circulant Hadamard matrix. □

Similarly, the following theorem holds.

Theorem 2.5. *If H_n is a circulant complex Hadamard matrix with entries $\pm i$ of order n, then $T(H_n)$ is a skew-Hermitian circulant complex Hadamard matrix of order n.*

If K_n is a skew-Hermitian circulant complex Hadamard matrix of order n, then $T(K_n)$ is a circulant complex Hadamard matrix with entries $\pm i$ of order n.

Proof. If H_n is a circulant complex Hadamard matrix with entries $\pm i$, then $T(H_n)$ is a circulant complex Hadamard matrix satisfying $T(H_n) = T(-\overline{H_n}) = -T(H_n)^*$. Hence $T(H_n)$ is a skew-Hermitian circulant complex Hadamard matrix.

Conversely, if K_n is a skew-Hermitian circulant complex Hadamard matrix, then $T(K_n)$ is a circulant complex Hadamard matrix satisfying $T(K_n) = T(-{K_n}^*) = -\overline{T(K_n)}$. Hence $T(K_n)$ is a circulant complex Hadamard matrix with entries $\pm i$. □

3 The nonexistence

Craigen and Kharaghani derived the following lemma [3, Lemma 4] from Ma's theorem [4, Theorem 3.1]. This lemma is our key tool.

Lemma 3.1 (Craigen and Kharaghani)**.** *Let $n > 4$. Then there is no circulant Hadamard matrix H_n of order n such that ${H_n}^m = n^{m/2} I_n$ for some $m > 0$.*

Using the notation in Section 2, we prove our previous result [5, Theorem 3.2].

Theorem 3.2. *Let $q \geq 2$ and $n > 4$. Then there is no Hermitian circulant q-Butson Hadamard matrix of order n.*

Proof. Suppose that

$$K_n = \mathrm{circ}(k_0, \ldots, k_{n-1}) = n^{1/2} F_n \mathrm{diag}(h_0, \ldots, h_{n-1}) {F_n}^*$$

is a Hermitian circulant q-Butson Hadamard matrix of order $n > 4$. By Theorem 2.4, $H_n = \mathrm{circ}(h_0, \ldots, h_{n-1})$ is a circlant Hadamard matrix. Since $T(T(K_n)) = T(H_n) = {K_n}^t$, we have $\sum_{j=0}^{n-1} h_j {\zeta_n}^{lj} = n^{1/2} k_{n-l}$, where $k_n = k_0$. Hence

$${F_n}^* {H_n}^q F_n = n^{q/2} \mathrm{diag}({k_0}^q, {k_{n-1}}^q, \ldots, {k_1}^q) = n^{q/2} I_n,$$

so that ${H_n}^q = n^{q/2} I_n$. However, this contradicts Lemma 3.1. □

If K_n is a skew-Hermitian circulant complex Hadamard matrix, then iK_n is a Hermitian circulant complex Hadamard matrix. Hence Theorem 3.2 implies the following theorem.

Theorem 3.3. *Let $q \geq 2$ and $n > 4$. Then there is no skew-Hermitian circulant q-Butson Hadamard matrix of order n.*

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Received: December 3, 2025; Published: December 12, 2025