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Decomposing $K_{n,n}$ into Hamiltonian Cycles and Small Stars

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Abstract

A decomposition of a graph G is a set consisting of edge-disjoint subgraphs of G whose union is G. A Hamiltonian cycle in G, is a cycle containing all of the vertices of G. A m-star is a star containing m edges and denoted by S_m, which is isomorphic to the complete bipartite graph $K_{1,m}$. In this paper, the necessary and sufficient conditions for decomposing the balanced complete bipartite graph $K_{n,n}$ into p copies of Hamiltonian cycles and q copies of 3-stars (4-stars) are given.

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1 Introduction

Let G and H be graphs. A *decomposition* of a graph G is a set consisting of edge-disjoint subgraphs of G whose union is G. An H*-decomposition* of G is a decomposition of G whose members are isomorphic to H. If there exists an H-decomposition of G, then G is referred to as H*-decomposable*.

Let m and n be positive integers. The *balanced complete bipartite graph*, denoted by $K_{n,n}$, is a complete bipartite graph with both parts of sizes n. A m-star is a star containing m edges and denoted by S_m, which is isomorphic to the complete bipartite graph $K_{1,m}$. A *m-cycle*, denoted by C_m, is a cycle with m edges. A *Hamiltonian cycle* in a graph G, is a cycle containing all vertices of G.

The problem of decomposing a graph into p copies of a graph G and q copies of a graph H where G and H are different types has attracted a fair share of interest. Jeevadoss and Muthusamy [1, 2] studied the decomposability of $K_{m,n}$ and $\lambda K_{m,n}$ into paths and cycles having k edges, and gave some necessary or sufficient conditions of the decompositions. In [3], Jeevadoss and Muthusamy obtained necessary and sufficient conditions for decomposing product graphs of complete graphs into paths and cycles with four edges. Shyu investigated problems of decomposing of K_n into paths and stars with 3 edges [5], paths and cycles having k edges with $k = 3$ or 4) [6, 7], and cycles and stars having 4 edges [8], and gave necessary and sufficient conditions for such decompositions to exist. In [9], Shyu obtained necessary and sufficient conditions for decomposing $K_{m,n}$ into paths and stars with 3 edges. Lee and Chen [4] considered the problem of decomposing K_n into p copies of Hamiltonian paths (cycles) and q copies of 3-stars, giving necessary and sufficient conditions of the decomposition.

In this paper, we investigate the problem of decomposing $K_{n,n}$ into p copies of Hamiltonian cycles and q copies of 3-stars (4-stars). The following results are obtained.

1. The necessary and sufficient conditions for decomposing $K_{n,n}$ into p copies of Hamiltonian cycles and q copies of 3-stars.

2. The necessary and sufficient conditions for decomposing $K_{n,n}$ into p copies of Hamiltonian cycles and q copies of 4-stars.

2 Preliminaries

In this section, essential terminology and notation used in our analysis are introduced. We also present some useful results for our discussions to follow.

Let H be a graph. The *degree* $\deg_H u$ of a vertex u in H is the number of edges incident with u. For $m \geq 2$, the vertex with degree m in the m-star is the *center* of the m-star, and any vertex with degree 1 is an *endvertex* of the m-star. We use $(x; y_1, \ldots, y_m)$ to denote the m-star with center x and endvertices $y_1, y_2, \ldots, y_m$. Furthermore, $(u_1, u_2, \ldots, u_m)$ denotes the m-cycle through vertices $u_1, u_2, \ldots, u_m, u_1$ in order. Suppose that $H_1, H_2, \ldots, H_r$ are graphs, $H_1 \cup H_2 \cup \cdots \cup H_r$ (or $\bigcup_{j=1}^{r} H_j$) is used to denoted the graph with vertex set $\bigcup_{j=1}^{r} V(H_j)$ and edge set $\bigcup_{j=1}^{r} E(H_j)$.

Let a_ib_j be an edge in $K_{n,n}$. The *label* of a_ib_j is $j - i \pmod{n}$. For example, in $K_{8,8}$ the labels of a_1b_6 and a_7b_3 are 5 and 4, respectively. For

$i \in \{0, 1, 2, \ldots, n-1\}$, there is exactly one edge with label i incident with vertex u for each u in $K_{n,n}$.

A *spanning subgraph* H of a graph G is a subgraph of G with $V(H) = V(G)$. A *1-factor* of G is a spanning subgraph of G in which $\deg_G u = 1$ for each vertex u in G.

Let $A = \{a_0, a_1, \ldots, a_{n-1}\}$ and $B = \{b_0, b_1, \ldots, b_{n-1}\}$. We use (A, B) to denote the bipartition of $K_{n,n}$ throughout the paper.

Lemma 2.1 *Let n be an even integer. Then there exists a C_{2n}-decomposition $\{Q_0, Q_1, \ldots, Q_{n/2-1}\}$ of $K_{n,n}$, where $Q_i = (b_{2i}, a_0, b_{2i+1}, a_1, \ldots, b_{2i+n-2}, a_{n-2}, b_{2i+n-1}, a_{n-1})$ for $i = 0, 1, \ldots, n/2-1$, where the subscripts of b's are taken modulo n.*

Lemma 2.2 *Let n be an odd integer. Then $K_{n,n}$ can be decomposed into $(n-1)/2$ copies of C_{2n}, $R_1, R_2, \ldots, R_{(n-1)/2}$, and a 1-factor M, where $E(M) = \{a_0b_0, a_1b_1, \ldots, a_{n-1}b_{n-1}\}$ and $R_i = (b_{2i-1}, a_0, b_{2i}, a_1, \ldots, b_{2i+n-3}, a_{n-2}, b_{2i+n-2}, a_{n-1})$ for $i = 1, 2, \ldots, (n-1)/2$, where the subscripts of b's are taken modulo n.*

The following result is trivial.

Lemma 2.3 *Let r be a positive integer, and let H be a bipartite graph with bipartition (A, B). If $\deg_H u \equiv 0 \pmod r$ for each $u \in A$ or $\deg_H v \equiv 0 \pmod r$ for each $v \in B$, then H has an S_r-decomposition such that there are $\deg_H u/r$ stars with center at u for each $u \in A$ or $\deg_H v/r$ stars with center at v for each $v \in B$, respectively.*

Let $|A|$ denote the cardinality of set A. By Lemma 2.3, we have the following result.

Lemma 2.4 *Let d and r be positive integers. Suppose that H is a d-regular bipartite graph with bipartition (A, B). If $d \equiv 0 \pmod r$, then H can be decomposed into $|A|d/r$ copies of r-stars.*

Let $W_1, W_2, \ldots, W_t$ be edge-disjoint Hamiltonian cycles of $K_{n,n}$. Since $\bigcup_{j=1}^{t} W_j$ is a $2t$-regular bipartite graph, the result bellow follows from Lemma 2.4

Lemma 2.5 *Let n, r and t be positive integers. Suppose that $W_1, W_2, \ldots, W_t$ are edge-disjoint Hamiltonian cycles of $K_{n,n}$. If $2t \equiv 0 \pmod r$, then $\bigcup_{j=1}^{t} W_j$ can be decomposed into $2nt/r$ copies of r-stars.*

3 Decomposing $K_{n,n}$ into Hamiltonian cycles and 3-stars

In this section, we investigate the problem of decomposing $K_{n,n}$ into p copies of Hamiltonian cycles and q copies of 3-stars, and give the complete solution of the problem. Before plunging into the proof of the main result, we need the following lemmas.

Lemma 3.1 *Let n be a positive even integer, and let p be a nonnegative integer. If $n^2-2np \geq 0$ and $n^2-2np \equiv 0 \pmod 3$, then*

$$p \in \begin{cases} \{0,1,\ldots,n/2\} & \text{if } n \equiv 0 \pmod 6, \\ \{n/2-3m \mid m=0,1,\ldots,\lfloor n/6 \rfloor\} & \text{otherwise.} \end{cases}$$

Proof. Since $n^2-2np \geq 0$ and n is even, $p \leq \lfloor n^2/2n \rfloor = n/2$. Let $p = n/2-(3m+i)$ where m is a nonnegative integer and $i \in \{0,1,2\}$. Since $n^2-2np = n^2-2n(n/2-(3m+i)) = 6mn+2ni$, $n^2-2np \equiv 2ni \pmod 3$. When $n \equiv 0 \pmod 6$, the condition $2ni \equiv 0 \pmod 3$ holds for any i. Thus $p \in \{0,1,\ldots,n/2\}$. When $n \equiv 2,4 \pmod 6$, the condition $2ni \equiv 0 \pmod 3$ holds if and only if $i=0$. Thus $p = n/2-3m$ for some integer m. Since p is a nonnegative integer, we have $m \leq \lfloor n/6 \rfloor$. This completes the proof. □

Lemma 3.2 *Let n be a positive odd integer, and let p be a nonnegative integer. If $n^2-2np \geq 0$ and $n^2-2np \equiv 0 \pmod 3$, then*

$$p \in \begin{cases} \{0,1,\ldots,(n-1)/2\} & \text{if } n \equiv 3 \pmod 6, \\ \{(n-3)/2-3m \mid m=0,1,\ldots,\lfloor (n-3)/6 \rfloor\} & \text{otherwise.} \end{cases}$$

Proof. Since $n^2-2np \geq 0$ and n is odd, $p \leq \lfloor n^2/2n \rfloor = (n-1)/2$. Let $p = (n-1)/2-(3m+i)$ where m is a nonnegative integer and $i \in \{0,1,2\}$. Since $n^2-2np = n^2-2n((n-1)/2-(3m+i)) = 6mn+n(2i+1)$, $n^2-2np \equiv n(2i+1) \pmod 3$. When $n \equiv 3 \pmod 6$, the condition $n(2i+1) \equiv 0 \pmod 3$ holds for any i. Thus $p \in \{0,1,\ldots,(n-1)/2\}$. When $n \equiv 1,5 \pmod 6$, the condition $n(2i+1) \equiv 0 \pmod 3$ holds if and only if $i=1$. Thus $p = (n-1)/2-(3m+1) = (n-3)/2-3m$ for some integer m. Since p is a nonnegative integer, we have $m \leq \lfloor (n-3)/6 \rfloor$. This completes the proof. □

Let $r = n/2-1$ for even n and $r = (n-1)/2$ for odd n. Suppose that $W_1, W_2, \ldots, W_r$ are edge-disjoint Hamiltonian cycles in $K_{n,n}$, and $H = K_{n,n} - \bigcup_{i=1}^{r} E(W_i)$. Since $\deg_H u = n-2r < 3$ for each vertex u, H is not S_3-decomposable. Hence we have the following result.

Lemma 3.3 *For $n \equiv 0 \pmod 3$, $K_{n,n}$ cannot be decomposed into $n/2-1$ copies of C_{2n} and $2n/3$ copies of S_3 for even n, and cannot be decomposed into $(n-1)/2$ copies of C_{2n} and $n/3$ copies of S_3 for odd n.*

Lemma 3.4 *For even n, the following results hold:*

(1) $K_{n,n}$ can be decomposed into $n/2$ copies of C_{2n}.

(2) $K_{n,n}$ can be decomposed into $n/2-2$ copies of C_{2n} and $4n/3$ copies of S_3 if $n \equiv 0 \pmod 6$ and $n \geq 6$.

(3) $K_{n,n}$ can be decomposed into $n/2-4$ copies of C_{2n} and $8n/3$ copies of S_3 if $n \equiv 0 \pmod 6$ and $n \geq 12$.

Proof. By Lemma 2.1, $K_{n,n}$ can be decomposed into $n/2$ copies of C_{2n}, $Q_0, Q_1, \ldots, Q_{n/2-1}$ with $Q_i = (b_{2i}, a_0, b_{2i+1}, a_1, \ldots, b_{2i+n-2}, a_{n-2}, b_{2i+n-1}, a_{n-1})$ for $i = 0, 1, \ldots, n/2-1$, where the subscripts of b's are taken modulo n. Thus we obtain (1).

(2) Let $G = Q_0 \cup Q_1$. Trivially $K_{n,n} - E(G) = \bigcup_{i=2}^{n/2-1} Q_i$, which can be decomposed into $n/2-2$ copies of C_{2n}. If G can be decomposed into $4n/3$ copies of S_3, then we have the result. Note that G is a 4-regular spanning subgraphs of $K_{n,n}$, which contains all edges with labels $0, 1, 2, 3$. For $i \in \{0, 1, \ldots, n/3-1\}$, let $S(i) = (b_{3i+2}; a_{3i}, a_{3i+1}, a_{3i+2})$. Clearly $S(i)$ is a 3-star in G containing edges with labels 0, 1, 2, and $\deg_{\bigcup_{i=0}^{n/3-1} S(i)} a_j = 1$ for each $a_j \in A$. Hence $\deg_{G-E(\bigcup_{i=0}^{n/3-1} S(i))} a_j = 4 - 1 = 3$ for each $a_j \in A$. Thus $G - E(\bigcup_{i=0}^{n/3-1} S(i))$ can be decomposed in to n copies of S_3. This implies that G can be decomposed into $n + n/3 = 4n/3$ copies of S_3, and settles (2).

(3) Let $H = \bigcup_{i=0}^{3} Q_i$. Trivially $K_{n,n} - E(H) = \bigcup_{i=4}^{n/2-1} Q_i$, which can be decomposed into $n/2-4$ copies of C_{2n}. If H can be decomposed into $8n/3$ copies of S_3, then we have the result. Note that $Q_2 \cup Q_3$ is a 4-regular spanning subgraphs of $K_{n,n}$ containing all edges with labels $4, 5, 6, 7$, which is isomorphic to $Q_0 \cup Q_1 = G$. Thus $Q_2 \cup Q_3$ can be decomposed into $4n/3$ copies of S_3, in turn, H can be decomposed into $8n/3$ copies of S_3. This settles (3). □

Lemma 3.5 *For odd n with $n \geq 3$, the following results hold:*

(1) $K_{n,n}$ can be decomposed into $(n-3)/2$ copies of C_{2n} and n copies of S_3.

(2) $K_{n,n}$ can be decomposed into $(n-5)/2-2$ copies of C_{2n} and $5n/3$ copies of S_3 if $n \equiv 3 \pmod 6$ and $n \geq 9$.

(3) $K_{n,n}$ can be decomposed into $(n-7)/2-4$ copies of C_{2n} and $7n/3$ copies of S_3 if $n \equiv 3 \pmod 6$ and $n \geq 15$.

Proof. By Lemma 2.2, $K_{n,n}$ can be decomposed into $(n-1)/2$ copies of C_{2n}, $R_1, R_2, \ldots, R_{(n-1)/2}$, and a 1-factor M, where $E(M) = \{a_0b_0, a_1b_1, \ldots, a_{n-1}b_{n-1}\}$ and $R_i = (b_{2i-1}, a_0, b_{2i}, a_1, \ldots, b_{2i+n-3}, a_{n-2}, b_{2i+n-2}, a_{n-1})$ for $i = 1, 2, \ldots, (n-1)/2$, the subscripts of b's are taken modulo n.

(1) Let $G = R_1 \cup M$. Trivially $K_{n,n} - E(G) = \bigcup_{i=2}^{(n-1)/2} R_i$, which can be decomposed into $(n-3)/2$ copies of C_{2n}. If G can be decomposed into n copies of S_3, then we have the result. Note that G is a 3-regular spanning subgraphs of $K_{n,n}$, which contains all edges with labels $0, 1, 2$. By Lemma 2.4, G can be decomposed into $n(3/3) = n$ copies of S_3. This settles (1).

(2) Let $G = R_1 \cup R_2 \cup M$. Trivially $K_{n,n} - E(G) = \bigcup_{i=3}^{(n-1)/2} R_i$, which can be decomposed into $(n-5)/2$ copies of C_{2n}. If G can be decomposed into $5n/3$ copies of S_3, then we have the result. Note that G is a 5-regular spanning subgraphs of $K_{n,n}$, which contains all edges with labels $0, 1, 2, 3, 4$. For $i \in \{0, 1, \ldots, n/3 - 1\}$, let $S(i) = (b_{3i+2}; a_{3i}, a_{3i+1}, a_{3i+2})$ and $S'(i) = (b_{3i+3}; a_{3i+1}, a_{3i+2}, a_{3i+3})$. It is easy to check that $S(i)$ and $S'(i)$ are 3-stars in G containing edges with labels 0, 1, 2, and $\deg_{\bigcup_{i=0}^{n/3-1}(S(i)\cup S'(i))} a_j = 2$ for each $a_j \in A$. Hence $\deg_{G-E(\bigcup_{i=0}^{n/3-1}(S(i)\cup S'(i)))} a_j = 5 - 2 = 3$ for each $a_j \in A$. Thus $G - E(\bigcup_{i=0}^{n/3-1}(S(i) \cup S'(i))$ can be decomposed in to n copies of S_3. This implies that G can be decomposed into $n + 2n/3 = 5n/3$ copies of S_3, and settles (2).

(3) Let $H = (\bigcup_{i=1}^{3} R_i) \cup M$. Trivially $K_{n,n} - E(H) = \bigcup_{i=4}^{(n-1)/2} R_i$, which can be decomposed into $(n-7)/2$ copies of C_{2n}. If H can be decomposed into $7n/3$ copies of S_3, then we have the result. Note that H is a 7-regular spanning subgraphs of $K_{n,n}$, which contains all edges with labels $0, 1, \ldots, 6$. For $i \in \{0, 1, \ldots, n/3 - 1\}$, let $S(i) = (b_{3i+2}; a_{3i}, a_{3i+1}, a_{3i+2})$. Obviously $S(i)$ is a 3-star in H containing edges with labels 0, 1, 2, and $\deg_{\bigcup_{i=0}^{n/3-1} S(i)} a_j = 1$ for each $a_j \in A$. Note that $\deg_{H-E(\bigcup_{i=0}^{n/3-1} S(i))} a_j = 7 - 1 = 6$ for each $a_j \in A$. Thus $H - E(\bigcup_{i=0}^{n/3-1} S(i))$ can be decomposed in to $2n$ copies of S_3. Hence H can be decomposed into $2n + n/3 = 7n/3$ copies of S_3. This settles (3). $\square$

By Lemma 2.5, the union of $3m$ copies of edge-disjoint C_{2n} can be decomposed into $2n(3m)/3 = 2mn$ copies of S_3. Thus we have the following result.

Theorem 3.6 *Suppose that n, α and m are positive integers with $\alpha \geq 3m$, and β is a nonnegative integer. If $K_{n,n}$ can be decomposed into α copies of C_{2n} and β copies of S_3, then $K_{n,n}$ can be decomposed into $\alpha - 3m$ copies of C_{2n} and $\beta + 2mn$ copies of S_3.*

Trivially, if $K_{n,n}$ can be decomposed into p copies of C_{2n} and q copies of S_3, then $n^2 = 2np + 3q$. By Theorem 3.6 as well as Lemmas 3.1 to 3.5, the main result of this section is obtained.

Theorem 3.7 *Let p and q be nonnegative integers, and let n be a positive integer. There exists a decomposition of $K_{n,n}$ into p copies of Hamiltonian cycles and q copies of 3-stars if and only if $2np+3q=n^2$ and $p \neq n/2-1$ for $n \equiv 0 \pmod 6$ and $p \neq (n-1)/2$ for $n \equiv 3 \pmod 6$.*

4 Decomposing $K_{n,n}$ into Hamiltonian cycles and 4-stars

In this section, we investigate the problem of decomposing $K_{n,n}$ into p copies of Hamiltonian cycles and q copies of 4-stars, and give the complete solution of the problem.

Lemma 4.1 *Let n be a positive even integer, and let p be a nonnegative integer. $n^2-2np \geq 0$ and $n^2-2np \equiv 0 \pmod 4$ if and only if $n \equiv 0 \pmod 2$ and $p \leq n/2$.*

Proof. Note that $n-2p$ is even if and only if n is even. This implies that $n^2-2np = n(n-2p) \equiv 0 \pmod 4$ if and only if $n \equiv 0 \pmod 2$. Since $n^2-2np \geq 0$, $p \leq n^2/(2n) = n/2$. This completes the proof. □

Let $t = n/2-1$ for even integer n with $n \geq 4$. Suppose that $W_1, W_2, \ldots, W_t$ are edge-disjoint Hamiltonian cycles in $K_{n,n}$, and $H = K_{n,n} - \bigcup_{i=1}^{t} E(W_i)$. Since $\deg_H u = n-2t = 2 < 4$ for each vertex u, H is not S_4-decomposable. Hence we have the following result.

Lemma 4.2 *For even n with $n \geq 4$, $K_{n,n}$ cannot be decomposed into $n/2-1$ copies of C_{2n} and $n/2$ copies of S_4.*

Lemma 4.3 *Let n be an even integer with $n \geq 6$. Then $K_{n,n}$ can be decomposed into $n/2-3$ copies of C_{2n} and $3n/2$ copies of S_4.*

Proof. Let $G = \bigcup_{i=0}^{2} Q_i$. Trivially $K_{n,n} - E(G) = \bigcup_{i=3}^{n/2-1} Q_i$, which can be decomposed into $n/2-3$ copies of C_{2n}. If G can be decomposed into $3n/2$ copies of S_3, then we have the result. Note that G is a 6-regular spanning subgraphs of $K_{n,n}$, which contains all edges with labels $0, 1, \ldots, 5$. For $i \in \{0, 1, \ldots, n/2-1\}$, let $S(i) = (a_{2i}; b_{2i}, b_{2i+1}, b_{2i+2}, b_{2i+3})$. It is not difficult to check that $S(i)$ is a 4-star in G containing edges with labels 0, 1, 2, 3, and $\deg_{\bigcup_{i=0}^{n/2-1} S(i)} b_j = 2$ for each $b_j \in B$. Hence $\deg_{G-E(\bigcup_{i=0}^{n/2-1} S(i))} b_j = 6-2 = 4$ for each $b_j \in B$. Thus $G - E(\bigcup_{i=0}^{n/2-1} S(i))$ can be decomposed in to n copies of S_4. This implies that G can be decomposed into $n + n/2 = 3n/2$ copies of S_4 and completes the proof. □

The union of $2m$ copies of edge-disjoint C_{2n} is a $4m$-regular bipartite graph, which can be decomposed into $2n(2m)/4 = mn$ copies of S_4 by Lemma 2.5. Thus we have the following result.

Theorem 4.4 *Suppose that n, α and m are positive integers with $\alpha \geq 2m$, and β is a nonnegative integer. If $K_{n,n}$ can be decomposed into α copies of C_{2n} and β copies of S_4, then $K_{n,n}$ can be decomposed into $\alpha - 2m$ copies of C_{2n} and $\beta + mn$ copies of S_4.*

Trivially, if $K_{n,n}$ can be decomposed into p copies of C_{2n} and q copies of S_4, then $n^2 = 2np + 4q$. By Theorem 4.4 as well as Lemmas 2.1, 4.1 to 4.3, the main result of this section is obtained.

Theorem 4.5 *Let p and q be nonnegative integers, and let n be a positive integer. There exists a decomposition of $K_{n,n}$ into p copies of Hamiltonian cycles and q copies of 4-stars if and only if $2np + 4q = n^2$ and $p \neq n/2 - 1$.*

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