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Z-Shen Square Change of Finsler Metric

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Abstract

The purpose of the present paper is to find the necessary and sufficient conditions under which a Z-Shen square change of Finsler metric becomes a projective change. The condition under which a Z-Shen square change of Finsler metric of Douglas space becomes a Douglas space have been also found.

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1 Introduction

Let $F^n = (M^n, L)$ is a Finsler space, where L is Finsler function of x and y and M^n is n-dimensional smooth manifold. The Z-Shen square change of Finsler metric is represented as

(1.1) $$\bar{L} = \frac{(L+\beta)^2}{L} .$$

Then Finsler Space $\bar{F}^n = (M^n, \bar{L})$ is said to be obtained from Finsler space $F^n = (M^n, L)$ by Z-Shen square change. The quantities corresponding to $\bar{F}^n$ is denoted by putting bar on those quantities. The basic tensor of F^n are given as follows-

$$g_{ij} = \frac{1}{2}\left(\frac{\partial^2 L^2}{\partial y^i \partial y^j}\right) , l_i = \frac{\partial L}{\partial y^i} = L_i \ and \ h_{ij} = g_{ij} - l_i l_j ,$$

where g_{ij} is fundamental metric tensor, l_i is normalized supporting element and h_{ij} is angular metric tensor.

The partial derivative with respect to x^i and y^i will be denoted by ∂_i and $\dot{\partial}_i$ respectively and derivatives are written as

(1.2) (a) $$L_i = \frac{\partial L}{\partial y^i} ,$$

(b) $$L_{ij} = \frac{\partial^2 L}{\partial y^i \partial y^j} ,$$

(c) $$L_{ijk} = \frac{\partial^3 L}{\partial y^i \partial y^j \partial y^k} .$$

The equation of geodesic of a Finsler space [1] is $\frac{d^2 x^i}{ds^2} + 2G^i\left(x, \frac{dx}{ds}\right) = 0$, where G^i is positively homogeneous functions of degree two in y^i and is given by

$$2G^i = \frac{1}{2} g^{ij}\left(y^r \dot{\partial}_j \partial_r L^2 - \partial_j L^2\right) .$$

The Berwald connection $B\Gamma = \left(G^i_{jk}, G^i_j, 0\right)$ of the Finsler space F^n is given by [2]

$$G^i_j = \frac{\partial G^i}{\partial y^j} \quad and \quad G^i_{jk} = \frac{\partial G^i_j}{\partial y^k} .$$

The Cartan connection $C\Gamma = \left(F^i_{jk}, G^i_j, C^i_{jk}\right)$ is constructed from L with the help of following axioms [3]-

(i) $g_{ij|k} = 0$,
(ii) $g_{ij}|_k = 0$,
(iii) $F^i_{jk} = F^i_{kj}$,
(iv) $F^i_{0k} = G^i_k$,
(v) $C^i_{jk} = C^i_{kj}$.

here $_{|k}$ and $|_k$ denote the h and v-covariant derivative with respect to Cartan connection.

We may put

(1.3) $$\bar{G}^i = G^i + D^i.$$

Then $\bar{G}^i_j = G^i_j + D^i_j$ and $\bar{G}^i_{jk} = G^i_{jk} + D^i_{jk}$ where $D^i_j = \dot{\partial}_j D^i$ and $D^i_{jk} = \dot{\partial}_k D^i_j$. The tensors D^i , D^i_j and D^i_{jk} are positively homogeneous in y^i of degree two , one and zero respectively.

2 The Difference Tensor

In view of (1.1) and (1.2), we get

(2.1) $$\bar{L}_i = \frac{(L^2-\beta^2)}{L^2} L_i + \frac{2(L+\beta)}{L} b_i \ .$$

(2.2) $$\bar{L}_{ij} = \frac{(L^2-\beta^2)}{L^2} L_{ij} + \frac{2\beta^2}{L^3} L_i L_j - \frac{2\beta}{L^2}(L_i b_j + L_j b_i) + \frac{2}{L} b_i b_j \ .$$

(2.3) $$\bar{L}_{ijk} = \frac{(L^2-\beta^2)}{L^2} L_{ijk} + \frac{2\beta^2}{L^3}(L_i L_{jk} + L_j L_{ki} + L_k L_{ij})$$
$$- \frac{2\beta}{L^2}(L_{ij} b_k + L_{jk} b_i + L_{ki} b_j) + \frac{4\beta}{L^3}(L_i L_j b_k + L_k L_j b_i + L_k L_i b_j)$$
$$- \frac{2}{L^2}(L_i b_j b_k + L_k b_j b_i + L_j b_i b_k) - \frac{6\beta^2}{L^4}(L_i L_j L_k) \ .$$

(2.4) $$\partial_j \bar{L}_i = \frac{(L^2-\beta^2)}{L^2} \partial_j L_i + \frac{2\beta}{L^3}(\beta L_i - L b_i)\partial_j L + \frac{2}{L^2}(L b_i - \beta L_i)\partial_j \beta$$
$$+2(1 + \frac{\beta}{L})\partial_j b_i \ .$$

(2.5) $\partial_k \bar{L}_{ij} = \frac{(L^2-\beta^2)}{L^2}\partial_k L_{ij} + \left\{\frac{2\beta^2}{L^3}L_{ij} - \frac{6\beta^2}{L^4}L_iL_j + \frac{4\beta}{L^3}(L_ib_j + L_jb_i) - \frac{2}{L^2}b_ib_j\right\}\partial_k L$

$$-2\left\{\frac{1}{L^2}(L_ib_j + L_jb_i) + \frac{\beta}{L^2}L_{ij} - \frac{2\beta}{L^3}L_iL_j\right\}\partial_k\beta + \frac{2\beta}{L^3}(\beta L_i - Lb_i)\partial_k L_j$$

$$+\frac{2\beta}{L^3}(\beta L_j - Lb_j)\partial_k L_i + \frac{2}{L^2}(Lb_i - \beta L_i)\partial_k b_j + \frac{2}{L^2}(Lb_j - \beta L_j)\partial_k b_i\ .$$

Also in $\bar{F}^n$ and F^n , we get

(2.6) $\bar{L}_{ij|k} = 0 \Rightarrow \partial_k\bar{L}_{ij} - \bar{L}_{ijr}\bar{G}^r_k - \bar{L}_{ir}\bar{F}^r_{jk} - \bar{L}_{jr}\bar{F}^r_{ik} = 0\ ,$

where $\bar{G}^r_k = G^r_k + D^r_k$, $\bar{F}^r_{ik} = F^r_{ik} + D^{*r}_{ik}$.

(2.7) $L_{ij|k} = 0 \Rightarrow \partial_k L_{ij} - L_{ijr}G^r_k - L_{ir}F^r_{jk} - L_{jr}F^r_{ik} = 0\ .$

Putting the values from (2.2), (2.3), (2.5) and (2.7) in (2.6) and contract the resulting equation by y^k, we get

(2.8) $2\left\{\frac{1}{L^2}(L_ib_j + L_jb_i) + \frac{\beta}{L^2}L_{ij} - \frac{2\beta}{L^3}L_iL_j\right\}r_{00} - \frac{2}{L^2}(Lb_i - \beta L_i)(r_{j0} + s_{j0})$

$$-\frac{2}{L^2}(Lb_j - \beta L_j)(r_{i0} + s_{i0}) + 2\bar{L}_{ijr}D^r + \bar{L}_{ir}D^r_j + \bar{L}_{jr}D^r_i = 0\ .$$

where '0' in suffices denotes contraction with y^k.

Now deal with following equations in $\bar{F}^n$ and F^n, we get

(2.9) $\bar{L}_{i|j} = 0 \Rightarrow \partial_j\bar{L}_i - \bar{L}_{ir}\bar{G}^r_j - \bar{L}_r\bar{F}^r_{ij} = 0\ ,$

(2.10) $L_{i|j} = 0 \Rightarrow \partial_j L_i - L_{ir}G^r_j - L_rF^r_{ij} = 0\ .$

Putting the values from (2.1), (2.2), (2.4) and (2.10) in (2.9), we get

(2.11) $2\left(1 + \frac{\beta}{L}\right)b_{i|j} = \bar{L}_{ir}D^r_j + \bar{L}_rD^{*r}_{ij} + \frac{2}{L^2}(\beta L_i - Lb_i)(r_{j0} + s_{j0})\ .$

Now

(2.12) $2r_{ij} = b_{i|j} + b_{j|i}\ .$

Putting the value from (2.11) in (2.12), we get

(2.13) $4\left(1 + \frac{\beta}{L}\right)r_{ij} = \bar{L}_{ir}D^r_j + \bar{L}_{jr}D^r_i + 2\bar{L}_rD^{*r}_{ij} + \frac{2}{L^2}(\beta L_i - Lb_i)(r_{j0} + s_{j0})$

$$+\frac{2}{L^2}\big(\beta L_j - Lb_j\big)(r_{i0}+s_{i0}) \; .$$

Subtract (2.8) from (2.13) and contract the resulting equation by $y^i y^j$, we get

(2.14) $$(L-\beta)L_r D^r + 2Lb_r D^r = Lr_{00} \; .$$

Also

(2.15) $$2s_{ij} = b_{i|j} - b_{j|i} \; .$$

Putting the value from (2.11) in (2.15), we get

(2.16) $$4\Big(1+\frac{\beta}{L}\Big)s_{ij} = \bar{L}_{ir}D_j^r - \bar{L}_{jr}D_i^r + \frac{2}{L^2}(\beta L_i - Lb_i)\big(r_{j0}+s_{j0}\big)$$
$$-\frac{2}{L^2}\big(\beta L_j - Lb_j\big)(r_{i0}+s_{i0}) \; .$$

Subtract (2.8) from (2.16) and contract the resulting equation by $y^j b^i$, we get

(2.17) $$Ltb_r D^r - \beta t L_r D^r = 2L^3(L+\beta)s_0 + L(L^2b^2-\beta^2)r_{00},$$

where $t = L^2(1+2b^2) - 3\beta^2$ and $s_0 = s_{i0}b^i$.

The solution of algebraic equation (2.14) and (2.17) is given by

(2.18) $$b_r D^r = \frac{1}{t}\big[(L^2b^2 - 2\beta^2 + L\beta)r_{00} + L^2(L-\beta)s_0\big] \; ,$$

(2.19) $$L_r D^r = \frac{1}{t}\big[L(L-\beta)r_{00} - 2L^3 s_0\big] \; .$$

Subtract (2.8) from (2.16) and contract the resulting equation by y^j, we get

(2.20) $$2\Big(1+\frac{\beta}{L}\Big)s_{i0} + \frac{1}{L^2}(Lb_i - \beta L_i)r_{00} = \bar{L}_{ir}D^r \; .$$

Putting the value from (2.2) in (2.20) and using $LL_{ir} = g_{ir} - L_i L_r$ and $L_i = l_i$ and contracting the resulting equation by g^{ij}, we get

(2.21) $$D^i = \frac{2L^2}{(L-\beta)}s_0^i + \frac{(L-\beta)r_{00} - 2L^3 s_0}{(L-\beta)t}\{L^2b^i + (L-2\beta)y^i\} \; .$$

Proposition (2.1): The difference tensor of Z-Shen square change of Finsler metric $\bar{L}$ is given by the equation (2.21).

3 Projective Change of Finsler Metric

The Finsler space $\bar{F}^n$ is said to be projective to Finsler space F^n if every geodesic of F^n is transformed to a geodesic of $\bar{F}^n$. It is well known that the change $L \to \bar{L}$ is projective if $\bar{G}^i = G^i + P(x,y)y^i$ where $P(x,y)$ is a homogeneous scalar function of degree one in y^i, called the projective factor [4].

Thus from (1.3) it follows that $L \to \bar{L}$ is projective iff $D^i = P(x,y)y^i$. Now we consider that the Z-Shen square change $L \to \bar{L} = \frac{(L+\beta)^2}{L}$ is projective. Then from (2.21) and contracting the resulting by y_j, we get

(3.1) $$P = \frac{(L-\beta)r_{00} - 2L^2 s_0}{t}.$$

Putting $D^i = Py^i$ in equation (2.21) and the value P from (3.1) in (2.21), we have

(3.2) $$\left\{\frac{(L-\beta)r_{00} - 2L^2 s_0}{t}\right\}(\beta y^i - L^2 b^i) = 2L^2 s_0^i.$$

Contracting (3.2) by b_i, we get

(3.3) $$r_{00} = \frac{2L^2(L^2 + L^2 b^2 - 2\beta^2)s_0}{(\beta^2 - L^2 b^2)(L-\beta)}.$$

Putting the value from (3.3) in (3.1), we get

(3.4) $$P = \frac{2L^2 s_0}{(\beta^2 - L^2 b^2)}.$$

Eliminating r_{00} from (3.3) and (3.2), we get

(3.5) $$s_0^i = \left(b^i - \frac{\beta}{L^2} y^i\right) \frac{L^2 s_0}{(L^2 b^2 - \beta^2)}.$$

The equation (3.3) and (3.5) are necessary condition for Z-Shen square change of Finsler metric to be projective.

Conversely, if the conditions (3.3) and (3.5) are satisfied, then putting these value in (2.21), we get $D^i = \frac{2L^2 s_0}{(\beta^2 - L^2 b^2)} y^i = Py^i$.

Thus, The Finsler space $\bar{F}^n$ is projective to Finsler space F^n.

Theorem (3.1): The Z-Shen square change of Finsler metric is projective iff equations (3.3) and (3.5) are satisfied, then the projective factor P is given by $= \frac{2L^2 s_0}{(\beta^2 - L^2 b^2)}$.

4 Douglas Space

The Finsler space F^n is called a Douglas space iff $G^i y^j - G^j y^i$ is homogeneous polynomial of degree three in y^i [5]. We shall write $hp(r)$ are the homogeneous polynomial in y^i of degree r.

If we write

(4.1) $$B^{ij} = D^i y^j - D^j y^i .$$

Then from (2.21) , we get

(4.2) $$B^{ij} = \frac{2L^2}{(L-\beta)}(s_0^i y^j - s_0^j y^i) + \frac{L^2\{(L-\beta)r_{00} - 2L^3 s_0\}}{(L-\beta)t}(b^i y^j - b^j y^i).$$

If a Douglas space is transformed to a Douglas space by Z-Shen square change of Finsler metric, B^{ij} must be $hp(3)$ and if B^{ij} is $hp(3)$ then Douglas space transformed by Z-Shen square change is Douglas space.

Theorem(4.1): The Z-Shen square change of Douglas space is Douglas space iff B^{ij} is given by (4.2), is $hp(3)$.

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