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Weak Module Amenability of Module Extension Banach Algebras

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Abstract

In this paper we study module and weak module amenability of the module extension Banach algebra $A \oplus X$ of a Banach algebra A by a Banach A-module X. As an example we show that for an inverse semigroup S with set of idempotents E, the module extension $\ell^1(E) \oplus \ell^1(S)$ is amenable as an $\ell^1(E)$-module iff S is amenable. We also study module biflatness and module biprojectivity of module extensions.

1 Introduction

The notion of amenability for Banach algebras was first introduced by B.E. Johnson in [19]. A linear map $D : \mathcal{B} \longrightarrow \varepsilon$ is a derivation if

$$D(ab) = D(a) \cdot b + a \cdot D(b) \quad (a, b \,\epsilon\, \mathcal{B}).$$

A Banach algebra $\mathcal{B}$ is amenable if every continuous derivation D from $\mathcal{B}$ into any dual Banach $\mathcal{B}$-bimodule ε' is inner, namely there exists $f \epsilon\, \varepsilon'$ such that

$$D(a) = a \cdot f - f \cdot a \quad (a \epsilon \mathcal{B}),$$

where the module actions on ε' are defined by

$$\langle f \cdot a, x\rangle = \langle f, a \cdot x\rangle \;\; and \;\; \langle a \cdot f, x\rangle = \langle f, x \cdot a\rangle \quad (a \,\epsilon\, \mathcal{B}, x \,\epsilon\, \varepsilon, f \,\epsilon\, \varepsilon').$$

In [20], B. E. Johnson proved that a Banach algebra $\mathcal{B}$ is amenable if and only if it has a bounded approximate diagonal, that is, a bounded net $(\mathrm{m}_a) \subseteq \mathcal{B}\widehat{\otimes}\,\mathcal{B}$ such that

$$a \cdot \mathrm{m}_a - \mathrm{m}_a \cdot a \longrightarrow 0 \;\; and \;\; \Pi_{\mathcal{B}}(\mathrm{m}_a)a \longrightarrow a \quad (a\epsilon\mathcal{B})$$

where $\Pi_{\mathcal{B}} : \mathcal{B}\widehat{\otimes}\,\mathcal{B} \longrightarrow \mathcal{B}$ is the multiplication map defined by $\Pi_{\mathcal{B}} : (a \otimes b) = ab$.

The notions of biflatness and biprojectivity for Banach algebras were introduced by A. YA. Helemskii in [17]. A Banach algebra $\mathcal{B}$ is biflat if there is a bounded $\mathcal{B}$-bimodule homomorphism $\rho : \mathcal{B} \longrightarrow (\mathcal{B}\widehat{\otimes}\,\mathcal{B})''$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "\mathcal{K}_{\mathcal{B}}"] \arrow[dr, "\rho"'] & \mathcal{B}'' \\
& (\mathcal{B}\hat{\otimes}\mathcal{B})'' \arrow[u, "\pi''_{\mathcal{B}}"']
\end{tikzcd}$$

where $k_{\mathcal{B}} : \mathcal{B} \longrightarrow \mathcal{B}''$ is the natural embedding of $\mathcal{B}$ into its second dual, and we regard $\mathcal{B}\widehat{\otimes}\,\mathcal{B}$ as a Banach $\mathcal{B}$-bimodule with module actions:

$$a \cdot (b \otimes c) = ab \otimes c, \;\; (b \otimes c) \cdot a = b \otimes ca \quad (a, b, c\,\epsilon\mathcal{B})$$

Similarly, $\mathcal{B}$ is biprojective if there is a bounded $\mathcal{B}$ -bimodule homomorphism $\sigma : \mathcal{B} \longrightarrow (\mathcal{B}\widehat{\otimes}\,\mathcal{B})$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "id"] \arrow[dr, "\pi_{\mathcal{B}}"'] & \mathcal{B}'' \\
& (\mathcal{B}\hat{\otimes}\mathcal{B})'' \arrow[u, "\sigma"']
\end{tikzcd}$$

It is known that $\mathcal{B}$ is amenable if and only if $\mathcal{B}$ is biflat and has a bounded approximate identity [16]. Biflatness and bprojectivity are studied for various classes of Banach algebras, including C^*-algebras, group algebras and Segal algebras [16, 32]. Consider the situation where $\mathcal{B}$ has an extra module structure as a Banach module over another Banach algebra $\mathcal{B}$ with compatible actions. The second author introduced and studied module amenability of $\mathcal{B}$ in [1]. The module versions of weak amenability [2] permanent amenability [8] super amenability [28] contractibility [5, 6] biflatness and biprojectivity [7] topological center [3] and Arens regularity [30] are studied by several authors.

The main motivating example in most of the works cited above was $\mathcal{B} = \ell^1(S)$ and $\mathcal{B} = \ell^1(E)$, where S is an inverse semigroup with set of idempotents E. In this paper we consider another class of examples coming from module extensions of Banach algebras, in which $\mathcal{B} = A \oplus X$ and $\mathcal{B} = A$, where A is a Banach algebra, X is a Banach A-module, and $A \oplus X$ is the module extension of A by X, considered as a Banach algebra and Banach A-module (compare with [27]). This class of Banach algebras first appeared in [11, 4] and contains the class of triangular Banach algebras [21]. The amenability and weak amenability of these algebras are studied in general in [23, 24] and in the special case of triangular Banach algebras in [13, 14, 21]. The biflatness and biprojectivity of these algebras are studied in [22].

2 Module Amenability

Through out this paper, $\mathcal{A}$ and $\mathfrak{A}$ are Banach algebras such that $\mathcal{A}$ is Banach $\mathfrak{A}$-bimodule with compatible actions, that is

$$\alpha \cdot (ab) = (\alpha \cdot a)b,\ (ab) \cdot \alpha = a(b \cdot \alpha) \qquad (a, b \in \mathcal{A},\ \alpha \in \mathfrak{A}).$$

We say that $\mathfrak{A}$ acts trivially on $\mathcal{A}$ from left if $\alpha \cdot a = f(\alpha)a$, for each $\alpha \in \mathfrak{A}$ and $a \in \mathcal{A}$, where $f \in \Phi_{\mathfrak{A}}$ is a character on $\mathfrak{A}$.

Let $\mathfrak{X}$ be a Banach $\mathcal{A}$-bimodule and a Banach $\mathfrak{A}$-bimodule with compatible actions, that is

$$\alpha\cdot(a\cdot x) = (\alpha\cdot a)\cdot x,\ a\cdot(\alpha\cdot x) = (a\cdot\alpha)\cdot x,\ (\alpha\cdot x).a = \alpha\cdot(x\cdot a) \quad (a \in \mathcal{A}, \alpha \in \mathfrak{A}, x \in \mathfrak{X}),$$

and the same for the right or two-sided actions. Then we say that $\mathfrak{X}$ is a Banach $\mathcal{A}$-$\mathfrak{A}$-module. If moreover

$$\alpha \cdot x = x \cdot \alpha \qquad (\alpha \in \mathfrak{A}, x \in \mathfrak{X})$$

$\mathfrak{X}$ is called a commutative $\mathcal{A}$-$\mathfrak{A}$-module.

If $\mathfrak{X}$ is a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module, then so is $\mathfrak{X}^*$, where the actions of $\mathcal{A}$ and $\mathfrak{A}$ and $\mathfrak{X}^*$ are defined by

$$\big\langle \alpha \cdot f, x \big\rangle = \big\langle f, x \cdot \alpha \big\rangle, \big\langle a \cdot f, x \big\rangle = \big\langle f, x \cdot a \big\rangle \qquad (a \in \mathcal{A}, \alpha \in \mathfrak{A}, x \in \mathfrak{X}, f \in \mathfrak{X}^*),$$

and the same for the right actions. Let be another $\mathcal{A}$-$\mathfrak{A}$-module then a $\mathcal{A}$-$\mathfrak{A}$-module morphism from $\mathfrak{X}$ to $\mathfrak{Y}$ is a norm-continuous map $\phi : \mathfrak{X} \longrightarrow$ with $\phi(x \pm y) = \phi(x) \pm \phi(y)$ and

$$\varphi(\alpha \cdot x) = \alpha \cdot \varphi(x), \varphi(x \cdot \alpha) = \varphi(x) \cdot \alpha, \varphi(a \cdot x) = a \cdot \varphi(x), \varphi(x \cdot a) = \varphi(x) \cdot a$$

for $x, y \in \mathfrak{X}, a \in mathcal A$ and $\mathfrak{A}$. Consider the projective tensor product $\mathcal{A}\hat{\otimes}\mathcal{A}$, which is a Banach algebra with respect to the canonical multiplication

$$(a \otimes b)(c \otimes d) = ac \otimes bd,$$

and extended by linearity and continuity [10]. Also $\mathcal{A}\hat{\otimes}\mathcal{A}$ is a Banach $\mathcal{A}$-$\mathfrak{A}$-module with canonical actions. Let I be the closed ideal of $\mathcal{A}\hat{\otimes}\mathcal{A}$ generated by elements of the form $(a\cdot\alpha)\otimes b - a\otimes(\alpha\cdot b)$ for $\alpha\in\mathfrak{A}, a, b\in\mathcal{A}$. Consider the multiplication map $\pi_{\mathcal{A}}:\mathcal{A}\hat{\otimes}\mathcal{A}\to\mathcal{A}$ defined by $\pi_{\mathcal{A}}(a\otimes b)=ab$, extended by linearity and continuity. Let $J_{\mathcal{A}}$ be the closed ideal of $\mathcal{A}$ generated by $\pi_{\mathcal{A}}(I)$. Then the module projective tensor product $\mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A}\cong\frac{\mathcal{A}\hat{\otimes}\mathcal{A}}{I}$ and the quotient Banach algebra $\frac{\mathcal{A}}{J_{\mathcal{A}}}$ are Banach $\mathfrak{A}$-modules with compatible actions. Also the map $\tilde{\pi}_{\mathcal{A}}:\mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A}\to\frac{\mathcal{A}}{J_{\mathcal{A}}}$ defined by $\tilde{\pi}_{\mathcal{A}}(a\otimes b+I)=ab+I$ extends to an $\mathfrak{A}$-module morphism. Let $\mathcal{A}$ and $\mathfrak{A}$ be as above and $\mathfrak{X}$ be a Banach $\mathcal{A}$-$\mathfrak{A}$-module. A bounded $\mathcal{A}$-$\mathfrak{A}$-module morphism $D:\mathcal{A}\longrightarrow\mathcal{X}$ is called a module derivation if

$$D(a\pm b)=D(a)\pm D(b)\ \ (a\,\epsilon\,\mathcal{A},\alpha\epsilon\,\mathfrak{A})$$

When $\mathcal{X}$ is commutative, each $x\epsilon\,\mathcal{X}$ defines a module derivation

$$D_x(a)=a.x-x.a\ \ (a\,\epsilon\,\mathcal{A}).$$

These are called inner module derivations. The Banach algebra $\mathcal{A}$ is called module amenable (as an $\mathfrak{A}$-module) if for any commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module $\mathcal{X}$, each module derivation D : $\mathcal{A}\longrightarrow\mathcal{X}^*$ is inner [1]. The Banach algebra $\mathcal{A}$ is called weakly module amenable (as an $\mathfrak{A}$-module) if $(\frac{\mathcal{A}}{J_{\mathcal{A}}})^*=J_A^1$ is commutative Banach $\mathfrak{A}$-module and each module derivation from $\mathcal{A}$ to $(\frac{\mathcal{A}}{J_{\mathcal{A}}})^*=J$ $\frac{1}{A}$ is inner [2]. For a Banach module $\mathcal{X}$ over $\mathcal{A}$, a net (a_α) in $\mathcal{A}$ is called a bounded approximate identity for $\mathcal{X}$ if

$$\parallel a_\alpha.X-X\parallel+\parallel X.a_\alpha\parallel\longrightarrow 0\ \ (x\epsilon\mathcal{X}).$$

The following results are proved in [3].

Theorem 2.1. *If $\mathfrak{A}$ has bounded approximate identity for $\mathcal{A}$, then amenability of $\frac{\mathcal{A}}{J_{\mathcal{A}}}$ implies module amenability of $\mathcal{A}$. Conversely if $\mathcal{A}$ is module amenable as an $\mathfrak{A}$-module with trivial left action, J_0 is a closed ideal of $\mathcal{A}$ such that $J_A\subseteq J_0$ and $\frac{\mathcal{A}}{J_0}$ has a left bounded approximate identity, then $\frac{\mathcal{A}}{J_0}$ is amenable.*

Theorem 2.2. *Let $\mathfrak{A}$ acts trivially on $\mathcal{A}$ from left and $\frac{\mathcal{A}}{J_{\mathcal{A}}}$ has a left or right bounded approxima identity, then weak module amenability of $\mathcal{A}$ implies weak amenability of $\frac{\mathcal{A}}{J_{\mathcal{A}}}$. The converse is true if $\mathcal{A}$ is a right essential $\mathfrak{A}-$module.*

3 Module Extension Banach Algebras

In this section we study module extensions of a Banach algebra $\mathcal{A}$ by a Banach $\mathcal{A}$-bimodule $\mathcal{X}$. This is the Banach algebra $A\oplus\,\mathcal{X}$, the l^1-direct sum of $\mathcal{A}$ and $\mathcal{X}$, with the algebra product

$$(a,x).(b,y)=(ab,a.y+x.b)(a,b\,\epsilon A,x,y\,\epsilon X).$$

We obtain the necessary and sufficient conditions for a module extension Banach algebra to be module amenable, weakly module amenable, module biflat, or module biprojective, as an A-module.

We consider the Banach algebra $\mathcal{A}$ as an $\mathcal{A}$ -bimodule with the following compatible module actions

$$b.a = ba,\ \ a.b = f(a,b) \qquad (a, b \,\epsilon\, \mathcal{A}, f \epsilon\, \Phi_A),$$

where Φ_A is the character space of $\mathcal{A}$. Then J_A is the closed ideal of $\mathcal{A}$ generated by the set $\{bac - f(a)bc : a, b, c \,\epsilon\, \mathcal{A}\}$. Consider the module extension $\mathcal{B} := \mathcal{A} \oplus \mathcal{X}$ as an $\mathcal{A}$ -bimodule with the following compatible module actions

$$(b,x).a = (ba, xa), a.(b,x) = (f(a)b, f(a)x) \qquad (a, b \,\epsilon\, \mathcal{A}, x \epsilon X, f \epsilon\, \Phi_{\mathcal{A}}),$$

then

$$\begin{aligned} J_{\mathcal{B}} &= \langle \{[(b,x).a](c,y) - (bx)[a.(c,y)] : a,b,c \,\epsilon\, \mathcal{A}, x, y \,\epsilon\, \mathcal{X}\}\rangle \\ &= \langle \{(ba, xa)(c,y) - (b,x)(f(a)c, f(a)y) : a,b,c \,\epsilon A, x, y \,\epsilon \mathcal{X}\}\rangle \\ &= \langle \{(bac - bf(a)c, (ba - f(a)b)y + x(ac - f(a)c)) : a,b,c \,\epsilon\, A, x, y \,\epsilon\, \mathcal{X}\}\rangle \\ &= J_{\mathcal{A}} \oplus (\mathcal{A}\mathcal{X} + \mathcal{X}\mathcal{A}) = J_{\mathcal{A}} \oplus \mathcal{X} \end{aligned}$$

when $\mathcal{X}$ is an essential $\mathcal{A}$ -bimodule.

Proposition 3.1. *If $\{e_\lambda\}$ is a bounded approximate identity for $\mathcal{A}$ then $\{e_\alpha\}$ is a bounded approximate identity for the module extension $\mathcal{A} \oplus \mathcal{X}$ when $\mathcal{X}$ is an essential $\mathcal{A}$-bimodule.*

Proof. For $(a,x) \,\epsilon\, \mathcal{A} \oplus \mathcal{X}$,

$$\begin{aligned} \|e_\alpha.(a,x) - (a,x)\| &= \|f(e_\alpha a, f(e_\alpha)x)(a.x)\| \\ &= |f(e_\alpha) - 1|\|(a,x)\| \longrightarrow 0, \end{aligned}$$

and if x = y.b, for some $b \,\epsilon\, A$ and $y \,\epsilon\, \mathcal{X}$, then

$$\begin{aligned} \|(a,x).e_\alpha - (a,x)\| &= \|(ae_\alpha, x.e_\alpha)(a,x)\| \\ &= \|ae_\alpha - a\| + \|x.e_\alpha - x\| \\ &= \|ae_\alpha - a\| + \|(y.b).e_\alpha - (y.b)\| \\ &\leq \|ae_\alpha - a\| + \leq \|y\|\|be_\alpha - b\| \longrightarrow 0. \end{aligned}$$

□

Theorem 3.2. *Let $\mathcal{A}$ has a bounded approximate identity and $\mathcal{X}$ is an essential $\mathcal{A}$-bimodule. Then module extension Banach algebra $\mathcal{A} \oplus \mathcal{X}$ is module amenable as $\mathcal{A}$-bimodule if and only if the Banach algebra $\frac{\mathcal{A}}{J_A}$ is amenable.*

Proof. We define the map $\varphi := \mathcal{A} \oplus \mathcal{X} \longrightarrow \frac{\mathcal{A}}{J_A}$ by $\varphi((a,x)) = a + J_A$. It is easy to see that φ is a well-defined $\mathcal{A}$-module morphism and an algebra homomorphism and

$$\ker\varphi = \{(a,x) : a \,\epsilon\, J_A, x \,\epsilon\, X = J_A \oplus X\}$$

and we have $\frac{\mathcal{A}\oplus\mathcal{X}}{J_A\oplus\mathcal{X}} \simeq \frac{\mathcal{A}}{J_A}$, that is $\frac{\mathcal{A}\oplus\mathcal{X}}{J_\mathcal{B}} \simeq \frac{\mathcal{A}}{J_A}$. Therefore, by Theorem 2.1 we get the result. □

Theorem 3.3. *Let $\mathcal{A}$ has a bounded approximate identity for itself and for $\mathcal{X}$. Then the module extension Banach algebra $\mathcal{A} \oplus \mathcal{X}$ is weakly module amenable as an $\mathcal{A}$-bimodule if and only if the Banach algebra $\frac{\mathcal{A}}{J_A}$ is weakly amenable.*

Proof. This follows from Theorem 2.3 and the fact that $\frac{\mathcal{A}\oplus\mathcal{X}}{J_\mathcal{B}} \simeq \frac{\mathcal{A}}{J_A}$. □

As an example, let S is an inverse semigroup with an upward directed set of idempotents E, then E satisfies condition D_1 of Duncan and Namoika [12], hence $l^1(E)$ has a bounded approximate identity. If $\{g_j\}$ is a bounded approximate identity of $l^1(E)$, then

$$g_j * \delta_s = g_j * \delta_{ss^*s} = g_j * \delta_{ss^*} * \delta_s \longrightarrow \delta_s \;\; (s\epsilon S),$$

and similarly for the right multiplication. Therefore $l^1(E)$ has a bounded approximate identity for $l^1(S)$. Consider A = $l^1(E)$, X = $l^1(S)$ and let $l^1(E)$ act on $l^1(S)$ by multiplication from right and trivially from left, that is

$$\delta_e.\delta_s = \delta_s, \delta_s.\delta_e = \delta_{se} = \delta_s * \delta_e \;\; (s\epsilon S, e\epsilon E).$$

It is easy to show that $l^1(S)$ is a Banach $l^1(E)$-module with compatible actions. Define an equivalence relation on E as follows:

$$e_1 \approx e_2 \Longleftrightarrow \delta_{e_1} - \delta_{e_2} \epsilon J_{l^1(E)} \;\; (e_1, e_2 \,\epsilon E).$$

Then the quotient $\frac{E}{\approx}$ is discrete group (see [1, 2]). As in [2], one may observe that $\frac{l^1(E)}{J_A} \cong l^1(\frac{E}{\approx})$. Thus by Theorems 3.2 and 3.3, the module extension $\mathcal{B} = l^1(E) \oplus l^1(S)$ is module amenable as an $l^1(E)$-bimodule if and only if the discrete group $\frac{E}{\approx}$ is amenable. Also it is always weakly module amenable as an $l^1(E)$-bimodule, since the group algebra $l^1(\frac{E}{\approx})$ is always weakly amenable.

As a negative result, consider the case $\mathcal{A} = \mathcal{C}, \mathcal{X} = \mathcal{C}$ and let $\mathcal{A}$ act on $\mathcal{X}$ by multiplication from both sides. Then the module extension $\mathcal{B} = \mathcal{C} \oplus \mathcal{C}$ is module amenable as an $\mathcal{C}$-bimodule by Theorem 3.2 (since $\frac{\mathcal{C}}{J_A} \simeq \mathcal{C}$ is amenable), but it is not even weak amenable [34].

Next we turn to module biflatness and module biprojectivity of module extension Banach algebras. First let us recall some definition and results from [7].

Definition 3.4. *The Banach algebra $\mathcal{A}$ is called module biprojective (as an A-module) if $\overrightarrow{\Pi}_{\mathcal{A}} : \mathcal{A}\widehat{\otimes}_{\mathcal{A}} \longrightarrow \frac{\mathcal{A}}{J_{\mathcal{A}}}$ has a bounded right inverse which is an $\frac{\mathcal{A}}{J_{\mathcal{A}}} - \mathfrak{A} -$ module morphism. It is called module biflat (as an $\mathfrak{A}$-module) if $\tilde{\Pi}^*_{\mathcal{A}} : (\frac{\mathcal{A}}{J_{\mathcal{A}}})^* \longrightarrow (\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A})^*$ has a bounded left inverse which is an $\frac{\mathcal{A}}{J_{\mathcal{A}}} - \mathfrak{A}$-module morphism.*

Proposition 3.5. *Assume that $\mathfrak{A}$ acts on $\mathcal{A}$ trivially form left and $\frac{\mathcal{A}}{J_{\mathcal{A}}}$ has a left identity and $\mathfrak{A}$ has a bounded approximate identity for $\mathcal{A}$. If $\mathcal{A}$ is module (biflat) biprojective, then $\frac{\mathcal{A}}{J_{\mathcal{A}}}$ is (biflat) biprojective.*

Now let $\mathcal{X}$ be an essential $\mathcal{A}$-bimodule and consider $\mathcal{B} = \mathcal{A} \oplus \mathcal{X}$ as an $\mathcal{A}$-bimodule with trivially left action and canonical right action.

Similar to the proof of Theorem 3.2, and using the above proposition, we get the following result.

Corollary 3.6. *Let $\mathcal{A}$ has a bounded approximate identity and $\frac{\mathcal{A}}{J_{\mathcal{A}}}$ has a left identity. If the module extension $\mathcal{A} = \mathcal{A} \oplus \mathcal{X}$ is module (biflat) biprojective then $\frac{\mathcal{A}}{J_{\mathcal{A}}}$ is (biflat) biprojective.*

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