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On Some Non MED Symmetric Numerical Semigroups

Ahmet Çelik

Adıyaman University
Technical Sciences Vocational School
Departmant of Technologies Computer
Adıyaman,Turkey
Orcid no.: 0000-0001-5980-0625

Abstract

The theory of numerical semigroups is important for its applications in algebra. Research on symmetric numerical semigroups, in particular, has attracted attention in recent years with its applications in various fields. In this original work, one of these studies, we consider a class of symmetric numerical semigroups with no maximal embedding dimension and half of this class.

Here, we will give some results on some non MED symmetric numerical semigroups such that $S=<3,x>$ and $\frac{S}{2}=<3,\frac{x}{2}>$. Also, we will examine the relationships between these semigroups.

Mathematics Subject Classification: 20M14

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Introduction

Let $\mathbb{N}$ and $\mathbb{Z}$ be the sets of nonnegative integers and integers, respectively. The subset S of $\mathbb{N}$ is a numerical semigroup if $0 \in S$, $x+y \in S, for\ all\ x,y \in S$, and $Card(\mathbb{N} \backslash S) < \infty$ (this condition is equivalent to $\gcd(S)=1$, $\gcd(S)=$ greatest common divisor the element of S) .

Let S be a numerical semigroup, then $F(S)=\max(\mathbb{Z}\backslash S)$ and $m(S)=\min\{s\in S: s>0\}$ are called Frobenius number and multiplicity of S, respectively. Also, $n(S)=Card(\{0,1,2,...,F(S)\}\cap S)$ is called the number number of S. If $F(S)-x\in S$ then is called symmetric numerical semigroup, for all $x\in\mathbb{Z}\backslash S$. It is known that $S=<a,b>$ is symmetric numerical semigroup ([1,7]). If S is a numerical semigroup such that $S=<a_1,a_2,...,a_v>$, then we observe that $S=<a_1,a_2,...,a_v>=\{s_0=0,s_,,s_2,...,s_{n-1},s_n=F(S)+1,\rightarrow...\}$ where $s_i<s_{i+1}$, $n=n(S)$, and the arrow means that every integer greater than $F(S)+1$ belongs to S, for $i=1,2,...,n=n(S)$. Here, the number $v\in\mathbb{N}$ is called embedding dimension of S, and denoted by $e(S)$. If $e(S)=m(S)$ then S is called maximal embedding dimension (MED) numerical semigroup.

If $a\in\mathbb{N}$ and $a\notin S$, then a is called gap of S. We denote the set of gaps of S, by $H(S)$, i.e, $H(S)=\mathbb{N}\backslash S$ and, the $G(S)=Card(H(S))$ is called the genus of S. Also, It is know that $G(S)=F(S)+1-n(S)$ and, if S is a symmetric numerical semigroup then $n(S)=G(S)=\dfrac{F(S)+1}{2}$ (for details see [4,5,6]). Let S be a numerical semigroup and $0\neq d\in\mathbb{N}$. The set $\dfrac{S}{d}=\{y\in\mathbb{N}: yd\in S\}$ is called the divisor set of S by d. If $d=2$ then the set $\dfrac{S}{2}$ is called half of S. It is clear that $S\subseteq\dfrac{S}{d}$ ([2,7]).

Let S be a numerical semigroup and $0\neq m\in S$. The Apery set of m in S is $Ap(S,m)=\{s\in S: s-m\notin S\}$. It can easily be proved that $Ap(S,m)$ is formed by the smallest elements of S belonging to the differenr congruence classes $\bmod m$. According to this, we $Card(Ap(S,m))=m$ and $\max(Ap(s,m))=F(S)+m$ ([11,12]).

A numerical semigroup S is called Arf if $a_1+a_2-a_3\in S$, for all $a_1,a_2,a_3\in S$ such that $a_1\geq a_2\geq a_3$. The smallest Arf numerical semigroup containing a numerical semigroup S is called the Arf closure of S, and it is denoted by $Arf(S)$ (for detail see [8,10]). If S is a numerical semigroup such that $S=<a_1,a_2,...,a_n>$, then $L(S)=\langle a_1,a_2-a_1,a_3-v_1,...,a_n-v_1\rangle$ is called Lipman numerical semigroup of S, and it is known that
$L_0(S)=S\subseteq L_1(S)=L(L_0(S))\subseteq L_2=L(L_1(S))\subseteq...\subseteq L_\alpha=L(L_{\alpha-1}(S))\subseteq...\subseteq\mathbb{N}$
(see [9]).

In this study, we will give some results about non med symmetric numerical semigroups S and $\frac{S}{2}$ such that $S=<3,x>$ and $\frac{S}{2}=<3,\frac{x}{2}>$ where x even integer and $(3,x)=1$.

Main Results

Theorem 1 ([8]). Let S be a numerical semigroup and f its Frobenius number, let $m_i = m(L_i)$ where L_i is the i th term of the Lipman sequence of semigroups of S for each $i \geq 0$. Let, $f(Arf(S)) = f^{(a)}$, $n(Arf(S) = n^{(a)}$. Then $n^{(a)} = \lambda(S) = \lambda$ and we have,

$$Arf(S) = \{s_0^{(a)} = 0, s_1^{(a)}, s_2^{(a)}, ..., s_{\lambda-1}^{(a)}, s_\lambda^{(a)} = f^{(a)} + 1, \rightarrow ...\}$$

where,

$$s_1^{(a)} = m_0 = m(S),$$
$$s_2^{(a)} = m_0 + m_1, ..., s_{\lambda-1}^{(a)} = m_0 + m_1 + m_2 + ... + m_{\lambda-2},$$
$$s_\lambda^{(a)} = m_0 + m_1 + m_2 + ... + m_{\lambda-2} + m_{\lambda-1}$$

and

$$f^{(a)} = m_0 + m_1 + m_2 + ... + m_{\lambda-2} + m_{\lambda-1} - 1.$$

Theorem 2. Let be $S=<3,x>$ a non MED symmetric numerical semigroup where x even integer and $(3,x)=1$. Then $\frac{S}{2}=<3,\frac{x}{2}>$.

Proof. Let be $S=<3,x>$ a non MED symmetric numerical semigroup where x even integer and $(3,x)=1$. Then, we write

$$\begin{aligned} t \in <3,\frac{x}{2}> &\Leftrightarrow \exists m_1, m_2 \in \mathbb{N},\ t = 3m_1 + (\frac{x}{2})m_2 \\ &\Leftrightarrow 2t = 3(\underbrace{2m_1}_{v\in\mathbb{N}}) + xm_2 = 3v + xm_2 \in <3,x> \\ &\Leftrightarrow 2t \in <3,x> = S \\ &\Leftrightarrow t \in \frac{S}{2} \ . \end{aligned}$$

Finally, we obtain that $\frac{S}{2}=<3,\frac{x}{2}>$.

Corollary 3. Let be $S=<3,x>$ are $\frac{S}{2}=<3,\frac{x}{2}>$ non MED symmetric numerical semigroups where x even integer and $(3,x)=1$. Then, we write

(a) $F(\frac{S}{2}) = x-3$ **(b)** $n(\frac{S}{2}) = \frac{x}{2}-1$ **(c)** $G(\frac{S}{2}) = \frac{x}{2}-1$.

Corollary 4 . Let be $S=<3,x>$ and $\frac{S}{2}=<3,\frac{x}{2}>$ non MED symmetric numerical semigroups where x even integer and $(3,x)=1$. Then, we have

(a) $F(S)=F(\frac{S}{2})+5$ **(b)** $n(S)=2n(\frac{S}{2})+2$

(c) $G(S)=G(\frac{S}{2})+2$.

Theorem 5. ([3]) Let be $S=<3,x>$ a non MED symmetric numerical semigroup where x even integer and $(3,x)=1$. For $k\geq 1,\ k\in\mathbb{Z}$,

(i) if $x=3k+1$ then $Arf(S)=\{0,3,6,9,...,3k,\rightarrow...\}$.

(ii) if $x=3k+2$ then $Arf(S)=\{0,3,6,9,...,3k,3k+2,\rightarrow...\}$.

Theorem 6. Let be $S=<3,x>$ and $\frac{S}{2}=<3,\frac{x}{2}>$ non MED symmetric numerical semigroups where x even integer and $(3,x)=1$. For $k\geq 1,\ k\in\mathbb{Z}$,

(i) if $x=3k+1$ then $Arf(\frac{S}{2})=\{0,3,6,9,...,\frac{3(k-1)}{2},\frac{3k+1}{2},\rightarrow...\}$

(ii) if $x=3k+2$ then $Arf(\frac{S}{2})=\{0,3,6,9,...,\frac{3k}{2},\rightarrow...\}$.

Proof. Let be $S=<3,x>$ and $\frac{S}{2}=<3,\frac{x}{2}>$ non MED symmetric numerical semigroup where x even integer and $(3,x)=1$. For $k\geq 1,\ k\in\mathbb{Z}$,

(i) if $x=3k+1$ then we write $\frac{S_k}{2}=<3,\frac{3k+1}{2}>$. So, we obtain

$L_0(\frac{S_k}{2})=\frac{S_k}{2}, m_0=3$ and $L_1(\frac{S_k}{2})=<3,\frac{3k+1}{2}-3>=<3,\frac{3k-5}{2}>$.

Here ,

(a) if $\frac{3k-5}{2}<3$ (i.e. $k=3$) then we write that

$\frac{S_3}{2}=<3,5>$, $L_1(\frac{S_3}{2})=L_1(<3,5>)=<3,2>=<2,3>$, $m_1=2$ and

$L_2(\frac{S_3}{2})=L(L_1(\frac{S_3}{2}))=L(<2,3>)=<2,1>=<1,2>=<1>=\mathbb{N}$, $m_2=1$.

Thus, we find that $Arf(\frac{S_3}{2})=\{0,3,5,\rightarrow...\}$.

(b) if $\frac{3k-5}{2}>3$ then we write $m_1=3$ from

$L_1(\frac{S_k}{2})=<3,\frac{3k+1}{2}-3>=<3,\frac{3k-5}{2}>$ and

$L_2(\frac{S_k}{2}) = <3, \frac{3k-5}{2} - 3> = <3, \frac{3k-11}{2}>$. In this case,

(1) if $\frac{3k-11}{2} < 3$ (i.e. $k = 5$) then we write that

$\frac{S_5}{2} = <3,8>$, $L_0(\frac{S_5}{2}) = L_0(<3,8>) = <3,8>$, $m_0 = 3$;

$L_1(\frac{S_5}{2}) = L_1(<3,8>) = <3,5>$, $m_1 = 3$

$L_2(\frac{S_5}{2}) = L(<3,5>) = <3,2> = <2,3>$, $m_2 = 2$ and

$L_3(\frac{S_5}{2}) = L_3(<3,8>) = L(L_2(<3,8>)) = L(<2,3>) = <2,1> = <1> = \mathbb{N}$, $m_3 = 1$

. Thus, we find that $Arf(\frac{S_3}{2}) = \{0,3,6,8,\rightarrow ...\}$.

(2) if $\frac{3k-11}{2} > 3$ then we write $m_2 = 3$ from $L_2(\frac{S_k}{2}) = <3, \frac{3k-11}{2}>$ and

$L_3(\frac{S_k}{2}) = <3, \frac{3k-11}{2} - 3> = <3, \frac{3k-17}{2}>$. Here,

(I) if $\frac{3k-17}{2} < 3$ (i.e. $k = 7$ since $17 < 3k < 23$) then we write that

$\frac{S_7}{2} = <3,11>$, $L_0(\frac{S_7}{2}) = L_0(<3,11>) = <3,11>$, $m_0 = 3$;

$L_1(\frac{S_7}{2}) = L(<3,11>) = <3,8>$, $m_1 = 3$;

$L_2(\frac{S_7}{2}) = L(<3,8>) = <3,5>$, $m_2 = 3$;

$L_3(\frac{S_7}{2}) = L_3(<3,11>) = L(L_2(<3,11>)) = L(<3,5>) = <3,2> = <2,3>$, $m_3 = 2$

and

$L_4(\frac{S_7}{2}) = L(L_3(\frac{S_7}{2})) = L(<2,3>) = <2,1> = <1,2> = <1> = \mathbb{N}$, $m_4 = 1$.

Thus, we find that $Arf(\frac{S_3}{2}) = \{0,3,6,9,11,\rightarrow ...\}$.

(II) if $\frac{3k-17}{2} > 3$ then we write $m_3 = 3$ from $L_3(\frac{S_k}{2}) = <3, \frac{3k-17}{2}>$ and we write $L_4(\frac{S_k}{2}) = <3, \frac{3k-17}{2} - 3> = <3, \frac{3k-23}{2}>$.

If we continue the operations, we get

$Arf(\frac{S}{2}) = \{0,3,6,9,..., \frac{3(k-1)}{2}, \frac{3k+1}{2}, \rightarrow ...\}$.

(ii) if $x = 3k+2$ then we write $\frac{S_k}{2} = <3, \frac{3k+2}{2}>$. So, we obtain $L_0(\frac{S_k}{2}) = \frac{S_k}{2}, m_0 = 3$ and $L_1(\frac{S_k}{2}) = <3, \frac{3k+2}{2} - 3> = <3, \frac{3k-4}{2}>$. For this situation, if we do operations similar to the above, we will obtain that $Arf(\frac{S}{2}) = \{0,3,6,9,..., \frac{3k}{2}, \rightarrow ...\}$.

Theorem 7. ([3]) Let be $S = <3, x>$ a non MED symmetric numerical semigroups where x even integer and $(3,x) = 1$. For $k \geq 1, k \in \mathbb{Z}$,

(1) if $x = 3k+1$ then $Arf(S) = \{0,3,6,9,...,3k, \rightarrow ...\}$. In this case, we have

a) $F(Arf(S)) = 3k-1$ **b)** $n(Arf(S)) = k$ **c)** $G(Arf(S)) = 2k$

(2) if $x = 3k+2$ then $Arf(S) = \{0,3,6,9,...,3k,3k+2, \rightarrow ...\}$. So, we write

a) $F(Arf(S)) = 3k+1$ **b)** $n(Arf(S)) = k+1$ **c)** $G(Arf(S)) = 2k$

Corollary 8. Let be $S = <3, x>$ a non MED symmetric numerical semigroups where x even integer and $(3,x) = 1$. For $k \geq 1, k \in \mathbb{Z}$,

(1) if $x = 3k+1$ then $Arf(\frac{S}{2}) = \{0,3,6,9,..., \frac{3(k-1)}{2}, \frac{3k+1}{2} \rightarrow ...\}$. In this case, we have,

i) $F(Arf(\frac{S}{2})) = \frac{3k-1}{2}$ **ii)** $n(Arf(\frac{S}{2})) = \frac{k+1}{2}$ **iii)** $G(Arf(\frac{S}{2})) = k$

(2) if $x = 3k+2$ then $Arf(\frac{S}{2}) = \{0,3,6,9,..., \frac{3k}{2}, \rightarrow ...\}$. So, we write

i) $F(Arf(\frac{S}{2})) = \frac{3k}{2} - 1$ **ii)** $n(Arf(\frac{S}{2})) = \frac{k}{2}$ **iii)** $G(Arf(\frac{S}{2})) = k$

Corollary 9. Let be $S = <3, x>$ and $\frac{S}{2} = <3, \frac{x}{2}>$ non MED symmetric numerical semigroups where x even integer and $(3,x) = 1$.

(i) If $x = 3k+1$ then

(1) $F(Arf(S)) = 2F(Arf(\frac{S}{2}))$ **(2)** $n(Arf(S)) = 2n(Arf(\frac{S}{2})) - 1$

(3) $G(Arf(S)) = 2G(Arf(\frac{S}{2}))$

(ii) If $x = 3k+2$ then

(1) $F(Arf(S)) = 2F(Arf(\frac{S}{2})) + 3$ **(2)** $n(Arf(S)) = 2n(Arf(\frac{S}{2})) + 1$

(3) $G(Arf(S)) = 2G(Arf(\frac{S}{2})) + 1$

Corollary 10. Let be $S =< 3, x >$ and $\frac{S}{2} =< 3, \frac{x}{2} >$ non MED symmetric numerical semigroup where x even integer and $(3, x) = 1$.

(1) If $x = 3k + 1$ then

(a) $F(\frac{S}{2}) = 2F(Arf(\frac{S}{2})) - 1$ **(b)** $n(\frac{S}{2}) = 3n(Arf(\frac{S}{2})) - 2$

(c) $G(\frac{S}{2}) = \frac{3}{2}G(Arf(\frac{S}{2})) - \frac{1}{2}$

(2) If $x = 3k + 2$ then

(i) $F(\frac{S}{2}) = 2F(Arf(\frac{S}{2})) + 1$ **(ii)** $n(\frac{S}{2}) = 3n(Arf(\frac{S}{2}))$

(iii) $G(\frac{S}{2}) = \frac{3}{2}G(Arf(\frac{S}{2}))$.

Theorem 11. Let be $S =< 3, x >$ and $\frac{S}{2} =< 3, \frac{x}{2} >$ non MED symmetric numerical semigroups where x even integer and $(3, x) = 1$. Then, we have $Ap(S,3) = \{0, x, 2x\}$ and $Ap(\frac{S}{2}, 3) = \left\{0, \frac{x}{2}, x\right\}$.

Proof. Let be $S =< 3, x >$ and $\frac{S}{2} =< 3, \frac{x}{2} >$ non MED symmetric numerical semigroup where x even integer and $(3, x) = 1$. We assume that $T = \left\{0, \frac{x}{2}, x\right\}$. If $a \in T$ then $a = 0$ or $a = \frac{x}{2}$ or $a = x$.

(i) If $a = 0$ then $a = 0 \in Ap(\frac{S}{2}, 3)$ since $0 - 3 \notin \frac{S}{2} =< 3, \frac{x}{2} >$.

(ii) If $a = \frac{x}{2}$ then $a = \frac{x}{2} \in Ap(\frac{S}{2}, 3)$ since $\frac{x}{2} - 3 \notin \frac{S}{2} =< 3, \frac{x}{2} >$.

Conversely, if $(\frac{x}{2} - 3) \in \frac{S}{2}$ then we obtain contradiction by

$2(\frac{x}{2} - 3) = x - 6 = (x - 3) - 3 = s \in S \Rightarrow x - 3 = s + 3 \in S$.

(iii) If $a = x$ then $2x \in S$ from $x \in \frac{S}{2}$. Therefore,

$a = x \in Ap(\frac{S}{2}, 3)$

since $x-3\notin \frac{S}{2}=<3,\frac{x}{2}>$. Conversely, if $(x-3)\in \frac{S}{2}$ then we obtain contradiction by $x-3=b\in \frac{S}{2}\Rightarrow x-3=2b\in S$. Because, $x-3\notin S$ since $x\in Ap(S,3)$. Thus, we find that $T=Ap(\frac{S}{2},3)$.

Example12. We put $x=16\,(for\ k=5\ in\ x=3k+1)$. Then we have $S=<3,16>=\{0,3,6,9,12,15,16,18,19,21,22,24,25,27,28,30,\rightarrow ...\}$. In this case, we obtain $F(S)=29,\ m(S)=3,\ e(S)=2$, $n(S)=\frac{F(S)+1}{2}=15$, $H(S)=\{1,2,4,5,7,8,10,11,13,14,17,20,23,26,29\}$ and $G(S)=Card(H(S))=15$. Thus, the numerical semigroup $S=<3,16>$ is not MED because $m(S)\neq e(S)$. On the other hand, we write

$\frac{S}{2}=\{x\in \mathbb{N}:\ 2x\in S\}=\{0,3,6,8,9,11,12,14,\rightarrow ...\}=<3,8>$,

$F(\frac{S}{2})=13,\ m(\frac{S}{2})=3$, $n(\frac{S}{2})=\frac{F(\frac{S}{2})+1}{2}=7$ and $G(\frac{S}{2})=Card(H(\frac{S}{2}))=n(\frac{S}{2})=7$ since $\frac{S}{2}=<3,8>$ is symmetric numerical semigroup. Thus, we find $Arf(S)=\{0,3,6,9,12,15,\rightarrow ...\}$ from Theorem 5/(i), and $Arf(\frac{S}{2})=\{0,3,6,8,\rightarrow ...\}$ from Theorem 6/(i). In this case, we write that $F(Arf(\frac{S}{2}))=\frac{3k-1}{2}=7$, $n(Arf(\frac{S}{2}))=\frac{k+1}{2}=3$, $G(Arf(\frac{S}{2}))=k=5$ from Corollary 8 /(1), and $F(Arf(S))=2F(Arf(\frac{S}{2}))=2.7=14$, $n(Arf(S))=2n(Arf(\frac{S}{2}))-1=3$ and $G(Arf(S))=2G(Arf(\frac{S}{2}))=10$ from Corollary 9/(i). Also,we have $F(\frac{S}{2})=2F(Arf(\frac{S}{2}))-1=13$, $n(\frac{S}{2})=3n(Arf(\frac{S}{2}))-2=7$ and $G(\frac{S}{2})=\frac{3}{2}G(Arf(\frac{S}{2}))-\frac{1}{2}=7$ from Corollary 10/(1).

Example 13. We put $x=14\,(for\ k=4\ in\ x=3k+2)$. Then we have $S=<3,14>=\{0,3,6,9,12,14,15,17,18,20,21,23,24,26,\rightarrow ...\}$.
In this case, we obtain $F(S)=25,\ m(S)=3,\ e(S)=2$,

$n(S)=G(S)=\dfrac{F(S)+1}{2}=13$. Thus, the numerical semigroup $S=<3,14>$ is not MED because $m(S)\neq e(S)$. On the other hand, we write

$\dfrac{S}{2}=\{u\in\mathbb{N}\colon 2u\in S\}=\{0,3,6,7,9,10,12,\rightarrow...\}=<3,7>$,

$F(\dfrac{S}{2})=x-3=14-3=11$, $n(\dfrac{S}{2})=G(\dfrac{S}{2})=\dfrac{x}{2}-1=7-1=6$.

Thus,we find $Arf(S)=\{0,3,6,9,12,14,\rightarrow...\}$ by Theorem 5/(ii), and

$Arf(\dfrac{S}{2})=\{0,3,6,\rightarrow...\}$ from Theorem 6/(ii) . In this case, we write that

$F(Arf(\dfrac{S}{2}))=\dfrac{3k}{2}-1=5$, $n(Arf(\dfrac{S}{2}))=\dfrac{k}{2}=2$, $G(Arf(\dfrac{S}{2}))=k=4$ from

Theorem 8 /(2), and $F(Arf(S))=2F(Arf(\dfrac{S}{2}))=2.5+3=13$,

$n(Arf(S))=2n(Arf(\dfrac{S}{2}))+1=5$ and $G(Arf(S))=2G(Arf(\dfrac{S}{2}))+1=9$ from

Corollary 9/(ii). Also, we have $F(\dfrac{S}{2})=2F(Arf(\dfrac{S}{2}))+1=11$,

$n(\dfrac{S}{2})=3n(Arf(\dfrac{S}{2}))=6$, $G(\dfrac{S}{2})=\dfrac{3}{2}G(Arf(\dfrac{S}{2}))=6$ from Corollary 10/(2).

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