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# On Maillet Conjecture for Even Integers

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**Abstract**

Maillet conjecture states that any even integer is the difference of two primes. In this paper, based on the characteristic function of odd primes, we construct an extreme value problem subject to constraints and use the method of Lagrange multipliers to prove Maillet conjecture. Here Bertrand postulate, the technique of "adding a new variable" and the infinitude of odd primes are used. Also the result for the existence of primes between $2n$ and $3n$ by El Bachraoui is applied.

**Mathematics Subject Classification:** 11N05

**Keywords:** Maillet conjecture, odd prime, extreme value problem, constraint

# 1 Introduction

An important and interesting problem in the number theory is Maillet conjecture proposed in 1905 (see [4]), which states that any even integer is the difference of two primes. We will confirm the conjecture.

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Let $\delta(i)$ be the characteristic function of odd primes, i.e.,

$$\delta(i) = 1, \text{ if } i \text{ is an odd prime;}$$

$$\delta(i) = 0, \text{ if } i = 1, 2 \text{ or composite number.}$$

For example, $\delta(1) = 0$, $\delta(2) = 0$, $\delta(3) = 1$, $\delta(4) = 0$, $\delta(5) = 1$, $\delta(6) = 0$, $\delta(7) = 1$, $\delta(8) = 0$, and $\delta(9) = 0, \cdots$. It sees easily

$$\delta(i)^2 = \delta(i).$$

The main result of the paper is

**Theorem 1.1.** *Any even integer is the difference of two primes.*

To prove Theorem 1.1, on the basis of the characteristic function of odd primes, we construct a conditional extreme values problem and solve it by using the method of Lagrange multipliers to obtain the conclusion. In the process, Bertrand postulate ([1,3]), the result for the existence of primes between $2n$ and $3n$ by El Bachraoui ([2]), the technique of "adding a new variable" and the infinitude of odd primes are used.

The proof of Theorem 1.1 is in Section 2.

## 2 Proof of Theorem 1.1

Let us first relate the method of Lagrange multipliers (e.g., refer to [5]) which will be used. For seeking the maximum and minimum values of $f(x)(x \in R^n)$ subject to constraints

$$g_i(x) = 0 \ (i = 1, 2, \cdots, k, k < n)$$

(assuming that these extreme values exist and the rank of Jacobian matrix

$$\frac{\partial(g_1, \cdots, g_k)}{\partial(x_1, \cdots, x_n)}$$

of $g_i(x)$ $(i = 1, 2, \cdots, k)$ is $k$):

(a) find all $x \in R^n, \lambda_1, \cdots, \lambda_k \in R$ such that

$$\frac{\partial f}{\partial x_i} + \lambda_1 \frac{\partial g_1}{\partial x_i} + \cdots + \lambda_k \frac{\partial g_k}{\partial x_i} = 0, \ i = 1, \cdots, n,$$

$$g_i(x) = 0, \ i = 1, 2, \cdots, k,$$

where $x$ is the stationary point and $\lambda_1, \cdots, \lambda_k$ are multipliers;

(b) evaluate $f$ at all the points $x$ that result from (a). The largest of these values is the maximum value of $f$ and the smallest is the minimum value of $f$.

**Proof of Theorem 1.1** For any even integers $2n > 6$ (for even integers $2n \leq 6$, one can see $2 = 5 - 3,\ 4 = 11 - 7,\ 5 = 11 - 5$), denote odd primes between 1 and $n$ by

$$p_1, p_2, \cdots, p_l,$$

where $l \geq 1$. Denote odd primes between $n$ and $2n$ by

$$q_1, q_2, \cdots, q_{l_1},$$

where $l_1 \geq 1$ by Bertrand postulate ([1,3]). Note that odd integers between $2n$ and $3n$ allow two forms. One form is

$$2n + p_i, i = 1, 2, \cdots, l,$$

which satisfy

$$\delta(2n + p_i) = 0 \text{ or } 1;$$

another form is $2n + k$ ($k \neq p_i$), in which odd primes are denoted by

$$r_1, r_2, \cdots, r_{l_2} (l_2 \geq 0).$$

Following to [2], there exist primes between $2n$ and $3n$, then if $l_2 = 0$, so there exists an odd prime of the type $2n + p_i$, and we obtain by combining $p_i$ being a prime that

$$2n = (2n + p_i) - p_i,$$

the conclusion is proved. Now we let $l_2 > 0$ and prove that there exists a prime $p_i$ such that $2n + p_i$ is an odd prime.

To do so, take a large odd prime $N > 3n$, i.e., $\delta(N) = 1$ (such $N$ can be chosen from the infinitude of odd primes). Denote the point with components

$$\delta(p_i)(i = 1, \cdots, l), \delta(q_j)(j = 1, \cdots, l_1), \delta(2n+p_i)(i = 1, \cdots, l), \delta(r_s)(s = 1, \cdots, l_2), \delta(N)$$

by $P \in \mathrm{R}^{2l+l_1+l_2+1}$($\mathrm{R}^{2l+l_1+l_2+1}$ is the $2l + l_1 + l_2 + 1$ dimensional Euclidean space). Then

$$\sum_{i=1}^{l} \delta(p_i) + \sum_{j=1}^{l_1} \delta(q_j) + \sum_{i=1}^{l} \delta(2n + p_i) + \sum_{s=1}^{l_2} \delta(r_s) = \pi(3n). \tag{2.1}$$

Denoting

$$x = (x_1, \cdots, x_l), y = (y_1, \cdots, y_{l_1}), u = (u_1, \cdots, u_l), v = (v_1, \cdots, v_{l_2}), z = z.$$

we introduce an objective function on $\mathrm{R}^{2l+l_1+l_2+1}$

$$f(x, y, u, v, z) = \sum_{i=1}^{l} \left(u_i^2 + u_i\right). \tag{2.2}$$

Using (2.1), the property $\delta(i)^2=\delta(i)$ and the fact of $P$ satisfying

$$\sum_{i=1}^{l}\delta(p_i)^2+\sum_{j=1}^{l_1}\delta(q_j)^2+\sum_{i=1}^{l}\delta(2n+p_i)^2+\sum_{s=1}^{l_2}\delta(r_s)^2=\pi(3n)\delta(N)$$

and

$$\sum_{i=1}^{l}\left(\delta(p_i)^2+\delta(p_i)+\delta(2n+p_i)^2\right)+\sum_{j=1}^{l_1}\left(\delta(q_j)^2+\delta(q_j)\right)+\sum_{s=1}^{l_2}\left(\delta(r_s)^2+\delta(r_s)\right)$$
$$=\pi(3n)+l+l_1+l_2,$$

we let two functions on $\mathrm{R}^{2l+l_1+l_2+1}$

$$g(x,y,,u,v,z)=\sum_{i=1}^{l}(x_i^2+u_i^2)+\sum_{j=1}^{l_1}y_j^2+\sum_{s=1}^{l_2}v_s^2-\pi(3n)z \tag{2.3}$$

and
(2.4)

$$h(x,y,,u,v,z)=\sum_{i=1}^{l}\left(x_i^2+x_i+u_i^2\right)+\sum_{j=1}^{l_1}\left(y_j^2+y_j\right)+\sum_{s=1}^{l_2}\left(v_s^2+v_s\right)-\left(\pi(3n)+l+l_1+l_2\right).$$

Let us investigate the extreme values of $f(x,y,u,v,z)$ subject to constraints

$$g(x,y,u,v,z)=0 \text{ and } h(x,y,u,v,z)=0.$$

Denote

$$A=\{(x,y,u,v,z)\in\mathrm{R}^{2l+l_1+l_2+1}|g(x,y,u,v,z)=0,h(x,y,u,v,z)=0\}. \tag{2.5}$$

Clearly,

$$P\in A.$$

Since $g(x,y,u,v,z)=0$ is the rotating paraboloid in $\mathrm{R}^{2l+l_1+l_2+1}$ and $h(x,y,u,v,z)=0$ is the ellipse cylinder in $\mathrm{R}^{2l+l_1+l_2+1}$, we see that $A$ is a bounded closed set in $\mathrm{R}^{2l+l_1+l_2+1}$ and the rank of Jacobian matrix on $A$ of $g(x,y,u,v,z)$ and $h(x,y,u,v,z)$ is 2. Then $f(x,y,u,v,z)$ allows the maximum value and minimum value on $A$.

Define the Lagrange function

$$Q(x,y,u,v,z,\lambda,\mu)=f(x,y,u,v,z)+\lambda g(x,y,u,v,z)+\mu h(x,y,u,v,z). \tag{2.6}$$

We will use the method of Lagrange multipliers to solve all stationary points of $f(x,y,u,v,z)$ on $A$.

Because of

$$Q_z = -\pi(3n)\lambda = 0,$$

so

$$\lambda = 0.$$

From

$$\begin{cases} Q_{x_i} = 2\lambda x_i + 2\mu x_i + \mu = 0, \\ Q_{y_j} = 2\lambda y_j + 2\mu y_j + \mu = 0, \\ Q_{u_i} = 2u_i + 1 + 2\lambda u_i + 2\mu u_i = 0, \\ Q_{v_s} = 2\lambda v_s + 2\mu v_s + \mu = 0, \end{cases}$$

it follows by combining $\lambda = 0$ that

$$\begin{cases} \mu(2x_i + 1) = 0, \\ \mu(y_j + 1) = 0, \\ (2 + 2\mu)u_i = -1, \\ \mu(2v_s + 1) = 0. \end{cases} \tag{2.7}$$

If $\mu = 0$, then we have from (2.7) that

$$x_i, y_j, v_s \text{ are arbitrary, } u_i = -\tfrac{1}{2},$$

then

$$f = \left(\frac{1}{4} - \frac{1}{2}\right) l < 0,$$

and

$$f_{\max} = \left(\frac{1}{4} - \frac{1}{2}\right) l < 0,$$

hence $f(P) \leq f_{\max} < 0$, but it contradicts to $f(P) \geq 0$.

If $\mu \neq 0$ (noting $\mu \neq -1$, otherwise, it has $0 \cdot u_i = -1$ from $(2+2\mu)u_i = -1$ in (2.7), a contradiction), then

$$x_i = -\frac{1}{2}, y_j = -\frac{1}{2}, u_i = -\frac{1}{2+2\mu}, v_s = -\frac{1}{2}.$$

Using

$$0 = h(x, y, u, v, z) = -\frac{l}{4} + \frac{l}{(2+2\mu)^2} - \frac{l_1}{4} - \frac{l_2}{4} - (\pi(3n) + l + l_1 + l_2),$$

we see

$$\frac{l}{(2+2\mu)^2} = \frac{5(l + l_1 + l_2)}{4} + \pi(3n),$$

and then

$$\frac{1}{2+2\mu} = \pm\sqrt{\frac{\pi(3n)}{l} + \frac{5(l + l_1 + l_2)}{4l}},$$

so

$$u_i = \mp\sqrt{\frac{\pi(3n)}{l} + \frac{5(l+l_1+l_2)}{4l}}.$$

Since $\pi(3n) > l$, we have

$$f_{\min} = l\left(\frac{\pi(3n)}{l} + \frac{5(l+l_1+l_2)}{4l} - \sqrt{\frac{\pi(3n)}{l} + \frac{5(l+l_1+l_2)}{4l}}\right) > 0.$$

Then for $P \in A$, it yields

$$f(P) = \sum_{i=1}^{l_1} \big(\delta(2n-p_i)^2 + \delta(2n-p_i)\big) \geq f_{\min} > 0,$$

so there exists $p_i$, such that

$$\delta(2n+p_i) = 1,$$

namely, $2n + p_i$ is an odd prime between $2n$ and $3n$. Theorem 1.1 is proved.

**Conflicts of Interest.** The authors declare that there is no conflict of interest.

**Data Availability.** Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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