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On Locally n-Open Sets and Locally n-Continuous Functions

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Abstract

The concept of a locally n-open set is introduced. The basic properties of these sets are established. Relationships between this class of sets and other classes of generalized open sets are developed. It is shown that the class of locally n-open sets properly contains the class of n-open sets and is closely related to the class of generalized n-closed sets. The notion of a locally n-continuous function is developed. The basic properties of these functions are investigated.

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Keywords: n-open, locally n-open, un-closed, gn-closed, n-continuous, locally n-continuous, contra gn-continuous

1 Introduction

The concept of a Q-set (a subset of a topological space on which the closure and interior operators commute) was investigated by Levine [5] in 1961. Recently the notion of a locally Q-set was studied by Gowri [4] in 2019. The concept of an n-open set (a set on which the closure and interior operators are not equal) was introduced by Baker [1] in 2021. It follows from the definitions that, if a set is not a Q-set, then the set is n-open. However, the converse implication does not hold. In this note we continue this line of inquiry by introducing the notion of a locally n-open set. Since the space itself is not locally n-open, the locally n-open sets do not form a minimal structure. The basic properties and relationships of these sets are investigated. For example, it is shown that this collection of sets is closed under intersection but not union. Additionally it is established that the locally n-open sets are closely related to the generalized n-closed sets and coincide with the union-n-closed sets. The notion of a locally n-continuous function is defined. It is established that local n-continuity is not necessarily preserved under either composition or restriction. Additionally it is shown that, if the domain is not discrete, then local-n continuity is strictly between n-continuity and contra generalized n-continuity.

2 Preliminaries

Unless otherwise stated, the symbols X, Y, and Z represent topological spaces (briefly spaces) with no separation properties assumed. All topological spaces are assumed to be nonempty. The closure and interior of a set A are signified by $Cl(A)$ and $Int(A)$, respectively.

Definition 2.1 *Let X be a nonempty set and $\mathcal{P}(X)$ the power set of X. A subfamily m_X of $\mathcal{P}(X)$ is called a minimal structure (briefly an m-structure) on X* [6]*, if $\emptyset \in m_X$ and $X \in m_X$.*

Definition 2.2 *A subset A of a space X is said to be n-open* [1] *if $Int(A) \neq Cl(A)$. A subset of X is called n-closed if its complement is n-open.*

Theorem 2.3 [1] *If A is a subset of a space X, then*

(a) *A is n-open if and only if A is not clopen.*

(b) *A is n-open if and only if $X - A$ is n-open.*

Thus the n-open sets coincide with the n-closed sets.

Remark 2.4 *Nether X nor $\emptyset$ is n-open. Therefore the collection of n-open sets does not form a minimal structure.*

Theorem 2.5 [1] *If a space X is not discrete, then for every $x \in X$ there exists an n-open set containing x.*

Remark 2.6 *A space is discrete if and only if there are no n-open sets.*

Definition 2.7 *Let A be a subset of a space X. The n-interior of A* [1] *is denoted by $nInt(A)$ and given by $nInt(A) = \cup\{U \subseteq X : U \subseteq A$ and U is n-open$\}$. The n-closure of A* [1] *is denoted by $nCl(A)$ and given by $nCl(A) = \cap\{F \subseteq X : A \subseteq F$ and $F$is n-closed$\}$.*

Theorem 2.8 [1] *If A is a subset of a space X, then*

(a) $nInt(X - A) = X - nCl(A)$.

(b) $nCl(X - A) = X - nInt(A)$.

(c) $x \in nCl(A)$ *if and only if* $U \cap A \neq \emptyset$ *for every n-open set* U *containing* x.

Theorem 2.9 [1] *If X is a space, then*

(a) $nCl(X) = X$.

(b) $nInt(\emptyset) = \emptyset$.

Theorem 2.10 [1] *If a space X is not discrete, then*

(a) $nInt(X) = X$.

(b) $nCl(\emptyset) = \emptyset$.

Theorem 2.11 [1] *If a space X is discrete, then*

(a) $nInt(A) = \emptyset$ *for every set* $A \subseteq X$.

(b) $nCl(A) = X$ *for every set* $A \subseteq X$.

Theorem 2.12 [2] *If A is a subset of a space X, then*

(a) $nCl(A) = A$ *or* $nCl(A) = X$.

(b) $nInt(A) = A$ *or* $nInt(A) = \emptyset$.

Definition 2.13 *A subset A of a space X is said to be generalized n-closed (briefly gn-closed)* [2], *if whenever $A \subseteq U$ and U is open, then $nCl(A) \subseteq U$. A subset of X is called generalized n-open (briefly gn-open) if its complement is gn-closed.*

Theorem 2.14 [2] *If A is a subset of a space X, then*

(a) *A is gn-closed if and only if* $nCl(A) = A$.

(b) *A is gn-open if and only if* $nInt(A) = A$.

(c) $nCl(A)$ *is gn-closed.*

(d) $nInt(A)$ *is gn-open.*

Definition 2.15 *A function $f : X \to Y$ is said to be n-continuous* [1] *if $f^{-1}(V)$ is n-open in X for every proper nonempty open set $V \subseteq Y$.*

Definition 2.16 *A function $f : X \to Y$ is said to be generalized n-continuous (briefly gn-continuous)* [2] *if $f^{-1}(F)$ is gn-closed in X for every closed set $F \subseteq Y$.*

Definition 2.17 *A subset A of a space X is said to be union-n-open (briefly un-open)* [3], *if A is the union of a nonempty collection of n-open sets.*

3 Locally n-open sets

Definition 3.1 *A subset A of a space X is said to be locally n-open if there exist an n-open set B and a open set U such that $A = B \cap U$.*

Theorem 3.2 *Every n-open set is locally n-open.*

Proof. Let A be an n-open subset of a space X. Since $A = A \cap X$, A is locally n-open.

Example 3.3 *Let $X = \{a, b, c\}$ have the topology $\tau = \{X, \emptyset, \{a, b\}, \{c\}\}$. The n-open sets are $\{a\}, \{b\}, \{a, c\}$, and $\{b, c\}$. Since $\{c\} = \{a, c\} \cap \{c\}$ and $\{a, c\}$ is n-open, $\{c\}$ is locally n-open. However $\{c\}$ is clopen and hence not n-open. Also $\{a, b\}$ is open but not locally n-open, since it is not contained in an n-open set, and $\{a\}$ is n-open hence locally n-open but not open. Therefore locally n-open does not imply n-open and locally n-open is independent of open. Also $\{a\}$ and $\{b\}$ are locally n-open, but their union is not locally n-open.*

If X is discrete, there are no n-open sets and hence no locally n-open sets. If X is not discrete, then by Theorem 2.5 there exist an n-open set A. Since $\emptyset = A \cap \emptyset$, the $\emptyset$ is locally n-open.

Theorem 3.4 *If A is a subset of a space X, then A is locally n-open if and only if there exists an n-open set B such that $A \subseteq B$.*

Proof. If A is locally n-open, then it follows from the definition that A is contained an n-open set.

Assume there exist an n-open set B such that $A \subseteq B$. If A is open, then, since $A = B \cap A$, A is locally n-open. If A is not open then A is not clopen and hence A is n-open and therefore also locally n-open.

Corollary 3.5 *Every subset of a locally n-open set is locally n-open.*

Corollary 3.6 *The locally n-open sets are closed under intersection.*

Corollary 3.7 *Let A be a subset of a space X. The set A is locally n-open if and only if $nCl(A) \neq X$.*

Proof. Recall that the n-open sets coincide with the n-closed sets. Therefore $nCl(A) \neq X$ if and only if there exists an n-closed set B such that $A \subseteq B$ if and only if there exists an n-open set B such that $A \subseteq B$ if and only if A is locally n-open.

The following result is a consequence if Theorem 2.12(a).

Corollary 3.8 *Let A be a subset of a space X. The set A is locally n-open if and only if $A \neq X$ and $nCl(A) = A$.*

The next corollary is follows from Theorem 2.14(a).

Corollary 3.9 *Let A be a subset of a space X. The set A is locally n-open if and only if $A \neq X$ and A is gn-closed.*

Lemma 3.10 *Let A be a subset of space X. Then A is un-closed if and only if A is the intersection of a nonempty collection of n-closed sets.*

Theorem 3.11 *A set is locally n-open if and only if it is un-closed.*

Proof. Assume A is un-closed. Since A is the intersection of a collection of n-closed sets, A is also the intersection of a collection of locally n-open sets and hence A is locally n-open.

Assume A is locally n-open. By Corollary 3.8 $A \neq X$ and $nCl(A) = A$. Thus it follows from the definition of the n-closure of a set that A is the intersection of a nonempty collection of n-closed sets and hence by Lemma 3.10 A is un-closed.

4 Locally n-continuous functions

Definition 4.1 *A function $f : X \to Y$ is said to be locally n-continuous if $f^{-1}(V)$ is locally n-open for every proper open set $V \subseteq Y$.*

Definition 4.2 *A function $f : X \to Y$ is said to be contra gn-continuous if $f^{-1}(V)$ is gn-closed for every open set $V \subseteq Y$.*

Theorem 4.3 *If X is not discrete and $f : X \to Y$ is n-continuous, then f is locally n-continuous.*

Proof. Assume $f : X \to Y$ is n-continuous. Let $V \subseteq Y$ be a proper open set. If $V = \emptyset$, then $f^{-1}(V) = \emptyset$, which, since X is not discrete, is locally n-open. If $V \neq \emptyset$, then V is a proper nonempty set and, since f is n-continuous, then $f^{-1}(V)$ is n-open and hence locally n-open. Therefore f is locally n-continuous.

Theorem 4.4 *If $f : X \to Y$ is locally n-continuous, then f is contra gn-continuous.*

Proof. Assume $f : X \to Y$ is locally n-continuous. Let $V \subseteq Y$ be an open set. If $V = Y$, then $f^{-1}(V) = X$ which is gn-closed. If $V \neq Y$, then V is a proper set and, since f is locally n-continuous, then $f^{-1}(V)$ is locally n-open and hence by Corollary 3.9 $f^{-1}(V)$ is gn-closed. Thus f is contra gn-continuous.

Example 4.5 *Let $X = \{a, b, c\}$ have the topologies $\tau = \{X, \emptyset, \{a, b\}, \{c\}\}$ and $\sigma = \{X, \emptyset, \{c\}\}$ and let $f : (X, \tau) \to (Y, \sigma)$ be the identity mapping. Since $f^{-1}(\{c\})$ is locally n-open but not n-open (see Example 3.3), f is locally n-continuous but not n-continuous.*

Example 4.6 *Let $X = \{a, b, c\}$ have the topology $\tau = \{X, \emptyset, \{c\}\}$ and let $f : (X, \tau) \to (Y, \tau)$ be the constant mapping given by $f(x) = c$ for every $x \in X$. Since $f^{-1}(\{c\}) = X$ which is gn-closed but not locally n-open, f is contra gn-continuous but not locally n-continuous.*

As the following example shows, the restriction of a locally n-continuous function is not necessarily locally n-continuous, even when restricted to an n-open set.

Example 4.7 *If $X = \{a, b, c\}$ has the topologies $\tau = \{X, \emptyset, \{a, b\}, \{c\}\}$ and $\sigma = \{X, \emptyset, \{a, c\}\}$, then the identity map $f : (X, \tau) \to (X, \sigma)$ is locally n-continuous, but, if $S = \{b, c\}$, then $f|_S : (S, \tau_S) \to (X, \sigma)$ is not locally n-continuous. Note that, since the subspace topology τ_S is discrete, there is no locally n-continuous function from (S, τ_S) to (X, σ).*

From the following example, we see that the composition of locally n-continuous functions is not necessarily locally n-continuous.

Example 4.8 *Let $X = \{a, b, c\}$ have the topologies $\tau = \{X, \emptyset, \{a, b\}, \{c\}\}$, $\sigma = \{X, \emptyset, \{a\}\}$, and $\delta = \{X, \emptyset, \{a, b\}\}$. Then the identity mappings $f : (X, \tau) \to (X, \sigma)$ and $g : (X, \sigma) \to (X, \delta)$ are locally n-continuous, but $g \circ f$ is not locally n-continuous. Note that $(g \circ f)^{-1}(\{a, b\})$ is not locally n-open in (X, τ).*

Theorem 4.9 *If $f : X \to Y$ is locally n-continuous and $g : Y \to Z$ is continuous and $g^{-1}(V) \neq Y$ for every proper open set $V \subseteq Z$, then $g \circ f$ is locally n-continuous.*

Corollary 4.10 *Let $f_\alpha : X \to Y_\alpha$ be a function for every $\alpha \in \Lambda$. If the product function $f : X \to \prod_{\alpha\in\Lambda} Y_\alpha$, given by $f(x) = (f_\alpha(x))_\alpha$, is locally n-continuous, then f_α is locally n-continuous for every $\alpha \in \Lambda$.*

Proof. Since $f_\alpha = p_\alpha \circ f$ for every $\alpha \in \Lambda$, where p_α is the projection onto Y_α, the desired result follows from Theorem 4.9.

Definition 4.11 *A function $f : X \to Y$ is said to be locally n-open if $f(U)$ is open for every locally n-open set $U \subseteq X$.*

Theorem 4.12 *Let $f : X \to Y$ and $g : Y \to Z$ be functions. If f is locally n-continuous and locally n-open and surjective and $g^{-1}(V) \neq Y$ for every proper open set $V \subseteq Z$, then $g \circ f$ is locally n-continuous if and only if g is continuous.*

Proof. The necessity follows from Theorem 4.9. For the sufficiency assume $g \circ f$ is locally n-continuous and let V be a proper open subset of Y. Then $f^{-1}(g^{-1}(V))$ is locally n-open and, since f is surjective and locally n-open, $g^{-1}(V) = f(f^{-1}(g^{-1}(V)))$ is open. Therefore g is continuous.

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