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Estimation of Parameters Using an Updated Vector Autoregressive Model

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Abstract

Many statistical models, be it deterministic or stochastic, usually contain a number of parameters that make up the model(s). Ordinarily, the maximum likelihood estimation (MLE) and least squares estimation (LSE) methods are the most applied methods of estimation. However, in the two approaches, the main focus is on the estimation of the parameters since parameter estimation is a key step that cannot be avoided as far as modelling or model building is concerned. In this paper, parameter estimation in the updated vector autoregressive model is shown. We consider estimation of the parameters by use of the dual estimation approach, precisely using joint estimation which can estimate both the state and the parameters, applied to some VAR models in one dimension and in two dimension. From the results, it is observed that there is convergence of the parameters to the true parameter values as time evolves.

Keywords: Joint estimation, Parameter estimation, Algorithm, convergence

1 Introduction

Many statistical models, be it deterministic or stochastic, usually contain a number of parameters that make up the model(s). In fact, parameter estimation is a key step that cannot be avoided as far as modeling or model building is concerned. Commonly, the mostly applied parameter estimation methods include the maximum likelihood estimation, Bayesian estimation and the least squares methods among others. For instance, the maximum likelihood method is usually applied to a density function $f(X|\theta)$ which depends on a set of parameters θ and data set $x_1, x_2, \cdots, x_n$ that are independent and identically distributed (i.i.d), [7][13]. The likelihood function is given by

$$L(\theta|X) = \prod_{i=1}^{n} fX|\theta$$

which is then maximised though mostly the log-likelihood function given by

$$l(\theta) = lnL(\theta|X)$$

is used, [7][3][9]. However, this work attends not to these estimation procedures but by considering the use of an algorithm in the estimation of parameters in the updated VAR models. In this work, we consider estimation of the parameters by dual estimation approach using the updated vector autoregressive model. The approach, dual estimation, involves the estimation of the state and the parameters simultaneously, [2][4]. However, dual estimation can be done in two ways namely: joint estimation and dual filter whereby joint estimation requires only one filter whereas dual filter requires two filters. Joint estimation is advantageous over the dual estimation in that the former allows for dependencies in parameters and states while the later assumes no autocorrelation, that is the cross covariances are zero, [4]. In this paper, joint estimation is considered.

The rest of the paper is arranged as follows: parameter estimation is discussed in section 2, model implementation with examples in section 3 and then conclusions are given in section 4.

2 Parameter Estimation

In state-space models, estimation of parameters and the state is key. Some researches have been done on the estimation of the parameters excluding the state while others have tried to estimate both the parameters and the states after deriving their algorithm. For instance, Kantas *et al* [8] did a study on comprehensive review on particle methods proposed to perform static parameter estimation in state space models. The work mainly focuses on the estimation of the parameters by use of algorithms on particle filter and fails to capture estimation of the state which is critical as far as state-space models are concerned. In addition [1] did a research about on-line parameter estimation in non-linear non-Gaussian state-space models with the aim to estimate static parameters by point estimation.

Ruifeng and Linfan [11], did a study on parameter and state estimation for state space models. In the work, a least squares parameter identification algorithm is derived which is then used to estimate the parameters. Thereafter, the estimated parameters are then used to compute the system states by incorporating input-output data. Indeed, it can be noted that the estimation of parameters and states in state space models has attracted interest for most researchers based on the algorithm derived depending on the nature of the state space model as seen in other works such as [5][10][6][12].

Dual estimation, as indicated earlier on, involves estimating the states and the parameters simultaneously. It is further classified into dual estimation and the joint estimation. Joint estimation involves augmenting the state vector with vector of parameters to form an extended state-space and then the algorithm is run forward in time to update both the state and the parameters with expectations of the algorithm converging to the optimal state and parameter values.
Consider the vector autoregressive model given by

$$Y_t = A_1 Y_{t-1} + \cdots + A_p Y_{t-p} + \beta_t \tag{1}$$

where $\beta_t \sim \mathcal{WN}(0, Q)$. When new information is obtained which is given by

$$X_t = P_t Y_t + \alpha_t \tag{2}$$

where $\alpha_t \sim \mathcal{WN}(0, R)$, equation 1 can be updated using the Bayesian approach. The model in equation 1 is treated as the prior, new information as the likelihood and the update is the posterior. This is done in two steps, namely, the prediction and the update steps. In the prediction step the state, which is represented by equation 1, and the covariance are predicted. The predicted state and the covariance are given by

$$\hat{Y}_{t|t-1} = A_{1,t-1}\hat{Y}_{t|t-1} + \cdots + A_{p,t-p}\hat{Y}_{t|t-p} \tag{3}$$

and

$$\hat{S}_{t|t-1} = A_{1,t-1}S_{t-1}A_{1,t-1}^T + \cdots + A_{p,t-p}S_{t-p}A_{p,t-p}^T + Q \tag{4}$$

respectively. After new information is received, then the update is done. The updated state and covariance are given by

$$\hat{Y}_{t|t} = A_{1,t-1}\hat{Y}_{t|t-1} + K_t\left(X_t - P_t\hat{Y}_{t|t-1}\right) \tag{5}$$

and

$$\hat{S}_{t|t} = S_{t|t-1} - K_t P_t S_{t|t-1} \tag{6}$$

Algorithm 1 Algorithm for updated vector autoregressive model

1: Predict the state: $\hat{Y}_{t|t-1} = A_{1,t-1}\hat{Y}_{t|t-1} + \cdots + A_{p,t-p}\hat{Y}_{t|t-p}$
2: Predict the error covariance: $\hat{S}_{t|t-1} = A_{1,t-1}S_{t-1}A^T_{1,t-1} + \cdots + A_{p,t-p}S_{t-p}A^T_{p,t-p} + Q$
3: Compute the gain: $K_t = \frac{S_{t|t-1}P_t^T}{P_t S_{t|t-1}P_t^T + R}$
4: Update the state: $\hat{Y}_{t|t} = A_{1,t-1}\hat{Y}_{t|t-1} + K_t\left(X_t - P_t\hat{Y}_{t|t-1}\right)$
5: Update the error covariance: $\hat{S}_{t|t} = S_{t|t-1} - K_t P_t S_{t|t-1}$

respectively where $K_t = \frac{S_{t|t-1}P_t^T}{P_t S_{t|t-1}P_t^T + R}$ is the gain. This steps are summarised as seen in algorithm 1.
Now, to estimate the parameters and the states, let

$$z_t = \varphi_t \tag{7}$$

where for VAR(1)

$$z_t = \begin{pmatrix} Y_t \\ \theta_t \end{pmatrix} \qquad \textit{and} \qquad \varphi_t = \begin{pmatrix} A_1 Y_{t-1} + \beta_t \\ \theta_{t-1} \end{pmatrix}$$

On the other hand, for the VAR(p) model

$$z_t = \begin{pmatrix} Y_t \\ \theta_t \end{pmatrix} \qquad \textit{and} \qquad \varphi_t = \begin{pmatrix} A_1 Y_{t-1} + A_2 Y_{t-2} + \cdots + A_p Y_{t-p} + \beta_t \\ \theta_{t-1} \end{pmatrix}$$

Algorithm 1 is then applied on the joint state space system given in Equation 7 to estimate the states and the parameters simultaneously. As the estimation is done, it should be noted that we assume that the parameters are time-invariant, that is, they are static.

3 Model Implementation with Examples

We consider the model given by

$$\begin{aligned} Y_t = A_1 Y_{t-1} + \cdots + A_p Y_{t-p} + \beta_t \\ X_t = P_t Y_t + \alpha_t \end{aligned} \tag{8}$$

where, first, we assume it is in scalar form and proceed to estimate the parameters through joint estimation. We assume the initial state, Y_0, and the values of $A_1, \cdots, A_p$, P_t, u_t and η_t are given and then run the algorithm to investigate if it converges to the true parameter value. Suppose that we have the model given by

$$\begin{aligned} Y_t &= A_1 Y_{t-1} + \beta_t \\ X_t &= P_t Y_t + \alpha_t \end{aligned} \tag{9}$$

We set $A_1 = -0.2$ so that the model in equation (9) becomes

$$\begin{aligned} Y_t &= -0.2 Y_{t-1} + \beta_t \\ X_t &= Y_t + \alpha_t \end{aligned} \tag{10}$$

where in equation 10, $A_1 = a = -0.2$. Using algorithm 1, we proceed to estimate parameter a. Using MATLAB, where setting state covariance, $Q = 0.01$ and measurement covariance, $R = 0.001$ in algorithm 1, we have the panels as given in Figure 1 which gives the estimates of the parameter a over time plus its corresponding Box-plot for 10000 iterations. From Figure 1 it can be observed that the algorithm yields converging results to the true parameter value as time evolves which are 0.2 but with some margin of error. In addition, the accompanying Box-plot for parameter a in Figure 1 shows the dispersion in the results with some outliers present on both the lower and upper sides of the Box-plot, with most of them on the lower side than on the upper side. Next, suppose that we have the model given by

$$\begin{aligned} Y_t &= A_1 Y_{t-1} + A_2 Y_{t-2} + \beta_t \\ X_t &= P_t Y_t + \alpha_t \end{aligned} \tag{11}$$

Suppose we set $A_1 = -0.2$ and $A_2 = 0.2$. Then the model can be written as

$$\begin{aligned} Y_t &= -0.2 Y_{t-1} + 0.2 Y_{t-2} + \beta_t \\ X_t &= Y_t + \alpha_t \end{aligned} \tag{12}$$

where in equation 12, $A_1 = a = -0.2$ and $A_2 = b = 0.2$. Running algorithm 1, we estimate the parameters a and b. Setting state covariance $Q = 0.01$, measurement covariance $R = 0.001$ and number of iterations to be 30,000 in algorithm 1 in MATLAB, we have the panels as given in Figure 2 which gives the estimates of the parameters a and b over time plus their corresponding Box-plots. From Figure 2 it is clear that the algorithm yields converging results to the true parameter values as time evolves which are -0.2 for a and 0.2 for b but with some margin of error. The accompanying Box-plots for parameters a and b in Figure 2 shows the dispersion in the results and it can be observed that there are few outliers present which appear on the lower side of the Box-plots for both parameters a and b.

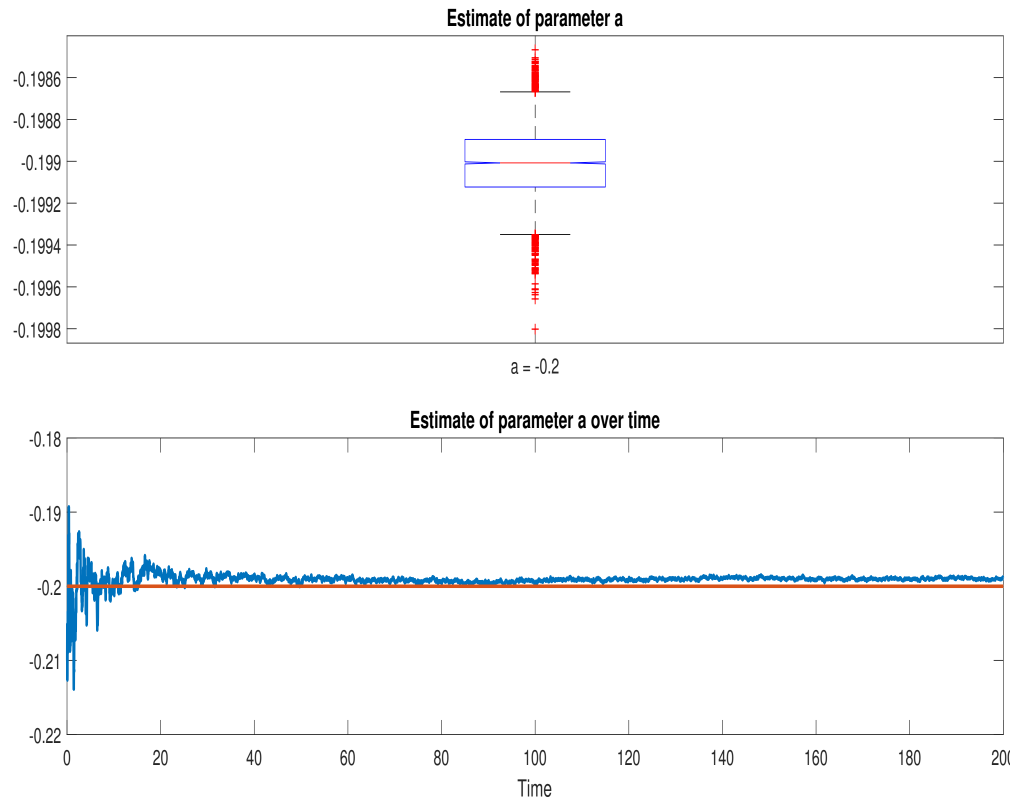

Figure 1: Parameter Estimation in AR(1)

Subplot one gives the corresponding box plot showing the dispersion while the second subplot shows the convergence of the parameter a to the true parameter value -0.2 as time evolves.

Next, we estimate the parameters in the two dimension model. Consider the model given by

$$\begin{aligned} \begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} &= \begin{pmatrix} -0.2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_{1,t-1} \\ y_{2,t-1} \end{pmatrix} + \begin{pmatrix} \beta_{1,t} \\ \beta_{2,t} \end{pmatrix} \\ \begin{pmatrix} x_{1,t} \\ x_{2,t} \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} + \begin{pmatrix} \alpha_{1,t} \\ \alpha_{2,t} \end{pmatrix} \end{aligned} \tag{13}$$

Using algorithm 1, we estimate the parameters, which are elements of matrix $a = \begin{pmatrix} -0.2 & 0 \\ 0 & 0 \end{pmatrix}$ where $a_{11} = -0.2$, $a_{12} = 0$, $a_{21} = 0$ and $a_{22} = 0$. If the the state and measurement covariance matrices, in algorithm 1 are set to be $Q = \begin{pmatrix} 0.001 & 0 \\ 0 & 0.001 \end{pmatrix}$, $R = \begin{pmatrix} 0.0101 & 0 \\ 0 & 0.0101 \end{pmatrix}$ respectively and having 32,000 iterations and using MATLAB, we have the panels as given in Figure 3 which

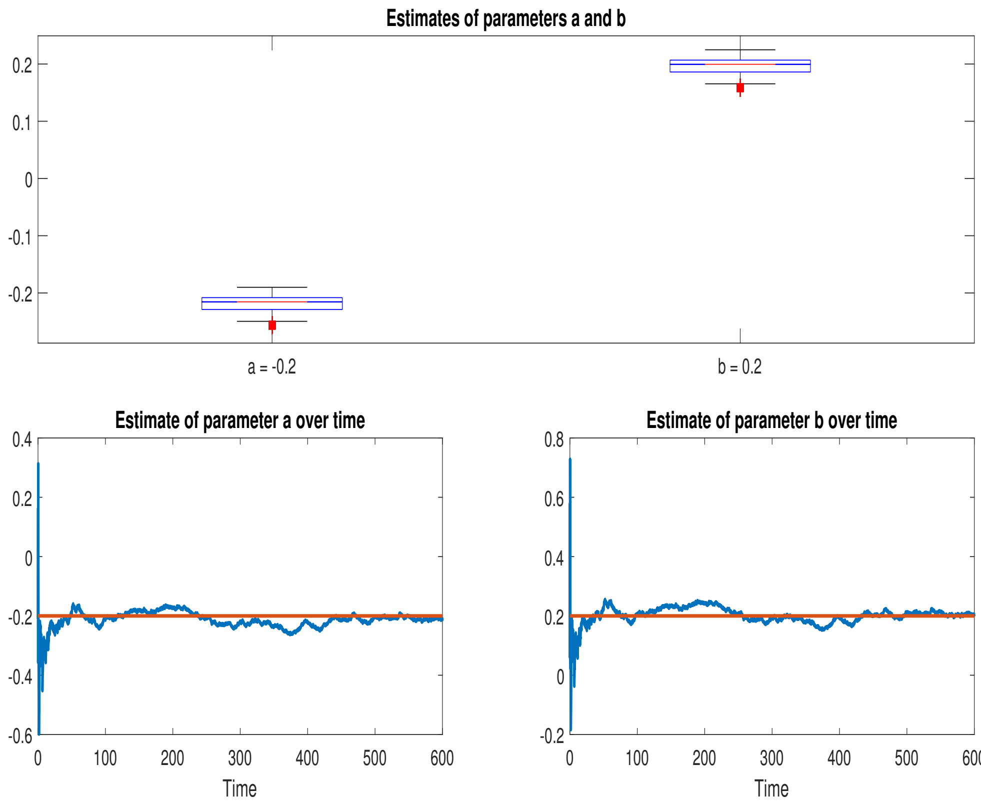

Figure 2: Parameter Estimation in AR(2)
The first subplot gives the corresponding box plots showing the dispersion in both parameters a and b, while the second subplot shows the convergence of the parameters a and b to the true parameter values -0.2 and 0.2, respectively as time evolves.

gives the estimate of the parameter a_{11} over time plus its corresponding Box-plot. From Figure 3 it is evident that the algorithm yields converging results to the true parameter value as time evolves which are -0.2. However, there is some margin of error present in the convergence. The accompanying Box-plot for parameter a_{11} in Figure 3 represents the dispersion in the results with some outliers present on both the lower and upper sides of the Box-plot. However, the outliers appear more on the lower side as compared to the upper side.

4 Conclusion

This paper intended to illustrate parameter estimation by joint estimation which can estimate both the state and the parameters using the updated vector

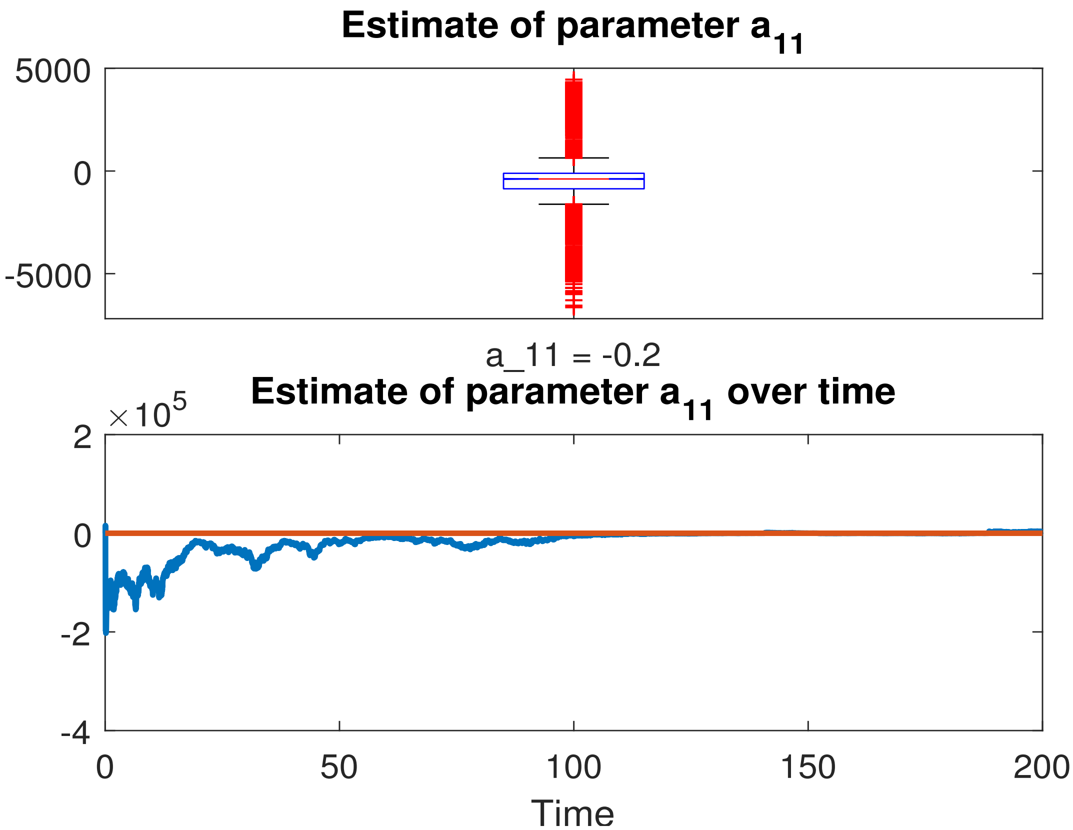

Figure 3: Parameter Estimation in VAR(1)
Subplot one shows the corresponding box plot showing the dispersion while the second subplot shows the convergence of the parameter to the true parameter value as time evolves.

autoregressive model. The state and the parameters were augmented together to form an extended state space after which the developed algorithm was run forward in time to estimate both the state and the parameters. From the results it is observed that their is convergence of parameters to the true parameter values as time evolves, though with some margin of error.

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