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Researching the Ranks in Elliptic Curves of the Forms $y^2 = x^3 \pm Ax$

Shin-Wook Kim

Deokjin-gu, Songcheon 54823
101-703, I-Park Apt
Jeonju, Jeonbuk, Korea

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Abstract

Assign $E_{\mp p}$ as elliptic curves $y^2 = x^3 \mp px$ then, we will treat the ranks of these curves. Assume that E_{-2p} is an elliptic curve $y^2 = x^3 - 2px$ then, we shall compute the ranks of it and compare the results with that of $y^2 = x^3 \mp px$.

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1 Introduction

In [6], the author verified that rank of $y^2 = x^3 - px$ is 1 when prime is such that $p = 200u^4 + 240u^2v^2 + 71v^4\ w.i.u.v.1$ and $p \equiv 15(mod\ 16)$ and the case $p = 6562u^4 + 10u^2v^2 + 25v^4\ w.i.u.v.1$ and $p \equiv 5(mod\ 16)$ and the case $p = 2u^4 + 52u^2v^2 + 337v^4\ w.i.u.v.1$ and $p \equiv 7(mod\ 16)$ and the case $p = 2u^4 + 20u^2v^2 + 49v^4\ w.i.u.v.1$ and $p \equiv 7(mod\ 16)$. In [7], the author showed that rank of curve $E_{-2p}: y^2 = x^3 - 2px$ is 1 where prime p is $p = 18t^4 + 8t^3u + 8tu^3 + 36t^2u^2 + 11u^4\ w.i.t.u.1$, $p \equiv 11(mod\ 16)$ and the case $p = 326t^4 - 8t^3u - 8tu^3 + 48t^2u^2 + 3u^4\ w.i.t.u.1$ and $p \equiv 3(mod\ 16)$ and the case $p = 66t^4 - 8t^3u - 8tu^3 + 60t^2u^2 + 11u^4\ w.i.t.u.1$, $p \equiv 11(mod\ 16)$. In [7], the author proved that rank of elliptic curve $y^2 = x^3 - 4px$ is 1 where prime is gotten as $p = 40t^4 + 16t^3u + 16tu^3 - 36t^2u^2 + 29u^4\ w.i.t.u.1$, $p \equiv 13(mod\ 16)$ and the case $p = 68t^4 + 16t^3u + 16tu^3 + 200t^2u^2 + 125u^4\ w.i.t.u.1$, $p \equiv 13(mod\ 16)$. In this paper, we will regard the rank of curve $E_{-2p}: y^2 = x^3 - 2px$.

In section 2, the ranks of $E_{\mp p}$ will be considered.

In section 3, we shall calculate the rank of E_{-2p} and compare the result with that of $E_{\mp p}$.

In section 4, the examples of results in $E_{\mp p}$ and E_{-2p} will be suggested.

First, we should note several notations in [9].

Define E as an elliptic curve $y^2 = x^3 + ax^2 + bx$ and Γ as the set of rational points on E. Then, Γ is a finitely generated abelian group.

And we have $\Gamma \cong E(Q)_{tors} \oplus Z^r$ with torsion subgroup $E(Q)_{tors}$ and Mordell-Weil rank r.

Take $Q^{\times}$ as multiplicative group whose components are non-zero rational numbers. Assume that $Q^{\times 2}$ is the subgroup of squares of elements of $Q^{\times}$.

Denote a homomorphism α as $\alpha: \Gamma \to Q^{\times}/Q^{\times 2}$ that satisfies $\alpha(O) = 1(mod\ Q^{\times 2})$ and $\alpha(0,\ 0) = b(mod\ Q^{\times 2})$ and $\alpha(x,\ y) = x(mod\ Q^{\times 2})$. Here O denotes infinity point and x is non-zero.

Suppose that $\bar{E}$ is the curve $y^2 = x(x^2 - 2ax + a^2 - 4b)$. Denote $\bar{\Gamma}$ as the set of rational points on curve $\bar{E}$.

Define a homomorphism $\bar{\alpha}$ as $\bar{\alpha}: \bar{\Gamma} \to Q^{\times}/Q^{\times 2}$ with $\bar{\alpha}(O) = 1(mod\ Q^{\times 2})$ and $\bar{\alpha}(0,\ 0) = a^2 - 4b(mod\ Q^{\times 2})$ and $\bar{\alpha}(x,\ y) = x(mod\ Q^{\times 2})$. Take O as infinity point and $x \neq 0$.

Take the relating equation for Γ as $N^2 = b_1M^4 + aM^2e^2 + b_2e^4$ that satisfies $b = b_1b_2$ with $b_1 \not\equiv 1, b(mod\ Q^{\times 2})$.

Let $N^2 = b_1M^4 - 2aM^2e^2 + b_2e^4$ be relating equation for $\bar{\Gamma}$ with $b_1b_2 = a^2 - 4b$ with $b_1 \not\equiv 1, a^2 - 4b(mod\ Q^{\times 2})$.

Suppose that $(M, e\ N)$ is an integral solution of above two equations with $1 = (M, N) = (M, e) = (N, e) = (b_1, e) = (b_2, M)$ and $M \neq 0, e \neq 0$.

There educed $2^r = \frac{\#\alpha(\Gamma)\#\bar{\alpha}(\Gamma)}{4}$ with rank r of E.

We take notations as follows:

$w.i.s.t.u.1$: with integers s and t and u and $(s, t, u) = 1$.

$w.i.u.v.1$: with integers u and v and $(u, v) = 1$ ([5]).

$r4.2$: $2^r = \frac{4\cdot 2}{4}$([5]).

$r2.4$: $2^r = \frac{2\cdot 4}{4}$([5]).

2 In $E_{\mp p}$

In this section, we will compute the ranks of curves $E_{\mp p}: y^2 = x^3 \mp px$.

Lemma 2.1. Let prime p be the form $p = 1002002u^4 + 18018u^2v^2 + 81v^4\, w.i.u.v.1$ and $p \equiv 5(mod\, 16)$ in curve E_{-p} then, we have that $rank(E_{-(1002002u^4+18018u^2v^2+81v^4)}(Q)) = 1$.

Proof. Due to [4], we must check the solvability of equation

$$2)N^2 = -M^4 + (1002002u^4 + 18018u^2v^2 + 81v^4)e^4 \text{ for } \Gamma.$$

There is $81v^4$ in coefficient of e^4, hence we can expect an inducement of square in resultant.

Next, we must note that $18018u^2v^2$.

It is given as $2 \cdot 1001 \cdot 9u^2v^2$.

Thus, there ought to be shown the square

$$1002001u^4.$$

In numerical value

$$-M^4 + 1002002u^4e^4$$

we take e as 1 then, there must be established that

$$-M^4 + 1002002u^4 = 1002001u^4.$$

So the value M is educed as u.

Besides, from the computation

$$-u^4 + 1002002u^4 + 18018u^2v^2 + 81v^4$$

$$= 1002001u^4 + 18018u^2v^2 + 81v^4$$

the triple $(u, 1, 1001u^2 + 9v^2)$ is deduced as the solution of equation.

Whence, we got the conclusions $\#\alpha(\Gamma) = 4$ and $r4.2$.

So we take the result $rank(E_{-(1002002u^4+18018u^2v^2+81v^4)}(Q)) = 1$. □

Lemma 2.2. If prime p is given as $p = 5u^4 + 96u^2v^2 + 576v^4\, w.i.u.v.1$ and $p \equiv 5(mod\, 16)$ in elliptic curve E_{-p} then, it is produced that $rank(E_{-(5u^4+96u^2v^2+576v^4)}(Q)) = 1$.

Proof. By [4], if we search the solution of

$$2)N^2 = -M^4 + (5u^4 + 96u^2v^2 + 576v^4)e^4 \text{ for } \Gamma$$

then, it is done the calculation of rank.
Replace u and 1 into M and e derives that

$$-u^4 + 5u^4 + 96u^2v^2 + 576v^4$$

$$= 4u^4 + 96u^2v^2 + 576v^4.$$

Thus, we get the solution as (u, 1, $2u^2 + 24v^2$).
It shows that $\#\alpha(\Gamma) = 4$ and $r4.2$.
Whence, the result $rank(E_{-(5u^4+96u^2v^2+576v^4)}(Q)) = 1$ is educed. □

Lemma 2.3. Suppose that prime p is the form $p = 6562u^4 + 50u^2v^2 + 625v^4$ $w.i.u.v.1$ and $p \equiv 5(mod\ 16)$ in elliptic curve E_{-p} then, we say that $rank(E_{-(6562u^4+50u^2v^2+625v^4)}(Q)) = 1$.

Proof. Because of [4], we only needed to look for the solution of equation for Γ:

$$2) N^2 = -M^4 + (6562u^4 + 50u^2v^2 + 625v^4)e^4.$$

There is a square $625v^4$ in coefficient of e^4.
The term $50u^2v^2$ is factored as

$$2 \cdot 25u^2v^2,$$

thus there must be deduced the square u^4.
Now we must notice arithmetical value

$$-M^4 + 6562u^4e^4.$$

Take e as 1 then, the relation

$$-M^4 + 6562u^4 = u^4$$

should be valid.
From above, we get that $M^4 = 6561u^4$, hence the integer M is given as $9u$.
Wherefore, the pair $(e, M) = (1, 9u)$ is derived.
Next from the computation

$$-(9u)^4 + 6562u^4 + 50u^2v^2 + 625v^4$$

$$= u^4 + 50u^2v^2 + 625v^4.$$

there comes that $N = u^2 + 25v^2$.

Resultantly, the triple $(9u, 1, u^2 + 25v^2)$ satisfies the solution.
And so we reach that

$$\#\alpha(\Gamma) = 4 \text{ and } r4.2.$$

Whence, $rank(E_{-(6562u^4+50u^2v^2+625v^4)}(Q)) = 1$ is derived. □

Lemma 2.4. Assign prime p as $p = 633u^4 + 8u^2v^2 + 2v^4\ w.i.u.v.1$ and $p \equiv 3(mod\ 16)$ in elliptic curve E_p then, $rank(E_{(633u^4+8u^2v^2+2v^4)}(Q)) = 1$ is gotten.

Proof. First take p as $p = 16k + 3$ with integer k then, relating equation for Γ is $1)N^2 = M^4 + pe^4$ and it is trivial that $\#\alpha(\Gamma) = 2$.
In the next step, the curve $\overline{E_p}$ is $y^2 = x^3 - 4(16k + 3)x$, thus there comes relating equations for $\bar{\Gamma}$ as

$$1)N^2 = M^4 - 4(16k + 3)e^4 \text{ and}$$

$$2)N^2 = -M^4 + 4(16k + 3)e^4 \text{ and}$$

$$3)N^2 = 2M^4 - 2(16k + 3)e^4 \text{ and}$$

$$4)N^2 = -2M^4 + 2(16k + 3)e^4 \text{ and}$$

$$5)N^2 = 4M^4 - (16k + 3)e^4 \text{ and}$$

$$6)N^2 = -4M^4 + (16k + 3)e^4.$$

We omit to say about solvability of 1), 5). For it refer to [3].
Cutting down on equations 2) and 3) by prime p gives that $2)N^2 \equiv -M^4(mod\ p)$ and $3)N^2 \equiv 2M^4(mod\ p)$ and there also given that 2) $\left(\frac{-M^4}{p}\right) = -1$ and 3) $\left(\frac{2M^4}{p}\right) = -1$ and these cannot exist simultaneously, thus a contradiction is educed in each case.
In modulo 4 in equation 6) yields that $1 \equiv N^2 \equiv 3e^4 \equiv 3(mod\ 4)$ and the sides are unmatched.
Finally, equation 4) is $N^2 = -2M^4 + 2(633u^4 + 8u^2v^2 + 2v^4)e^4$. To search the solution of this equation, treating the coefficient of e^4 is essential. In coefficient of e^4, there is a square $4v^4$ in it, hence one condition for being appeared the polynomial's square is gotten. Next, from the factorization $16u^2v^2 = 2 \cdot 4 \cdot 2u^2v^2$ there ought to be emerged the term $16u^4$. Namely, after substituting some integers into M and e the square $16u^4$ has to be appeared as a component of square. Now the left thing which must be examined is $-2M^4 + 1266u^4e^4$. Because our aim was $16u^4$ the relation $-2M^4 + 1266u^4e^4 =$

$16u^4 \cdots\cdots (YT)$ should be valid. Now determining the values M and e which satisfies this equality become significant matter. And it is sufficient that we only find one triple of integers. In equality $16u^4 = -2M^4 + 1266u^4e^4$, there is u^4 in two sides and in RHS it is in $1266u^4e^4$. Thus, we determine the integer e earlier. Suppose that $e = 1$ then, we confront to $-2M^4 + 1266u^4 = 16u^4$. Now from the computation $1266u^4 - 16u^4 = 2M^4$ the integer M is educed as $5u$. In other worlds, the pair $(M, e) = (5u, 1)$ satisfies the relation (YT). Moreover, because of calculation $-2(5u)^4 + 1266u^4 + 16u^2v^2 + 4v^4 = 16u^4 + 16u^2v^2 + 4v^4$ there comes that $N = 4u^2 + 2v^2$. Henceforth, the triple $(5u, 1, 4u^2 + 2v^2)$ satisfies the solution of equation.

For this reason, we reach that $\#\bar{\alpha}(\bar{\Gamma}) = 4$.

Thus, we get that $r2.4$.

Hence, the result $rank(E_{(633u^4+8u^2v^2+2v^4)}(Q)) = 1$ is educed. □

Lemma 2.5. Take prime p as $p = 72u^4 + 72u^2v^2 + 19v^4\ w.i.u.v.1$ and $p \equiv 3(mod\ 16)$ in E_p then, we have that $rank(E_{(72u^4+72u^2v^2+19v^4)}(Q)) = 1$.

Proof. Due to in lemma 2.4, there is only remained equation for $\bar{\Gamma}$ as

$$4)N^2 = -2M^4 + 2(72u^4 + 72u^2v^2 + 19v^4)e^4$$

to calculate the rank.

Let M and e as v and 1 respectively then, we gain

$$-2v^4 + 144u^4 + 144u^2v^2 + 38v^4$$

$$= 144u^4 + 144u^2v^2 + 36v^4.$$

Whence, the triple $(v, 1, 12u^2 + 6v^2)$ is derived as the solution.

It submits the conclusions $\#\bar{\alpha}(\bar{\Gamma}) = 4$ and $r2.4$

So we say that $rank(E_{(72u^4+72u^2v^2+19v^4)}(Q)) = 1$. □

Lemma 2.6. Assume that prime p is $p = 325u^4 + 12u^2v^2 + 36v^4\ w.i.u.v.1$ and $p \equiv 5(mod\ 16)$ in elliptic curve E_p then, there deduced that $rank(E_{(325u^4+12u^2v^2+36v^4)}(Q)) = 1$.

Proof. Take p as $p = 16k + 5$ with integer k then, $1)N^2 = M^4 + pe^4$ is only relating equation for Γ. And it is clear that $\#\alpha(\Gamma) = 2$.

Next, the curve $\overline{E_p}$ is induced as $y^2 = x^3 - 4(16k + 5)x$, hence it were educed relating equations for $\bar{\Gamma}$ as $1)N^2 = M^4 - 4(16k + 5)e^4$ and $2)N^2 = -M^4 + 4(16k + 5)e^4$ and $3)N^2 = 2M^4 - 2(16k + 5)e^4$ and $4)N^2 = -2M^4 + 2(16k + 5)e^4$ and $5)N^2 = 4M^4 - (16k + 5)e^4$ and $6)N^2 = -4M^4 + (16k + 5)e^4$.

Here, we will not mention about solvability of 1). See [3] for it.
In modulo 4 in equation 2) shows that $1 \equiv N^2 \equiv -M^4 \equiv 3(mod\ 4)$ and the sides RHS, LHS do not match, thus we gain a contradiction.
Cutting down on equations 3) and 4) by p reveals that 3)$N^2 \equiv 2M^4(mod\ p)$ and 4)$N^2 \equiv -2M^4(mod\ p)$ but at the same time there deduced 3) $\left(\frac{2M^4}{p}\right) = -1$ and 4) $\left(\frac{-2M^4}{p}\right) = -1$. Doing a pair in each case leaves a contradiction.
In modulo 4 in relating equation 5) implies the congruence $1 \equiv N^2 \equiv -5e^4 \equiv 3(mod\ 4)$ and this is unmatched relation, thereby no solution exists.
In the last case, equation 6) is $N^2 = -4M^4 + (325u^4 + 12u^2v^2 + 36v^4)e^4$.
Set M and e as $3u$ and 1 then, there derived

$$-4(3u)^4 + 325u^4 + 12u^2v^2 + 36v^4$$

$$= u^4 + 12u^2v^2 + 36v^4.$$

So the solution is educed as $(3u, 1, u^2 + 6v^2)$.
It presents that $\#\bar{\alpha}(\bar{\Gamma}) = 4$ and $r2.4$
Eventually, we conclude that $rank(E_{(325u^4+12u^2v^2+36v^4)}(Q)) = 1$. □

3 Curve E_{-2p}

In section 3, we shall manage the rank of curve E_{-2p}: $y^2 = x^3 - 2px$.

Theorem 3.1. (1). Set E_{-2p} as an elliptic curve $y^2 = x^3 - 2px$ that satisfies $p = 1531u^4 + 592u^2v^2 + 64v^4\ w.i.u.v.1$, $p \equiv 11(mod\ 16)$ then, we acquire that

$$rank(E_{-2(1531u^4+592u^2v^2+64v^4)}(Q))$$

$$= rank(E_{-(1002002u^4+18018u^2v^2+81v^4)}(Q)).$$

(2). If E_{-2p} is given as an elliptic curve $y^2 = x^3 - 2px$ where prime is gotten as $p = 1606u^4 + 684u^2v^2 + 81v^4\ w.i.u.v.1$ and $p \equiv 3(mod\ 16)$ then, we take that

$$rank(E_{-2(1606u^4+684u^2v^2+81v^4)}(Q))$$

$$= rank(E_{-(5u^4+96u^2v^2+576v^4)}(Q)).$$

(3). Let E_{-2p} be the curve $y^2 = x^3 - 2px$ with prime as $p = 1286s^4 + 16t^4 + u^4 - 48s^2t^2 + 12s^2u^2 - 8t^2u^2\ w.i.s.t.u.1$ and $p \equiv 11(mod\ 16)$ then, we gain

$$rank(E_{-2(1286s^4+16t^4+u^4-48s^2t^2+12s^2u^2-8t^2u^2)}(Q))$$

$$= rank(E_{-(6562u^4+50u^2v^2+625v^4)}(Q)).$$

(4). Take E_{-2p} as an elliptic curve $y^2 = x^3 - 2px$ where prime is $p = 402s^4 + 16t^4 + u^4 - 160s^2t^2 - 40s^2u^2 + 8t^2u^2\ w.i.s.t.u.1$ and $p \equiv 3(mod\ 16)$ then, we say that

$$rank(E_{-2(402s^4+16t^4+u^4-160s^2t^2-40s^2u^2+8t^2u^2)}(Q))$$

$$= rank(E_{(633u^4+8u^2v^2+2v^4)}(Q)).$$

(5). Set E_{-2p} as an elliptic curve $y^2 = x^3 - 2px$ with prime as $p = 146s^4 + t^4 + 4u^4 + 24s^2t^2 + 48s^2u^2 + 4t^2u^2\ w.i.s.t.u.1$ and $p \equiv 3(mod\ 16)$ then, we gain next result:

$$rank(E_{-2(146s^4+t^4+4u^4+24s^2t^2+48s^2u^2+4t^2u^2)}(Q))$$

$$= rank(E_{(72u^4+72u^2v^2+19v^4)}(Q)).$$

(6). If E_{-2p} is defined as an elliptic curve $y^2 = x^3 - 2px$ where prime is such that $p = 486s^4 + 16t^4 + u^4 - 176s^2t^2 - 44s^2u^2 + 8t^2u^2\ w.i.s.t.u.1$ and $p \equiv 3(mod\ 16)$ then, we face the following:

$$rank(E_{-2(486s^4+16t^4+u^4-176s^2t^2-44s^2u^2+8t^2u^2)}(Q))$$

$$= rank(E_{(325u^4+12u^2v^2+36v^4)}(Q)).$$

Proof. (1). Owing to [3], we only needed to treat the solvability of relating equation $4) N^2 = -2M^4 + (1531u^4 + 592u^2v^2 + 64v^4)e^4$ for Γ. In coefficient of e^4, there exists a square term $64v^4$, thus there is a probability that form of square in resultant will be shown after determining the values M and e. Since we have the factorization $592u^2v^2 = 2 \cdot 37 \cdot 8u^2v^2$ there must be deduced the term $1369u^4$. Next, it is necessary to take into account the numerical value $-2M^4 + 1531u^4e^4$. Due to our purpose in the above, the relation $-2M^4 + 1531u^4e^4 = 1369u^4 \cdots\cdots (AZ)$ must be established. First of all, assign the value e as 1 then, we are faced with $2M^4 = 162u^4$. On this account, we attain that $M = 3u$. For that reason, we got the pair $(M, e) = (3u, 1)$ which satisfies (AZ). Namely, it is gotten as a part of solution of equation. In addition, because of the computation $-2(3u)^4 + 1531u^4 + 592u^2v^2 + 64v^4 = 1369u^4 + 592u^2v^2 + 64v^4$ we get tha $N = 37u^2 + 8v^2$. Consequently, the solution of equation is produced as ($3u$, $1, 37u^2 + 8v^2$). On that account, there educed that $\#\alpha(\Gamma) = 4$. For this reason, we are confronted with $rank(E_{-2(1531u^4+592u^2v^2+64v^4)}(Q)) = 1$ on account of $r4.2$. Now from lemma 2.1 we achieved the proof.

(2). Because of [3], if we find the solution of equation

$$4) N^2 = -2M^4 + (1606u^4 + 684u^2v^2 + 81v^4)e^4 \text{ for } \Gamma$$

then, computing the rank is completed.

Substitute $3u$ and 1 into M and e then, we confront to

$$-2(3u)^4 + 1606u^4 + 684u^2v^2 + 81v^4$$

$$= 1444u^4 + 684u^2v^2 + 81v^4.$$

So we obtain the solution as

$$(3u, 1, 38u^2 + 9v^2).$$

It presents the conclusions $\#\alpha(\Gamma) = 4$ and $r4.2$.

Whence, the next result is induced:

$$rank(E_{-2(1606u^4+684u^2v^2+81v^4)}(Q)) = 1.$$

Now by lemma 2.2 we accomplished the proof.

(3). There is only left $4) N^2 = -2M^4 + (1286s^4 + 16t^4 + u^4 - 48s^2t^2 + 12s^2u^2 - 8t^2u^2)e^4$ for Γ that is needed to consider the solvability from [3]. In coefficient of e^4, the squares $16t^4$ and u^4 are appeared. For getting square in resultant, it is necessary to obtain more square. Suppose that $16t^4$ and $u^4 \cdots\cdots (SF)$ composes part of resultant. Next, we must note the terms $-48s^2t^2$ and $12s^2u^2$ and $-8t^2u^2$. We also assume that these are components of resultant. Then, we confront to the factorizations $-48s^2t^2 = -2 \cdot 6 \cdot 4s^2t^2$ and $12s^2u^2 = 2 \cdot 6s^2u^2$ and $-8t^2u^2 = -2 \cdot 4t^2u^2$. Whence, there ought to be given the squares $36s^4$, $16t^4$ and $36s^4$, u^4 and $16t^4$, u^4. Namely, the terms $36s^4$ and $16t^4$ and u^4 has to be emerged. The latter two squares are already induced from (SF). Thus, we have to find the first square $36s^4$. There is remained arithmetical value $-2M^4 + 1286s^4e^4$. Take e as 1 then, the relation $-2M^4 + 1286s^4 = 36s^4$ should be hold. Now from $2M^4 = 1250s^4$ there comes that $M = 5s$. Whence, if M and e are chosen as $5s$ and 1 then, we can attain square $36s^4$. Namely, the pair $(M, e) = (5s, 1)$ is induced as a part of solution. In addition, owing to $-2(5s)^4 + 1286s^4 + 16t^4 + u^4 - 48s^2t^2 + 12s^2u^2 - 8t^2u^2$ we get the integer N as $6s^2 - 4t^2 + u^2$. Wherefore, the solution of above relating equation is deduced as ($5s$, 1,

$6s^2 - 4t^2 + u^2$). On this account, we arrive at that $\#\alpha(\Gamma) = 4$. For this reason, we conclude that $r4.2$. Accordingly, there derived the result as $rank(E_{-2(1286s^4+16t^4+u^4-48s^2t^2+12s^2u^2-8t^2u^2)}(Q)) = 1$. Moreover, on account of lemmas 2.3 we complete the proof.

(4). If there is found the solution of relating equation $4) N^2 = -2M^4 + (402s^4 + 16t^4 + u^4 - 160s^2t^2 - 40s^2u^2 + 8t^2u^2)e^4$ for Γ then, treating the rank is done by [3]. There are two squares $16t^4$ and u^4 in coefficient of e^4. To obtain square in resultant, we must find another square term. We appoint that above two squares composed of resultant. In the next step, we have to notice the terms $-160s^2t^2$ and $-40s^2u^2$ and $8t^2u^2$. Assume that these comprised of resultant. Now we have factorizations $-160s^2t^2 = -2 \cdot 20 \cdot 4s^2t^2$ and $-40s^2u^2 = -2 \cdot 20s^2u^2$ and $8t^2u^2 = 2 \cdot 4t^2u^2$. Therefore, the squares $400s^4$, $16t^4$ and $400s^4$, u^4 and $16t^4$, u^4 should be educed. In sum, the terms $400s^4$ and $16t^4$ and u^4 ought to be shown. Since the latter two squares are already emerged, here we only needed to find the square $400s^4$. Now we confront to numerical value $-2M^4 + 402s^4e^4$. Let e be 1 then, the equality $-2M^4 + 402s^4 = 400s^4$ must be valid. Because we get that $2M^4 = 2s^4$ there derived that $M = s$. Now the pair $(M, e) = (s, 1)$ is deduced as a part of solution. Furthermore, from $-2s^4 + 402s^4 + 16t^4 + u^4 - 160s^2t^2 - 40s^2u^2 + 8t^2u^2$ we attain the value N as $20s^2 - 4t^2 - u^2$. Thus, the triple $(s, 1, 20s^2 - 4t^2 - u^2)$ is gotten as the solution of equation. Henceforth, we take the conclusion $\#\alpha(\Gamma) = 4$. It follows that $rank(E_{-2(402s^4+16t^4+u^4-160s^2t^2-40s^2u^2+8t^2u^2)}(Q)) = 1$ from $r4.2$. Besides, by lemma 2.4 the proof is done.

(5). The result in [3] requires to check only the solvability of equation

$$4) N^2 = -2M^4 + (146s^4 + t^4 + 4u^4 + 24s^2t^2 + 48s^2u^2 + 4t^2u^2)e^4 \text{ for } \Gamma .$$

Replace s and 1 into M and e leads to

$$-2s^4 + 146s^4 + t^4 + 4u^4 + 24s^2t^2 + 48s^2u^2 + 4t^2u^2$$

$$= 144s^4 + t^4 + 4u^4 + 24s^2t^2 + 48s^2u^2 + 4t^2u^2.$$

Therefore, we obtain the solution

$$(s, 1, 12s^2 + t^2 + 2u^2).$$

Whence, there induced $\#\alpha(\Gamma) = 4$.
And so we get that $r4.2$.
Consequentially, the result

$$rank(E_{-2(146s^4+t^4+4u^4+24s^2t^2+48s^2u^2+4t^2u^2)}(Q)) = 1$$

is derived.

Due to lemmas 2.5 the proof is done.
(6). Because of result in [3] we only needed to find the solution of

4)$N^2 = -2M^4 + (486s^4 + 16t^4 + u^4 - 176s^2t^2 - 44s^2u^2 + 8t^2u^2)e^4$ for Γ .

Take s and 1 as M and e then, we reach that

$$-2s^4 + 486s^4 + 16t^4 + u^4 - 176s^2t^2 - 44s^2u^2 + 8t^2u^2$$
$$= 484s^4 + 16t^4 + u^4 - 176s^2t^2 - 44s^2u^2 + 8t^2u^2.$$

Thus, the solution is produced as follows:

$$(s, 1, 22s^2 - 4t^2 - u^2).$$

So we get both $\#\alpha(\Gamma) = 4$ and $r4.2$.
Hence, next result is educed:

$$rank(E_{-2(486s^4+16t^4+u^4-176s^2t^2-44s^2u^2+8t^2u^2)}(Q)) = 1.$$

Now we accomplish the proof by lemma 2.6. □

Remark 3.2. In [(2.5) Theorem of Sec 2.Chap.8, 2], Husem$\ddot{o}$ller suggested that rank of $E_{-p}: y^2 = x^3 - px$ is 0 when p is $p \equiv 3(mod\ 8)$. But he didn't verified detaily. Here we treat about this. Put $p = 8k + 3$ with integer k. The relating equations for Γ are gotten as 1)$N^2 = M^4 - (8k + 3)e^4$ and 2)$N^2 = -M^4 + (8k + 3)e^4$. Equation 2) cannot possess a solution since in modulo p shows that $N^2 \equiv -M^4(mod\ p)$ but we have that $\left(\frac{-M^4}{p}\right) = -1$ and these cannot coexist. Therefore, we have that $\#\alpha(\Gamma) = 2$. Now there comes $\overline{E_{-p}}$ as $y^2 = x^3 + 4px$. There are relating equations for $\bar{\Gamma}$ as 1)$N^2 = M^4 + 4(8k + 3)e^4$ and 2)$N^2 = 2M^4 + 2(8k + 3)e^4$ and 3)$N^2 = 4M^4 + (8k + 3)e^4$. Cutting down on equations 2) and 3) by 16 and 4 yields that 2)0 , $4 \equiv N^2 \equiv 2M^4 + 6e^4 \equiv 8(mod\ 16)$ and 3)$1 \equiv N^2 \equiv 3e^4 \equiv 3(mod\ 4)$ and so we gain $\#\bar{\alpha}(\Gamma) = 2$, whence the rank is derived as 0 from $2^r = \frac{2\cdot 2}{4} = 1$.

Remark 3.3. In [8], the author considered ranks of curves $y^2 = x^3 \pm 3px$ and $y^2 = x^3 \pm 6px$ with Mersenne prime. There were added more conditions to Mersenne prime in each case. The ranks were 1 or 0 or at most 1.

Remark 3.4. In $E_{4pq}: y^2 = x^3 + 4pqx$, we can confront to much results of rank at most 1. In elliptic curve E_{2pq} we attain many results of rank 0. These are consequences of calculations. Even though, the rank is not certain, at most rank 1 is meaningful since we can obtain generalized rank 0 or 1 easily in relative.

Remark 3.5. Generally, in computing rank of elliptic curve there used three kinds of methods were used often. First is using the formula $2^r = \frac{\#\alpha(\Gamma)\#\bar{\alpha}(\Gamma)}{4}$ which were used in this article. Second is 2-descent and the third is linearly independent. The first method is most widely used.

4 Examples

In section 4, we will treat examples of previous results. Using by [1], we can check the primality.

Examples (p, u, v) from lemma 2.1 to 2.6 are next things:

(1020101, 1, 1), (9156677, 1, 15), (18062501, 1, 19), (93122501, 1, 31);

(677, 1, 1), (6101, 5, 1), (148997, 1, 4);

(7237, 1, 1), (57637, 1, 3), (17865637, 1, 13);

(643, 1, 1), (5827, 1, 7), (51347, 3, 1), (169987, 1, 17);

(163, 1, 1), (1459, 2, 1), (19603, 4, 1), (95923, 6, 1), (299539, 8, 1);

(373, 1, 1), (9733, 1, 4), (148549, 1, 8).

In above, ; differentiate the lemmas.

Examples (p, u, v) and $(p, s, t, u)$from theorem 3.1(1) to (6) are induced as follows:

(p, u, v)(from theorem 3.1(1) to (2)):

(12043, 1, 3), (129403, 3, 1), (146347, 3, 2);

(2371, 1, 1), (1020931, 5, 1).

(p, s, t, u)(from theorem 3.1(3) to (6)):

(1259, 1, 1, 1), (118411, 3, 1, 9); (227, 1, 1, 1), (20611, 3, 3, 1);

(227, 1, 1, 1), (1523, 1, 5, 1), (14323, 3, 3, 1);

(26083, 3, 3, 1), (3170851, 9, 1, 1).

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