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Calculation of Ranks in Curves $y^2 = x^3 \pm Apx$

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Abstract

We take that $y^2 = x^3 \pm Apx$ are elliptic curves then, we will compute the ranks of curves and present examples.

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1 Introduction

Assume that E is an elliptic curve $y^2 = x^3 + ax^2 + bx$ and Γ is the set of rational points on E. Then, Γ is a finitely generated abelian group and we have that $\Gamma \cong E(Q)_{tors} \bigoplus Z^r$ with torsion subgroup $E(Q)_{tors}$ and Mordell-Weil rank r. Let $Q^\times$ be the multiplicative group which is comprised of non-zero rational numbers. Furthermore, take $Q^{\times 2}$ as the subgroup of squares of elements of $Q^\times$. Set $\bar{E}$ as the curve $y^2 = x(x^2 - 2ax + a^2 - 4b)$ and $\bar{\Gamma}$ as the set of rational points on $\bar{E}$. Take $N^2 = b_1M^4 + aM^2e^2 + b_2e^4$ as elating equation for Γ where b_1 and b_2 are divisors of b such that $b = b_1b_2$ with $b_1 \not\equiv 1,\ b (mod\ Q^{\times 2})$. Suppose that $N^2 = b_1M^4 - 2aM^2e^2 + b_2e^4$is relating equation for $\bar{\Gamma}$ where b_1 and b_2 are divisors of $a^2 - 4b$ as $b_1b_2 = a^2 - 4b$ with $b_1 \not\equiv 1,\ a^2 - 4b (mod\ Q^{\times 2})$. Let (M, e, N) be an integral solution of above equations with (M, N)=(M, e)=(N, e) =$(b_1, e) =$ (b_2, M)= 1 and $M \neq 0, e \neq 0$. Lastly, we gain $2^r = \frac{\#\alpha(\Gamma)\#\bar{\alpha}(\bar{\Gamma})}{4}$ where r is rank of E. In this paper, we will treat ranks of various elliptic curves $y^2 = x^2 \pm Apx$.

We assign some notations as follows:

$w.i.t.u.1$: with integers t and u and $(t, u) = 1$([7]).

$w.i.u.v.1$: with integers u and v and $(u, v) = 1$([4]).

$r4.2$: $2^r = \frac{4\cdot 2}{4}$([4]).

2 Numeration of Ranks

Now we consider the ranks of elliptic curves by the method in [8].

Theorem 2.1.(1). Let the prime p be the form $p = 2t^4 + 32t^3u + 8tu^3 + 12t^2u^2 + 17u^4$ $w.i.t.u.1$ and $p \equiv 7(mod\ 16)$ in curve $E_{4p}: y^2 = x^3 + 4px$ then, we confront to

$$rank(E_{4(2t^4+32t^3u+8tu^3+12t^2u^2+17u^4)}(Q)) = 1.$$

(2). Assign the prime p as $p = 349t^4 - 432t^3u - 48tu^3 + 236t^2u^2 + 8u^4$ $w.i.t.u.1$ and $p \equiv 13(mod\ 16)$ in curve E_{-4p}: $y^2 = x^3 - 4px$ then, we say that

$$rank(E_{-4(349t^4-432t^3u-48tu^3+236t^2u^2+8u^4)}(Q)) = 1.$$

(3). If prime p is $p = 20t^4 - 16t^3u - 16tu^3 - 160t^2u^2 + 533u^4$ $w.i.t.u.1$ and $p \equiv 5(mod\ 16)$ in E_{-4p}: $y^2 = x^3 - 4px$ then, we get that

$$rank(E_{-4(20t^4-16t^3u-16tu^3-160t^2u^2+533u^4)}(Q)) = 1.$$

(4). We appoint that prime p is $p = 1060t^4 - 1024t^3u - 64tu^3 + 540t^2u^2 + 173u^4$ $w.i.t.u.1$ and $p \equiv 13(mod\ 16)$ in E_{-4p}: $y^2 = x^3 - 4px$ then, we gain

$$rank(E_{-4(1060t^4-1024t^3u-64tu^3+540t^2u^2+173u^4)}(Q)) = 1.$$

(5). If prime p is such that $p = 5112u^4 + 156u^2v^2 + v^4$ $w.i.u.v.1$ and $p \equiv 5(mod\ 16)$ in E_{-3p}: $y^2 = x^3 - 3px$ then, we say that

$$rank(E_{-3(5112u^4+156u^2v^2+v^4)}(Q)) = 1.$$

(6). If prime p is taken as the form $p = 32500u^4 + 1600u^2v^2 + v^4$ $w.i.u.v.1$ and $p \equiv 5(mod\ 16)$ in E_{-3p}: $y^2 = x^3 - 3px$ then, next result is derived:

$$rank(E_{-3(32500u^4+1600u^2v^2+v^4)}(Q)) = 1.$$

Proo$f$. (1). Owing to [2], there is only left relating equation

$$2) N^2 = 2M^4 + 2(2t^4 + 32t^3u + 8tu^3 + 12t^2u^2 + 17u^4)e^4 \text{ for } \Gamma.$$

If we take the integers M and e as $2t - u$ and 1 respectively then, we gain

$$2(2t-u)^4 + 2(2t^4 + 32t^3u + 8tu^3 + 12t^2u^2 + 17u^4)$$

$$= 2(16t^4 - 32t^3u - 8tu^3 + 24t^2u^2 + u^4)$$

$$+4t^4 + 64t^3u + 16tu^3 + 24t^2u^2 + 34u^4$$

$$= 32t^4 - 64t^3u - 16tu^3 + 48t^2u^2 + 2u^4$$

$$+4t^4 + 64t^3u + 16tu^3 + 24t^2u^2 + 34u^4$$

$$= 36t^4 + 72t^2u^2 + 36u^4.$$

Thereby, we attain the triple $(2t - u, 1, 6t^2 + 6u^2)$ as the solution equation.
It yields that $\#\alpha(\Gamma) = 4$ and $r4.2$ and so the result

$$rank(E_{4(2t^4+32t^3u+8tu^3+12t^2u^2+17u^4)}(Q)) = 1$$

is induced.
(2). Due to the result in [5], it is sufficient that we only find the solution of next relating equation

$$6) N^2 = -4M^4 + (349t^4 - 432t^3u - 48tu^3 + 236t^2u^2 + 8u^4)e^4 \text{ for } \Gamma.$$

Suppose that $M = 3t - u$ and $e = 1$ then, we get the numeration

$$-4(3t-u)^4 + (349t^4 - 432t^3u - 48tu^3 + 236t^2u^2 + 8u^4)$$

$$= -324t^4 + 432t^3u + 48tu^3 - 216t^2u^2 - 4u^4$$

$$+349t^4 - 432t^3u - 48tu^3 + 236t^2u^2 + 8u^4$$

$$= 25t^4 + 20t^2u^2 + 4u^4.$$

So the solution is induced as follows:

$$(3t - u, 1, 5t^2 + 2u^2).$$

And so we gain $\#\alpha(\Gamma) = 4$ and $r4.2$.
Thus, there derived the result

$$rank(E_{-4(349t^4-432t^3u-48tu^3+236t^2u^2+8u^4)}(Q)) = 1.$$

(3). From [3], it is enough that we only should consider the solvability of relating equation

$$6)N^2 = -4M^4 + (20t^4 - 16t^3u - 16tu^3 - 160t^2u^2 + 533u^4)e^4 \text{ for } \Gamma.$$

Replace $t - u$ and 1 into both M and e produces the calculation

$$-4(t-u)^4 + (20t^4 - 16t^3u - 16tu^3 - 160t^2u^2 + 533u^4)$$
$$= -4t^4 + 16t^3u + 16tu^3 - 24t^2u^2 - 4u^4$$
$$+20t^4 - 16t^3u - 16tu^3 - 160t^2u^2 + 533u^4$$
$$= 16t^4 - 184t^2u^2 + 529u^4.$$

Thereby, the solution is induced as

$$(t-u, 1, 4t^2 - 23u^2).$$

Eventually, from $\#\alpha(\Gamma) = 4$ and $r4.2$ we conclude that

$$rank(E_{-4(20t^4-16t^3u-16tu^3-160t^2u^2+533u^4)}(Q)) = 1.$$

(4). By [5], there is only left equation

$$6)N^2 = -4M^4 + (1060t^4 - 1024t^3u - 64tu^3 + 540t^2u^2 + 173u^4)e^4 \text{ for } \Gamma$$

which requires to check the solvability.
Replace $4t - u$ and 1 into M and e gives that

$$-4(4t-u)^4 + 1060t^4 - 1024t^3u - 64tu^3 + 540t^2u^2 + 173u^4$$
$$= -1024t^4 + 1024t^3u + 64tu^3 - 384t^2u^2 - 4u^4$$
$$+1060t^4 - 1024t^3u - 64tu^3 + 540t^2u^2 + 173u^4$$
$$= 36t^4 + 156t^2u^2 + 169u^4.$$

Thereby, the solution is educed as

$$(4t-u,\,1,\,6t^2+13u^2).$$

Hence, there derived $\#\alpha(\Gamma) = 4$ and $r4.2$.
Finally, the result

$$rank(E_{-4(1060t^4-1024t^3u-64tu^3+540t^2u^2+173u^4)}(Q)) = 1$$

is gotten.

(5). Because of [6], the remaining equation that is necessary to find the solution is $6)N^2 = 12M^4 + (5112u^4 + 156u^2v^2 + v^4)e^4$ for $\bar{\Gamma}$. There is a square term v^4 in coefficient of e^4. Suppose that both v^4 and $156u^2v^2$ consisted of resultant then, there must be given that $6084u^4$. Now we face arithmetical value $12M^4 + 5112u^4e^4$. Take $e = 1$ then, the relation $12M^4 + 5112u^4 = 6084u^4$ must be valid. Since we take that $12M^4 = 972u^4$ the value M is deduced as $3u$. That is the pair $(e, M) = (1, 3u)$ is gotten as the part of solution. Moreover from the numeration $12(3u)^4 + 5112u^4 + 156u^2v^2 + v^4 = 6084u^4 + 156u^2v^2 + v^4$ the integer N is educed as $78u^2 + v^2$. Wherefore, the solution of above relating equation is produced as $(3u, 1, 78u^2 + v^2)$. Eventually, we say that $\#\bar{\alpha}(\bar{\Gamma}) = 4$. To conclude we attain that $rank(E_{-3(5112u^4+156u^2v^2+v^4)}(Q)) = 1$ from $r2.4$.

(6). Due to the result in [6], it is sufficient that if we find the solution of equation $6)N^2 = 12M^4 + (32500u^4 + 1600u^2v^2 + v^4)e^4$ for $\bar{\Gamma}$ then, calculating rank of curve is done. A square v^4 is appeared in coefficient of e^4, hence we can expect an emergence of square in resultant. We appoint that $1600u^2v^2$ and v^4 are components of resultant then, there ought to be appeared that $640000u^4$. We should note arithmetical value $12M^4 + 32500u^4e^4$. Take $e = 1$ in this relation then, there must be induced that $640000u^4 = 12M^4 + 32500u^4$. Now we have that $12M^4 = 607500u^4$. Wherefore, we attain that $M^4 = 50625u^4$. In RHS, there is 25 in first and second places, hence if we factor of it by 25 then, there comes that $50625 = 25 \cdot 2025$. The number 2025 is a square of 45, thus we attain that $50625 = 5^2 \cdot 45^2 = 5^2 \cdot 3^4 \cdot 5^2 = 15^4$. Consequently, the integer M is induced as $15u$. Besides, from the numeration $12(15u)^4 + 32500u^4 + 1600u^2v^2 + v^4 = 607500u^4 + 32500u^4 + 1600u^2v^2 + v^4$ the value N is deduced as $800u^2 + v^2$. On that account, we attaint the solution of above equation as $(15u, 1, 800u^2 + v^2)$. As a result, we are confronted with $\#\bar{\alpha}(\bar{\Gamma}) = 4$. For that reason, we gain the conclusion $r2.4$ and it gives the result $rank(E_{-3(32500u^4+1600u^2v^2+v^4)}(Q)) = 1$. □

In above (5), we treated rank of curve E_{-3p}: $y^2 = x^3 - 3px$ where prime is the form $p = Hu^4 + Iu^2v^2 + Kv^4$. In [6], the author managed ranks of elliptic curves $y^2 = x^3 \mp 3px$. As the other forms, in curves $y^2 = x^3 \pm Apx$ generalized rank 1 were gotten easily more in curve $y^2 = x^3 - Apx$ than $y^2 = x^3 + Apx$. It is treatment of computation. In [7], the author calculated the rank of $y^2 = x^3 - 3px$ where prime is the form $p = As^4 + Bt^4 + Cu^4 + Ds^2t^2 + Fs^2u^2 + Gt^2u^2$. It were $p = 400s^4 + 4t^4 - 11u^4 + 80s^2t^2 - 40s^2u^2 - 4t^2u^2\ w.i.s.t.u.1$, $p \equiv 13(mod\ 16)$ and the case $p = 400s^4 + 4t^4 - 11u^4 - 80s^2t^2 + 40s^2u^2 - 4t^2u^2\ w.i.s.t.u.1, p \equiv 13(mod\ 16)$ and the case $p = 196s^4 + 16t^4 - 11u^4 - 112s^2t^2 + 28s^2u^2 - 8t^2u^2\ \ w.i.s.t.u.1$, $p \equiv 13(mod\ 16)$ and the case $p = 100s^4 + 4t^4 - 11u^4 + 40s^2t^2 + 20s^2u^2 + 4t^2u^2\ w.i.s.t.u.1$, $p \equiv 13(mod\ 16)$ and the case $p = 100s^4 + 4t^4 - 11u^4 + 40s^2t^2 - 20s^2u^2 - 4t^2u^2\ w.i.s.t.u.1$, $p \equiv 13(mod\ 16)$ and the case $p = 24s^4 + 4t^4 + u^4 - 24s^2t^2 + 12s^2u^2 - 4t^2u^2\ w.i.s.t.u.1, p \equiv 13(mod\ 16)$([7]).

3 Examples

Now we treat some examples of previous computations.

We can examine the primality from [1].

There exist examples (p, t, u) and (p, u, v) from theorem 2.1(1) to (6) are the followings:

(p, t, u)(from theorem 2.1(1) to (4)):

(71, 1, 1), (20807, 3, 5); (8429, 3, 4), (12413, 3, 2); (364373, 12, 1),

(1954613, 18, 1); (22381, 2, 3), (547853, 5, 1).

(p, u, v)(from theorem 2.1(5) to (6)):

(415477, 3, 1), (449797, 3, 5), (4210309, 3, 37); (2646901, 3, 1),

(213362101, 9, 1).

In above, ; differentiate the theorems.

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