International Journal of Contemporary Mathematical Sciences
Vol. 18, 2023, no. 1, 9 - 17
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ijcms.2023.91865

Union-n-Continuous Functions

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Abstract

The concept of a union-n-continuous function is introduced. The basic properties of these functions are developed. It is established that, if the domain is not discrete, then this class of functions is strictly between the classes of n-continuous functions and generalized n-continuous functions. A useful characterization of generalized n-open sets is obtained.

Mathematics Subject Classification: 54C10, 54D10

Keywords: n-open, un-open, gn-open, n-0-dimensional, n-continuous, un-continuous, gn-continuous

1 Introduction

Reilly and Vamanamurthy [5] introduced the class of clopen continuous functions characterized by the property that inverse images of open sets are unions of clopen sets. In [6] Singh continued the development of these functions under the name cl-supercontinuous. In this note we investigate the class of functions characterized by the property that inverse images of open sets are unions of non-clopen sets. Non-clopen sets have been developed in the literature under the name n-open [1]. The functions introduced in this note are called union-n-continuous (briefly un-continuous). It is established that, if the domain is not discrete, then this class of functions is strictly between the classes of n-continuous functions and generalized n-continuous (briefly gn-continuous) functions. It is shown that a function $f : X \to Y$ is un-continuous if and only if $f^{-1}(V) \neq \emptyset$ for every nonempty open set $V \subseteq Y$ and f is gn-continuous. Additionally the basic properties of these functions are developed.

2 Preliminaries

Unless otherwise stated, the symbols X, Y, and Z represent topological spaces (briefly spaces) with no separation properties assumed. All topological spaces are assumed to be nonempty. The closure and interior of a set A are signified by $Cl(A)$ and $Int(A)$, respectively.

Definition 2.1 *Let X be a nonempty set and $\mathcal{P}(X)$ the power set of X. A subfamily m_X of $\mathcal{P}(X)$ is called a minimal structure (briefly an m-structure) on X [4], if $\emptyset \in m_X$ and $X \in m_X$.*

Definition 2.2 *A subset A of a space X is said to be n-open [1] if $Int(A) \neq Cl(A)$. A subset of X is called n-closed if its complement is n-open.*

Theorem 2.3 [1] *If A is a subset of a space X, then*

(a) *A is n-open if and only if A is not clopen.*

(b) *A is n-open if and only if $X - A$ is n-open.*

Thus the n-open sets coincide with the n-closed sets.

Remark 2.4 *Nether X nor the $\emptyset$ is n-open. Therefore the collection of n-open sets does not form a minimal structure.*

Theorem 2.5 [1] *If X is not discrete, then for every $x \in X$ there exists an n-open set containing x.*

Remark 2.6 *A space is discrete if and only if there are no n-open sets.*

Definition 2.7 *Let A be a subset of a space X. The n-interior of A [1] is denoted by $nInt(A)$ and given by $nInt(A) = \cup\{U \subseteq X : U \subseteq A$ and U is n-open$\}$. The n-closure of A [1] is denoted by $nCl(A)$ and given by $nCl(A) = \cap\{F \subseteq X : A \subseteq F$ and $F$is n-closed$\}$.*

Theorem 2.8 [1] *The following statements hold for every set $A \subseteq X$:*

(a) *$nInt(X - A) = X - nCl(A)$.*

(b) *$nCl(X - A) = X - nInt(A)$.*

(c) *$x \in nCl(A)$ if and only if $U \cap A \neq \emptyset$ for every n-open set U containing x.*

Theorem 2.9 [1] *If X is a space, then*

(a) $nCl(X) = X$.

(b) $nInt(\emptyset) = \emptyset$.

Theorem 2.10 [1] *If X is not discrete, then*

(a) $nInt(X) = X$.

(b) $nCl(\emptyset) = \emptyset$.

Theorem 2.11 [1] *If X is a discrete space, then*

(a) $nInt(A) = \emptyset$ *for every set* $A \subseteq X$.

(b) $nCl(A) = X$ *for every set* $A \subseteq X$.

Theorem 2.12 [1] *If A is a subset of a space X, then*

(a) $nCl(A) = A$ *or* $nCl(A) = X$.

(b) $nInt(A) = A$ *or* $nInt(A) = \emptyset$.

Definition 2.13 *A subset A of a space X is said to be generalized n-closed (briefly gn-closed)* [2], *if whenever $A \subseteq U$ and U is open, then $nCl(A) \subseteq U$. A subset of X is called generalized n-open (briefly gn-open) if its complement is gn-closed.*

Theorem 2.14 [1] *If A is a subset of a space X, then*

(a) *A is gn-closed if and only if $nCl(A) = A$.*

(b) *A is gn-open if and only if $nInt(A) = A$.*

(c) *$nCl(A)$ is gn-closed.*

(d) *$nInt(A)$ is gn-open.*

Definition 2.15 [1] *A function $f : X \to Y$ is said to be n-continuous* [1] *if $f^{-1}(V)$ is n-open in X for every proper nonempty open set $V \subseteq Y$.*

Definition 2.16 *A function $f : X \to Y$ is said to be generalized n-continuous (briefly gn-continuous)* [2] *if $f^{-1}(F)$ is gn-closed in X for every closed set $F \subseteq Y$.*

Theorem 2.17 [2] *The following conditions are equivalent for a function $f : X \to Y$:*

(a) *f is gn-continuous.*

(b) *$f^{-1}(V)$ is gn-open for every open set $V \subseteq Y$.*

(c) *$f^{-1}(Int(B)) \subseteq nInt(f^{-1}(B))$ for every set $B \subseteq Y$.*

(d) *$nCl(f^{-1}(B)) \subseteq f^{-1}(Cl(B))$ for every set $B \subseteq Y$.*

See [1], [2], or [3] for additional properties and notation concerning n-open sets.

3 un-Continuous Functions

Definition 3.1 *A function $f : X \to Y$ is said to be union-n-continuous (briefly un-continuous) if for every nonempty open set $V \subseteq Y$, $f^{-1}(V)$ is a union of a nonempty collection of n-open sets.*

If U is a union of a nonempty collection of n-open sets, we will call U un-open. Since the $\emptyset$ is not un-open, the un-open sets do not form a minimal structure. The un-open sets are obviously closed under union but, as we see in the following example, not closed under intersection.

Example 3.2 *Let $X = \{a, b, c\}$ have the topology $\tau = \{X, \emptyset, \{a, b\}, \{c\}\}$. The sets $\{a, c\}$ and $\{b, c\}$ are un-open, but their intersection is not un-open.*

If X is a discrete space, then there are no n-open sets and hence there is no un-continuous function defined on X. If the domain is not discrete, then obviously n-continuity implies un-continuity. The following example shows that the converse implication does not hold.

Example 3.3 *Let $X = \{a, b, c\}$ have the topologies $\tau = \{X, \emptyset, \{a, b\}, \{c\}\}$ and $\sigma = \{X, \emptyset, \{a, b\}\}$. The identity mapping $f : (X, \tau) \to (Y, \sigma)$ is un-continuous but not n-continuous. Note that $f^{-1}(\{a, b\})$ is un-open but not n-open.*

Theorem 3.4 *Let A be subset of a space X.*

(a) *If A is un-open, then A is gn-open.*

(b) *The set A is un-open if and only if $A \neq \emptyset$ and A is gn-open.*

Proof. (a) Since A is un-open, A is a union of a nonempty collection of n-open sets. Therefore $A \subseteq nInt(A)$. Hence $A = nInt(A)$ and thus A is gn-open (Theorem 2.14(b)).

(b) Assume A is un-open. Then A is union of a nonempty collection of n-open sets, which are nonempty. Therefore $A \neq \emptyset$. It follows from (a) that A is gn-open.

Assume that $A \neq \emptyset$ and that A is gn-open. By Theorem 2.14(b) $A = nInt(A)$. Since $A \neq \emptyset$, $nInt(A) \neq \emptyset$. Thus A is the union of a collection of n-open sets and is therefore un-open.

Corollary 3.5 *A nonempty set is gn-open if and only if it is a union of n-open sets.*

Theorem 3.6 *The following statements are equivalent for a function $f : X \to Y$:*

(a) *f is un-continuous.*

(b) *For every nonempty open set $V \subseteq Y$, $f^{-1}(V) \neq \emptyset$ and for every $x \in f^{-1}(V)$ there exists an n-open set A such that $x \in A \subseteq f^{-1}(V)$.*

Theorem 3.7 *The following statements are equivalent for a function $f : X \to Y$:*

(a) *f is un-continuous.*

(b) *For every nonempty open set $V \subseteq Y$, $f^{-1}(V) \neq \emptyset$ and f is gn-continuous.*

(c) *For every nonempty open set $V \subseteq Y$, $f^{-1}(V) \neq \emptyset$ and for every $B \subseteq Y$, $nCl(f^{-1}(B)) \subseteq f^{-1}(Cl(B))$.*

(d) *For every nonempty open set $V \subseteq Y$, $f^{-1}(V) \neq \emptyset$ and for every $B \subseteq Y$, $f^{-1}(Int(B)) \subseteq nInt(f^{-1}(B))$.*

Proof. (a) $\Leftrightarrow$ (b) follows from Theorem 3.4(b).
(b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) follows from Theorem 2.17.

Definition 3.8 *A function $f : X \to Y$ is said to be weakly gn-continuous if $nCl(f^{-1}(V)) \subseteq f^{-1}(Cl(V))$ for every open set $V \subseteq Y$.*

It follows immediately from Theorem 2.17(d) that gn-continuity implies weak gn-continuity and hence un-continuity implies weak gn-continuity. The following example shows that the converse implication does not hold.

Example 3.9 *Let $X = \{a, b, c\}$ have the topologies $\tau = \{X, \emptyset, \{a, b\}, \{c\}\}$ and $\sigma = \{X, \emptyset, \{c\}\}$. The identity mapping $f : (X, \tau) \to (X, \sigma)$ is weakly gn-continuous since $f^{-1}(Cl(\{c\})) = X$ but not gn-continuous because $f^{-1}(\{a, b\})$ is not gn-closed.*

Assuming the domains are not discrete, the following implications, none of which are reversible, hold:
n-continuity $\Rightarrow$ un-continuity $\Rightarrow$ gn-continuity. $\Rightarrow$ weak gn-continuity

4 Properties of un-Continuous Functions

As we see in the following example, the composition of un-continuous functions is not necessarily un-continuous.

Example 4.1 *Let $X = \{a, b, c\}$ have the topologies $\tau = \{X, \emptyset, \{a, b\}, \{c\}\}$, $\sigma = \{X, \emptyset, \{a\}\}$, and $\delta = \{X, \emptyset, \{c\}\}$. The identity mappings $f : (X, \tau) \to (X, \sigma)$ and $g : (X, \sigma) \to (X, \delta)$ are un-continuous, but $g \circ f$ is not un-continuous. Note that $(g \circ f)^{-1}(\{c\})$ is not un-open in (X, τ).*

Theorem 4.2 *If $f : X \to Y$ is un-continuous and $g : Y \to Z$ is continuous and $g^{-1}(V) \neq \emptyset$ for every nonempty open set $V \subseteq Z$, then $g \circ f$ is un-continuous.*

Corollary 4.3 *Let $f_\alpha : X \to Y_\alpha$ be a function for every $\alpha \in \Lambda$. If the product function $f : X \to \prod_{\alpha \in \Lambda} Y_\alpha$, given by $f(x) = (f_\alpha(x))_\alpha$, is un-continuous, then f_α is un-continuous for every $\alpha \in \Lambda$.*

Proof. Since $f_\alpha = p_\alpha \circ f$ for every $\alpha \in \Lambda$, where p_α is the projection onto Y_α, the desired result follows from Theorem 4.2.

Definition 4.4 *A space X is said to be n-0-dimensional if every nonempty open set is a union of n-open sets or equivalently every nonempty open set is un-open.*

Corollary 4.5 *Let $f : X \to Y$ be a function and let $g : X \to X \times Y$ be the graph function of f given by $g(x) = (x, f(x))$ for every $x \in X$. If g is un-continuous, then f is un-continuous and X is n-0-dimensional.*

Proof. Assume g is un-continuous and let $p_Y : X \times Y \to Y$ be the projection onto Y. Since $f = p_Y \circ g$, it follows from Theorem 4.2 that f is un-continuous. To see that X is n-0-dimensional, let U be a nonempty open set in X. Then $U \times Y$ is a nonempty open set in $X \times Y$. Since g is un-continuous, $U = g^{-1}(U \times Y)$ is un-open. Therefore U is a union of n-open sets and hence X is n-0-dimensional.

Definition 4.6 *A function $f : X \to Y$ is said to be un-open if $f(U)$ is open for every un-open set $U \subseteq X$.*

Theorem 4.7 *Let $f : X \to Y$ and $g : Y \to Z$ be functions. If f is un-open, un-continuous, and surjective and $g^{-1}(V) \neq \emptyset$ for every nonempty open set $V \subseteq Z$, then $g \circ f$ is un-continuous if and only if g is continuous.*

Proof. If $g : Y \to Z$ is continuous, then $g \circ f$ is un-continuous by Theorem 4.2. Assume $g \circ f$ is un-continuous and let $V \subseteq Z$ be an open set. Then $f^{-1}(g^{-1}(V))$ is un-open and hence $g^{-1}(V) = f(f^{-1}(g^{-1}(V)))$ is open. Therefore g is continuous.

The restriction of a un-continuous function, even to an n-open set, is not necessarily un-continuous.

Example 4.8 *If $X = \{a, b, c\}$ has the topologies $\tau = \{X, \emptyset, \{a, b\}, \{c\}\}$ and $\sigma = \{X, \emptyset, \{a, c\}\}$, then the identity map $f : (X, \tau) \to (X, \sigma)$ is un-continuous, but, if $S = \{b, c\}$, then $f|_S : (S, \tau_S) \to (X, \sigma)$ is not un-continuous. Note that, since the subspace topology τ_S is discrete, there is no un-continuous function defined on (S, τ_S).*

Lemma 4.9 *Assume $U \subseteq A \subseteq X$. If U is n-open as a subset of A, then U is n-open as a subset of X.*

Remark 4.10 *The converse of Lemma 4.9 does not hold.*

Theorem 4.11 *Let $f : X \to Y$ be a function. If $\mathcal{C} = \{C_\alpha : \alpha \in \Lambda\}$ is a cover of X by n-open sets such that $f|_{C_\alpha} : C_\alpha \to Y$ is un-continuous for every $\alpha \in \Lambda$, then $f : X \to Y$ is un-continuous.*

Proof. Let $V \subseteq Y$ be a nonempty open set. Since for every $\alpha \in \Lambda$, $f|_{C_\alpha} : C_\alpha \to Y$ is un-continuous, $f|_{C_\alpha}^{-1}(V)$ is the union of a nonempty collection of sets that are n-open as subsets of C_α for every $\alpha \in \Lambda$. By Lemma 4.9 for every $\alpha \in \Lambda$, $f|_{C_\alpha}^{-1}(V)$ is the union of a nonempty collection of sets that are n-open as subsets of X. Since $f^{-1}(V) = \bigcup_{\alpha \in \Lambda} f|_{C_\alpha}^{-1}(V)$, $f^{-1}(V)$ is the union of a nonempty collection of sets that are n-open in X. Hence f is un-continuous.

Definition 4.12 *A function $f : X \to Y$ is said to be a un-homeomorphism if f is bijective and both f and f^{-1} are un-continuous.*

Theorem 4.13 *If $f : X \to Y$ is a homeomorphism, then X is n-0-dimensional if and only if f is a un-homeomorphism.*

Proof. Assume $f : X \to Y$ is a homeomorphism.

Suppose X is n-0-dimensional. Since n-0-dimensional is a topological property, Y is also n-0-dimensional. Thus all nonempty open sets in either space are un-open. Since f and $f^{-1}$are continuous, f and f^{-1} are also un-continuous and therefore f is a un-homeomorphism.

Suppose $f : X \to Y$ is a un-homeomorphism. Let U be a nonempty open set in X. Since f is a homeomorphism and a un-homeomorphism, $U = f^{-1}(f(U))$ is un-open. Hence X is n-0-dimensional.

The following example shows that the converse of Theorem 4.13 does not hold.

Example 4.14 *Let X and Y be non-homeomorphic connected spaces with the same cardinality. Since there are no proper, nonempty clopen sets in ether space, all proper nonempty sets in either space are n-open and hence un-open. Since neither X nor Y is discrete, it follows from Theorem 2.5 that both X and Y are un-open. Therefore any bijection from X onto Y is a un-homeomorphism. Also X is n-0-dimensional.*

Corollary 4.15 *If X is n-0-dimensional and $f : X \to Y$ is a homeomorphism, then f is a un-homeomorphism.*

Remark 4.16 *Being a homeomorphism is independent of being a un-homeomorphism.*

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Received: June 14, 2023; Publsihed: June 29, 2023