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A Note on Spectral Analysis of a Problem in Quantum Mechanics

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Abstract

The objective is to study a spectral problem which arises from physics by using results from functional analysis. Spectral properties for the principle operators are studied. It is shown how the spectral properties of physically important operators such as the energy operator which define the problem can be obtained. Spectral decomposition for the Hamiltonian is constructed and an application to coherent states is presented.

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1 Introduction

The creation of quantum mechanics has led to the development of a great deal of new mathematics and techniques to employ it. It is the intention here to employ some of these concepts and methods that have arisen in this area to study a spectral problem in nonrelativistic quantum mechanics, in particular, functional analysis [**1-3**].

Suppose $(X, ||\cdot||)$ is a Banach space over $\mathbb{C}$ and $\Omega \subset \mathbb{C}$ is a non-empty open set. A function $f : \Omega \to X$ is called analytic if for any $z_0 \in \Omega$, there exists a $\delta > 0$ such that $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ for every $z \in B_\delta(z_0)$, where $B_\delta(z_0) \subset \Omega$ and $a_n \in X$ for any integer $n \geq 0$, the series converges in norm.

Let X be a vector space, then A an operator on X signifies $A : D(A) \to X$, where the domain $D(A) \subset X$ is a subspace, usually not closed in X. The resolvent set $\rho(A)$ of numbers $\lambda \in \mathbb{C}$ satisfies (i) $\mathrm{Ran}(A - \lambda I) = X$ (ii) $(A - \lambda I) : D(A) \to X$ is injective (iii) $(A - \lambda I)^{-1} : \mathrm{Ran}(A - \lambda I) \to X$ is bounded. The spectrum of operator A is the set $\sigma(A) = \mathbb{C} \setminus \rho(A)$ and is made up of the disjoint union of the point spectrum $\sigma_p(A)$, $\lambda \in \mathbb{C}$ for which $A - \lambda I$ is not injective, the continuous spectrum $\sigma_c(A)$ such that $A - \lambda I$ is injective and $\mathrm{Ran}(A - \lambda I) = X$ but $(A - \lambda I)^{-1}$ is not bounded, and finally, $\sigma_r(A)$, $\lambda \in \mathbb{C}$ for which $A - \lambda I$ is injective but $\overline{\mathrm{Ran}(A - \lambda I)} \neq X$ [**4-6**].

Let A be an operator on the Hilbert space H, then a vector $\psi \in D(A)$ such that $A^n \psi \in D(A)$ for any $n \in \mathbb{N}$ is called a C^∞ vector for A, and $C^\infty(A)$ is the subspace of C^∞ vectors for A. A vector $\psi \in C^\infty(A)$ is an analytic vector for A if for aome $t > 0$

$$\sum_{n=0}^{\infty} \frac{||A^n \psi||}{n!} t^n < +\infty. \tag{1.1}$$

A vector $\psi \in C^\infty(A)$ is a vector of uniqueness for A if $A|_{D_\psi}$ is an essentially self-adjoint operator on the Hilbert space $H_\psi = \overline{D_\psi}$, where $D_\psi \subset H$ is the span of $A^n \psi$, for $n = 0, 1, 2, \ldots$. If ψ is an analytic vector for A, the series

$$\sum_{n=0}^{\infty} \frac{||A^n \psi||}{n!} x^n$$

converges for some $x > 0$. Convergence of power series results guarantee the complex series

$$\sum_{n=0}^{\infty} \frac{||A^n \psi||^n}{n!} z^n$$

converges absolutley for any $z \in \mathbb{C}$, $|z| < x$ and uniformly on $\{z \in \mathbb{C} | |z| < r\}$ for every $r < x$. Also the series of derivatives of any order

$$\sum_{n=0}^{\infty} \frac{||A^{n+p} \psi||}{n!} z^n,$$

converges for any given $p = 1, 2, 3, \ldots$ [**7-9**].

Theorem 1.1 Let A be a symmetric operator on the Hilbert space H. If $D(A)$ has a set of vectors of uniqueness whose linear span is dense in H, A is essentially self-adjoint.

Proof: It suffices to prove that the spaces $\mathrm{Ran}(A \pm iI)$ are dense, $\overline{\mathrm{Ran}(A + iI)} = H$. Given $\phi \in H$ and $\epsilon > 0$, there is a finite linear combination of vectors of uniqueness ψ_i such that $||\phi - \sum_{i=1}^n \alpha_i \psi_i|| < \epsilon/2$. Since $\psi_i \in H_\psi$ and $A|_{D_\psi}$ is essentially self-adjoint on this space $\overline{\mathrm{Ran}(A \pm iI)} = H$ implies there exists

$\eta_i \in H_\psi$ with $(A|_\psi + iI)\eta_i - \psi_i|| < (\epsilon/2)(\sum_{j=1}^{\infty} |\alpha_j|)^{-1}$. Setting $\eta = \sum_{i=1}^{N} \alpha_i \eta_i$ and $\psi = \sum_{i=1}^{N} \alpha_i \psi_i$, we have $\eta \in D(A)$ and

$$||(A + iI)\eta - \phi|| \leq ||(A|_{D_\psi} + iI)\eta - \psi|| + ||\phi - \psi|| < \epsilon.$$

As $\epsilon > 0$ is arbitrary, $\mathrm{Ran}(A + iI)$ is dense. The similar claim concerning $\mathrm{Ran}(A - iI)$ is similar, so A is essentially self-adjoint since $\overline{\mathrm{Ran}(A + iI)} = H$. $\square$

Theiorem 1.1 is used to prove Nelson's analytic vector theorem: Let A be a symmetric operator on the Hilbert space H. If $D(A)$ contains a set of analytic vectors for A whose span is dense in H, A is essentially self-adjoint.

Introduce the Schwarz space $S(\mathbb{R})$ of $\mathbb{R}$ which is the space of smooth complex functions that vanish at infinity along with any derivative, faster than any negative power of x. This acts as the domain of the operator to be defined. Let the complex Hilbert space $L^2(\mathbb{R}, d\mu)$ be given where $d\mu$ is Lebesgue measure on $\mathbb{R}$. The main object of interest is to consider the problem posed by the following operator H_0 defined on the space $S(\mathbb{R}) = D(H_0)$ and given by

$$H_0 = \frac{1}{2m} P^2 + \frac{1}{2} m\omega^2 X^2. \tag{1.2}$$

The operators X and P are related to the physical variables of position and momentum and can be interpreted as position and momentum for a particle moving in one-dimension.

A typical quantization scheme identifies P with the operator $-i\hbar\partial_x$ and X with multiplication by the coordinate x. This puts H_0 in the form **[3]**

$$H_0 = -\frac{\hbar^2}{2m}\,\partial_x^2 + \frac{1}{2} m\omega^2\, x^2, \tag{1.3}$$

where x^2 is multiplication by $\mathbb{R} \ni x \to x^2$ and $\hbar$, m and ω are real constants. In fact, H_0 is not an observable since it is not self-adjoint. It has closure $\bar{H}_0$ which is self-adjoint and can be considered the energy observable of the system.

Denote by $(|)$ the inner product throughout, such as the L^2 used below, but other Borel measures can be considered. If X is a locally compact Hausdorff space, a Borel measure on X is a positive σ-additive measure on the Borel sets of X. If $u, v \in C_0^2$ are C^2 functions with compact support, it follows that

$$(H_0 u|v) - (u|H_0 v) =$$

$$\int_{-\infty}^{\infty} [-\frac{\hbar^2}{2m} u''(x) + \frac{1}{2} m\omega^2 u(x)]\bar{v}(x) - u(x)[-\frac{\hbar^2}{2m}\bar{v}''(x) + \frac{1}{2} m\omega^2 \bar{v}(x)]\, dx$$

$$= \int_{-\infty}^{\infty} [-u''(x)\bar{v}(x) + u(x)\bar{v}''(x)]\, dx = 0.$$

Thus, the operator H_0 is symmetric as it is Hermitian and the Schwarz space is dense in $L^2(\mathbb{R}, d\mu)$. As it commutes with anti-unitary complex conjugates of L^2, it admits self-adjoint extensions by Von Neumann's criterion: Let A be a symmetric operator on the Hilbert space H. If there exists a conjugation $C : H \to H$ such that $CA \subset AC$, then A admits self-adjoint extensions. One of the objectives is to show H_0 is essentially self-adjoint and to obtain the spectrum and provide an explicit expression for it in terms of the spectral expansion of its unique self-adjoint extension $\bar{H}_0$. The Hamiltonian can be written in terms of of two operators which are usually called creation and annihilation operators. These operators are studied in detail. It is shown that the spectrum of the Hamiltonian is a discrete point spectrum, and an application of these operators to construct coherent states is given [**10-11**].

2 Basic Operators and Their Properties

Introduce the following pair of operators which are defined as [**4-5**]

$$a^* = \sqrt{\frac{m\omega}{2\hbar}}\big(x - \frac{\hbar}{m\omega}\partial_x\big), \qquad a = \sqrt{\frac{m\omega}{2\hbar}}\big(x + \frac{\hbar}{m\omega}\partial_x\big), \qquad \mathcal{N} = a^*\, a. \tag{2.1}$$

In the context of a physical model, these operators are called creation, annihilation operators and $\mathcal{N}$ the number operator. respectively. It can be assumed the operators are densely defined on the Schwarz space $S(\mathbb{R})$, so $D(a) = D(a^*) = D(\mathcal{N}) = S(\mathbb{R})$, where $S(\mathbb{R})$ is dense and invariant under H_0, a and a^*. A basis for the space $L^2(\mathbb{R}, d\mu)$ can be constructed from eigenvectors of $\mathcal{N}$ and H_0 by using a and a^*. These will all be analytic vectors by Nelson's criterion: Let A be a symmetric operator on the Hilbert space H, If $D(A)$ contains a set of analytic vectors for A where their span is dense in H, then A is essentially self-adjoint. Thus H_0 and $\mathcal{N}$ are essentially self-adjoint on $S(\mathbb{R})$. By (2.1), the commutation relation

$$[a, a^*] = I, \tag{2.2}$$

holds, where each side acts on the dense, invariant space $S(\mathbb{R})$. Clearly, it holds that

$$a^* a = \frac{m\omega}{2\hbar}\big(x^2 - \frac{\hbar}{m\omega} - x\frac{\hbar}{m\omega}\partial_x + x\frac{\hbar}{m\omega}\partial_x + \frac{\hbar^2}{(m\omega)^2}\partial_x^2\big)$$

and

$$\hbar\omega\,\mathcal{N} = -\frac{\hbar^2}{2m}\partial_x^2 + \frac{1}{2}m\omega^2\,x^2 - \frac{1}{2}\hbar\omega I. \tag{2.3}$$

Consequently, H_0 can be expressed in terms of the operator $\mathcal{N}$ as follows

$$H_0 = \hbar\omega\,(a^*a + \frac{1}{2}I) = \hbar\omega(\mathcal{N} + \frac{1}{2}I). \tag{2.4}$$

Consider the equation defined on $S(\mathbb{R})$ given by

$$a\,\psi_0 = 0. \tag{2.5}$$

Using (2.1) for a, this can be interpreted as a first order differential equation. A solution can be obtained for it,

$$\psi_0(x) = \frac{1}{\pi^{1/4}\sqrt{\sigma}} e^{-x^2/2\sigma^2}, \qquad \sigma = \sqrt{\frac{\hbar}{m\omega}}. \tag{2.6}$$

The constant in this solution enforces the constraint $||\psi_0|| = 1$. The function ψ_0 is a Hermite function in terms of the variable $y = x/\sigma$. A sequence of vectors can be constructed by applying the operator a^* repeatedly to ψ_0. The the functions which result are denoted ψ_n

$$\psi_n = \frac{1}{\sqrt{n!}} (a^*)^n\,\psi_0, \qquad n = 1, 2, \dots. \tag{2.7}$$

Theorem 2.1:

$$a\psi_n = \sqrt{n}\psi_{n-1}, \qquad a^*\psi_n = \sqrt{n+1}\,\psi_{n+1}, \qquad (\psi_n|\psi_m) = \delta_{nm}. \tag{2.8}$$

Proof: With (2.5) and the commutation relation (2.2), (2.8) can be proved by induction on n. Consider the first of these in (2.8).

$$a\,\psi_n = \frac{1}{\sqrt{n!}} a(a^*)^n\psi_0 = \frac{1}{\sqrt{n!}}[a, (a^*)^n]\psi_0 + \frac{1}{\sqrt{n!}}(a^*)^n a\psi_0. = \frac{1}{\sqrt{n!}}[a, (a^*)^n]\psi_0. \tag{2.9}$$

The bracket at the end of (2.9) is

$$[a, (a^*)^n] = a(a^*)^n - (a^*)^n a - aa^*(a^*)^{n-1} + a^*a(a^*)^{n-1} - a^*a(a^*)^{n-1} - (a^*)^n a. \tag{2.10}$$

Putting (2.10) into (2.9), we get

$$a\,\psi_n = \frac{1}{\sqrt{n!}} n(a^*)^{n-1}\psi_n = \frac{n}{\sqrt{n}}\frac{1}{\sqrt{(n-1)!}}(a^*)^{n-1}\,\psi_0 = \sqrt{n}\,\psi_{n-1}. \tag{2.11}$$

The second identity proceeds the same way. The third follows from

$$\begin{aligned}(\psi_m|\psi_n) &= \frac{1}{n!m}(\psi_{m-1}|a(a^*)^n\psi_0)\\ &= \frac{1}{\sqrt{n!m}}(\psi_{m-1}|[a, (a^*)^n]|\psi_0) = \frac{n}{\sqrt{n!m}}(\psi_{m-1}|(a^*)^{n-1}\psi_0)\\ &= \sqrt{\frac{n}{m}}(\psi_{m-1}|\psi_{n-1}) = \cdots = \sqrt{\frac{n!}{m!(n-m)!}}(\psi_0|\psi_{n-m}).\end{aligned} \tag{2.12}$$

If $m = n$, the result is one and zero otherwise, since $(\psi_0|\psi_{n-m}) = (n-m)^{1/2}(\psi_0|a^*\psi_{n-m-1}) = (n-m)^{-1/2}(a\psi_0|\psi_{n-m-1}) = 0$, using (2.5).
□

Since the ψ_n up to a constant and change of variable are Hermite functions, they form a basis of the Hilbert space. The inner product result in Theorem 2.1 implies that the set $\{\psi_n\}_{n\geq 0}$ forms an orthonormal system in $L^2(\mathbb{R}, d\mu)$ such that, by the first two results in (2.8),

$$\mathcal{N}\,\psi_n = n\,\psi_n. \tag{2.13}$$

Hence from (2.4) where H_0 is expressed in terms of $\mathcal{N}$, the $\{\psi_n\}$ are a Hilbert basis of eigenvectors of H_0 which satisfy the eigenvalue equation

$$H_0\psi_n = \hbar\omega(n+\frac{1}{2})\,\psi_n. \tag{2.14}$$

Since the set $\{||H_0\psi||\,|\psi \in D(H_0), ||\psi_0|| = 1\}$ consists of all numbers $\hbar\omega(n+1/2)$, both the operator H_0 and by (2.4) $\mathcal{N}$ are unbounded. By Nelson's theorem, the symmetric operators H_0 and $\mathcal{N}$ are both essentially self-adjoint, since their domains contain a set $\{\psi_n\}$ of analytic vectors spanning a dense subset in $L^2(\mathbb{R}, d\mu)$.

3 The Spectral Problem

The spectral decomposition of the operator $\bar{H}_0$ is determined. To obtain this, let us construct a spectral measure on $\mathbb{R}$ which has support on n with $n \in \mathbb{N}$ such that if $\mathcal{B}(X) = \mathcal{B}(X, X)$ is the subset of continuous operators,

$$\pi_F = s - \sum_{n\in F\cap\mathbb{N}} \psi_n\,(\psi_n|\cdot), \qquad F \in \mathcal{B}(\mathbb{R}). \tag{3.1}$$

The topology induced by the seminorms p_x with $p_x(T) = ||T(x)||_Y$ for $T \in \mathcal{L}(X, Y)$ or $\mathcal{B}(X, Y)$ is the strong topology on these spaces. To distinguish strong limits from weak limits in operator spaces, it is customary to write s, for example $T = s - \lim T_n$. Observables can be introduced by means of projector-valued measures (PVM). This concept resides at the heart of the mathematical foundation of quantum mechanics. The PMV obtained here cen be reinterpreted as a PVM defined on $\mathbb{N}$ identified with the collection $\{\psi_n\}$. Thus for any measurable map $f : \mathbb{R} \to \mathbb{C}$, it holds that

$$\int_{\mathbb{R}} f(x)\,d\pi(x) = \int_{\mathbb{N}} f(\phi(x))\,d\pi(x) = s - \sum_{n\geq 0} f(\hbar\omega(n+\frac{1}{2}))\,\psi_n(\psi_n|\cdot). \tag{3.2}$$

The last equality follows from the definition that is, if $\sum_{z\in N} |f(z)|^2|(z|\psi)|^2 < +\infty$,

$$\int_N f(x)\, d\pi(x) = s - \sum_{z\in N} f(z) z(z|\cdot),$$

on a basis N of a separable Hilbert space H. If the function f in (3.2) is taken to be $\mathbb{R} \ni x \to x$, so the following self-adjoint operator results

$$H = \int_{\mathbb{R}} x\, d\pi(x) = s - \sum_{n\geq 0} \hbar\omega(n+\frac{1}{2})\, \psi_n(\psi_n|\cdot). \tag{3.3}$$

At this point, it can be shown that $H = \overline{H_0}$. Thus, let $\langle N\rangle$ be the dense subspace spanned by finite linear combinations of the ψ_n. By Nelson's theorem, $H_0|_{\langle N\rangle}$ is still essentially self-adjoint, so

$$\overline{H_0} = \overline{H_0}|_{\langle N\rangle}. \tag{3.4}$$

It may be concluded that H_0 and $H_0|_{\langle N\rangle}$ have the same unique self-adjoint extension, their closure. However, H is certainly a self-adjoint extension of $H_0|_{\langle N\rangle}$ because (3.3) implies that

$$H\,\psi_n = (n+\frac{1}{2})\,\omega\psi_n = H_0\,\psi_n, \tag{3.5}$$

for any n, so $H|_{\langle N\rangle} = H_0|_{\langle N\rangle}$. This means H must be the unique self-adjoint extension of $H_0|_{\langle N\rangle}$, hence of H_0, which means that $H = \overline{H_0}$.

By the spectral theorem, decomposition of unbounded self-adjoint operators, if T is a self-adjoint, possibly unbounded, operator on the Hilbert space H, there exists a unique projector-valued measure such that $T = \int \lambda d\pi^{(T)}(\lambda)$. Thus, the spectral measure associated to $\bar{H}_0$ is $\mathcal{B}(\mathbb{R}) \ni F \to \pi_F$, and there exists the spectral decomposition of $\overline{H_0}$ given by

$$\overline{H_0} = s - \sum_{n\geq 0} \hbar\omega(n+\frac{1}{2})\, \psi_n(\psi_n|\cdot). \tag{3.6}$$

Moreover, the spectrum $\sigma(\bar{H}_0)$ is obtained as

$$\sigma(\overline{H_0}) = \sigma_p(\overline{H_0}) = \{\hbar\omega(n+\frac{1}{2})|n\geq 0\}. \tag{3.7}$$

The spectrum of $\overline{H_0}$ is therefore a point spectrum and the eigenspaces are all finite-dimensional, even though the operator is not compact, as it is unbounded, although the first and second inverse powers of $\overline{H_0}$ are compact.

Theorem 3.1. The operators a and a^* are closable and have the spectral properties $\sigma_p(\overline{a}) = \mathbb{C}$, $\sigma(\overline{a}) = \mathbb{C}$ and $\sigma_c(\overline{a}) = \sigma_r(\overline{a}) = \emptyset$.

Proof: The operators are closable because they admit closed extensions, as each is defined on a dense set and $a \subset (A^*)^*$, $a^* \subset a^*$. The Hilbert basis $\{\psi\}$ which was obtained in (2.7) can be used to construct explicitly an eigenvector ψ of $\overline{a}$ which satisifes the eigenvalue equation

$$\overline{a}\psi = \lambda\,\psi, \tag{3.8}$$

for every $\lambda \in \mathbb{C}$. To carry this out, define the expansion over $\{\psi_n\}$ to be

$$\psi = \sum_{n=0}^{\infty} b_n\,\psi_n. \tag{3.9}$$

Apply the operator a to both sides of (3.9) and substitute (2.8) so eigenvalue equation (3.8) becomes

$$\sum_{n=1}^{\infty} \sqrt{n}\,b_n\psi_{n-1} = \lambda \sum_{n=0}^{\infty} b_n\psi_n.$$

The sum can be reindexed and we obtain

$$\sum_{n=0}^{\infty} \sqrt{n+1}\,b_{n+1}\,\psi_n = \lambda \sum_{n=0}^{\infty} b_n\psi_n. \tag{3.10}$$

The following recursion relation for the b_n can be extracted from (3.10)

$$b_{n+1} = \frac{\lambda}{\sqrt{n+1}}\,b_n. \tag{3.11}$$

The recursion in (3.11) can be solved by iteration

$$b_n = \frac{\lambda}{\sqrt{n}}\,b_{n-1} = \frac{\lambda^2}{\sqrt{n(n-1)}}\,b_{n-2} = \cdots = \frac{\lambda^n}{\sqrt{n!}}\,b_0. \tag{3.12}$$

For real constant $b_0 \neq 0$, solution (3.9) takes the form,

$$\psi = b_0 \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}}\,\psi_n. \tag{3.13}$$

To verify that this satisfies the eigenvalue problem substitute (3.13) into (3.8)

$$\overline{a}\psi = b_0 \sum_{n=1}^{\infty} \frac{\lambda^n}{\sqrt{n!}}\sqrt{n}\psi_{n-1} = b_0 \sum_{n=1}^{\infty} \frac{\lambda^{n+1}}{\sqrt{n!}}\psi_n = \lambda\,\psi. \tag{3.14}$$

The series converges to a non-zero element of H which resides in $D(\overline{a})$ for $b_0 \neq 0$ and satisfies the eigenvalue equation. The fact that $\bar{a}\psi_0 = 0$ has been used, and $0 \in \sigma_p(\overline{a})$ as well. Thus the resolvent set is empty. The conclusion is then: $\sigma_c(\overline{a}) = \sigma_r(\overline{a}) = \emptyset$ and so $\sigma(\overline{a}) = \mathbb{C}$. □

Theorem 3.2. With respect to the operators a, a^*, let $\{\psi_n\}$ be the Hilbert basis constructed in (2.7). Then the following system of equalities holds: $a^* = \bar{a^*} = \bar{a}^* = a^{***}$. In particular, the identities are true when a symmetric. Morever,

$$\bar{a}\Big(\sum_{n=0}^{\infty} b_n\psi_n\Big) = \sum_{n=0}^{\infty} \sqrt{n+1}\, b_{n+1}\, \psi_n, \qquad a^*\Big(\sum_{n=0}^{\infty} b_n\psi_n\Big) = \sum_{n=1}^{\infty} \sqrt{n}\, b_{n+1}\, \psi_n. \tag{3.15}$$

The domains of the two operators $\bar{a}$ and a^* are given by

$$D(\overline{a}) = \{\psi \in H | \sum_{n=0}^{\infty} (n+1)|(\psi|\psi_{n+1})|^2 < +\infty\},$$

$$D(a^*) = \{\psi \in H | \sum_{n=1}^{\infty} n|(\psi|\psi_{n-1})|^2 < +\infty\}. \tag{3.16}$$

and $D(\overline{a}) = D(a^*)$.

Proof: These results follow from the definition of adjoint and the following theorem: If $(H, (,))$ is a Hilbert space, c an operator on H, then if $D(c)$ and $D(c^*)$ are dense, it follows that $c^* = \overline{c}^* = \bar{c^*} = c^{***}$. If $D(c)$ and $D(c^*)$ are dense, the operators c^*, c^{**}, c^{***} exist and in particular, $D(c) \subset D(c^{**})$ is dense. Since c^* is closed, $\bar{c^*} = c^*$. Take c to be operator a in these statements. Since it has been shown that a dense basis for H exists, for any $\psi \in H$, write the expansion $\psi = \sum_0^{\infty} c_n\psi_n$ respect to basis $\{\psi_n\}$ from (2.7). It follows that

$$\bar{a} \sum_{n=0}^{\infty} c_n\psi_n = \sum_{n=1}^{\infty} c_n\sqrt{n}\psi_{n-1} = \sum_{n=0}^{\infty} \sqrt{n+1}\, c_{n+1}\psi_n. \tag{3.17}$$

This series converges if and only if the sum $\sum_0^{\infty} (n+1)|(\psi_n|\psi_{n+1})|^2 < +\infty$, which characterizes $D(\bar{a})$. Using the definition of adjoint, with $\eta = (a^*)^*$

$$(\eta\psi|\psi) = ((a^*)^*\, \psi|\psi) = (\psi|a^*\psi). \tag{3.18}$$

the operator a^* can be identified

$$a^*\, \psi = \sum_{n=1}^{\infty} \sqrt{n}c_n\, \psi_{n-1}. \tag{3.19}$$

Moreover, this converges provided that $\sum_{n=1}^{\infty} n\, |(\psi|\psi_{n-1})|^2 < +\infty$. The relation $D(\bar{a}) = D(a^*)$ is also evident if one simply rearranges the expansions $D(\bar{a})$ and $D(a^*)$ for $\psi \in H$.

□

Corollary 3.1. For the operator $\mathcal{N}$ defined in terms of a and a^* by (2.1), it holds that

$$\mathcal{N} = a^* \, \overline{a} = \overline{a^* a}, \tag{3.20}$$

is the unique self-adjoint extension of the symmetric operator $\mathcal{N}$ defined on the space of vectors $\{\psi_n\}$ which satisfy the eigenvalue problem $\mathcal{N}\psi_n = n\psi_n$ for $n \in \mathbb{N}$.

Proof: The operator $a^*\overline{a}$ is the same as (2.1) on $S(\mathbb{R})$ and extends to all of the Hilbert space in the same way as in the proof of theorem 2. Moreover, $\overline{a^*} = a^*$, so it is the case that $\overline{a^* a} = a^* \overline{a}$.

□

4 Relationship of the Basic Operators to Coherent States

Applications of harmonic oscillators deal with the concept of coherent states. Coherent states are over-complete and non-orthogonal system of Hilbert space vectors. Thus at least one vector exists in the set which may be removed, leaving the system to remain complete.

An important subset of such wave functions was considered, related to the regular cell partition of the phase plane of a one-dimensional dynamical system and was given by von Neumann [**8-9**]. States which are eigenfunctions of the annihilation operator have been determined in (3.13).

Suppose an initial wavefunction of the form (3.13) called $\psi(x,0)$ here is considered. The wave function at time t is obtained by using the time-evolution operator on $\psi(x,0)$

$$\psi(x,t) = b_0 \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}} \psi_n \, e^{-i(n+1/2)\omega t} = b_0 e^{-i\omega t/2} \sum_{n=0}^{\infty} \frac{(\lambda e^{-i\omega t})^n}{\sqrt{n!}} \, \psi_n. \tag{4.1}$$

Up to the inessential phase factor, $\psi(x,t)$ and $\psi(x,0)$ have the same functional form. This leads to the statement that $\psi(x,t)$ is an eigenfunction of the annihilation operator with eigenvalue $\lambda e^{-i\omega t}$.

Let us now use (2.1) for the annihilation operator in terms of the position and momentum operators. Set $\mu = \sqrt{\hbar/2m\omega}$, then

$$a = \frac{1}{2\mu} \, x + \mu \, \frac{\partial}{\partial x}. \tag{4.2}$$

The eigenvalue equation for such an operator has been studied at $t = 0$, and so at t,

$$\bar{a}\psi(x,t) = \gamma \, \psi(x,t). \tag{4.3}$$

The complex eigenvalue γ is $\lambda\, e^{-i\omega t}$ and writing $\lambda = \rho e^{i\varphi}$, where ρ is real , γ becomes $\rho e^{i\vartheta}$ where $\vartheta = k - \omega t$. Given (4.2), (4.3) can be written in a separated form,

$$\psi(x,t) = C\, \exp\big(- \frac{x^2}{4\mu} + \frac{\gamma}{\mu} x \big). \tag{4.4}$$

The square of the modulus of $\psi(x,t)$ is given by

$$|\psi(x,t)|^2 = |C|^2 e^{-Q}, \tag{4.5}$$

where Q is upon completing the square

$$Q = \frac{x^2}{2\mu^2} - \frac{x}{\mu}(\gamma + \gamma^*) = \frac{1}{2\mu^2}\,(x - 2\mu\rho\cos\vartheta)^2 - 2\rho^2\cos^2\vartheta. \tag{4.6}$$

This produces the result for (4.5),

$$|\psi(x,t)|^2 = |C|^2 \cdot e^{2\rho^2\cos^2\vartheta} e^{-(x-x_0)^2/2\mu^2}, \qquad x_0 = 2\mu\rho\cos\vartheta. \tag{4.7}$$

As usual, the constant C which appears here can be evaluated by means of the normalization condition. The eigenfunction $\psi(x,t)$ is usually called a coherent state. The annihilation operator does not correspond to any physical observable, is not self-adjoint, hence its eigenvalues are in general complex. The eigenfunctions of the creation operator does not have L^2 eigenfunctions because solutions of the eigenvalue equation increase exponentially for large x, unlike this previous case. Coherent states are minimum-uncertainty states.

5 Conclusions

A variety of concepts from modern functional analysis have been introduced and applied to analyze a particular problem in quantum mechanics. A particular example appeared in the previous section, and further investigations of this type related to other systems can also be carried out along similar lines. The process demonstrates the influence and concurrent development of these two areas on each other. This is a process which often takes place in mathematical physics.

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