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A Note on the n-Closure and n-Interior Operators

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Abstract

The n-closure operator (nCl) and the n-interior operator ($nInt$) are used to develop several new classes of sets related to the n-open sets. These operators are used to prove a new characterization of discrete spaces. Additionally a new property of these operators is proved. Also the algebraic properties of these operators are investigated.

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1 Introduction

The concept of an n-open subset of a topological space was introduced in [1] and developed further in [2] and [3]. In this note the n-closure operator (nCl) and the n-interior operator ($nInt$) are used to define the semi-n-open sets, the pre-n-open sets, the regular n-open sets, and the regular n-closed sets. The basic properties of these sets are developed. A new characterization of discrete spaces is developed in terms of these operators. Specifically it is proved that a space X is discrete if and only if it has a subset A such that $nCl(A) = X$ and $nInt(A) = \emptyset$. A new property of these operators is proved. It is shown that for a subset A of a non-discrete space X, $nCl(nInt(A)) = nInt(nCl(A))$. Finally, these operators are considered from an algebraic viewpoint.

2 Preliminaries

Unless otherwise stated the symbol X represents a topological space (briefly a space). All topological spaces are nonempty with no separation properties assumed unless explicitly stated. The closure and interior of a set A are signified by $Cl(A)$ and $Int(A)$, respectively.

Definition 2.1 *Let X be a nonempty set and $\mathcal{P}(X)$ the power set of X. A subfamily m_X of $\mathcal{P}(X)$ is called a minimal structure (briefly an m-structure) on X [4], if $\emptyset \in m_X$ and $X \in m_X$.*

Definition 2.2 *A subset A of a space X is said to be n-open [1] if $Int(A) \neq Cl(A)$. A subset of X is called n-closed if its complement is n-open.*

Theorem 2.3 [1] *The following statements are equivalent for every set $A \subseteq X$:*

(a) *A is n-open.*

(b) *A is not clopen.*

(c) *$X - A$ is n-open.*

Thus the n-open sets coincide with the n-closed sets.

Remark 2.4 *Nether X nor $\emptyset$ is n-open. Therefore the collection of n-open sets does not form a minimal structure.*

Definition 2.5 *Let A be a subset of a space X. The n-interior of A [1] is denoted by $nInt(A)$ and given by $nInt(A) = \cup\{U \subseteq X : U \subseteq A$ and U is n-open$\}$. The n-closure of A [1] is denoted by $nCl(A)$ and given by $nCl(A) = \cap\{F \subseteq X : A \subseteq F$ and $F$is n-closed$\}$.*

Theorem 2.6 [1] *The following statements hold for every set $A \subseteq X$:*

(a) *$nInt(X - A) = X - nCl(A)$.*

(b) *$nCl(X - A) = X - nInt(A)$.*

(c) *$x \in nCl(A)$ if and only if $U \cap A \neq \emptyset$ for every n-open set U containing x.*

Theorem 2.7 [1] *If X is a space, then*

(a) *$nCl(X) = X$.*

(b) *$nInt(\emptyset) = \emptyset$.*

Theorem 2.8 [1] *If X is not discrete, then*

(a) $nInt(X) = X$.

(b) $nCl(\emptyset) = \emptyset$.

Theorem 2.9 [1] *If X is a discrete space, then*

(a) $nInt(A) = \emptyset$ *for every set* $A \subseteq X$.

(b) $nCl(A) = X$ *for every set* $A \subseteq X$.

Theorem 2.10 [2] *Let A be a subset of a space X. The following statements hold:*

(a) $nCl(A) = A$ *or* $nCl(A) = X$.

(b) $nInt(A) = A$ *or* $nInt(A) = \emptyset$.

(c) $nCl(nCl(A)) = nCl(A)$.

(d) $nInt(nInt(A)) = nInt(A)$.

Definition 2.11 *A subset A of a space X is said to be generalized n-closed (briefly gn-closed)* [2], *if whenever $A \subseteq U$ and U is open, then $nCl(A) \subseteq U$. A subset of X is called generalized n-open (briefly gn-open) if its complement is gn-closed.*

Theorem 2.12 [2] *Let A be a subset of a space X. The following statements hold:*

(a) *A is gn-closed if and only if $nCl(A) = A$.*

(b) *A is gn-open if and only if $nInt(A) = A$.*

(c) *$nCl(A)$ is gn-closed.*

(d) *$nInt(A)$ is gn-open.*

3 Semi-n-Open Sets and Pre-n-Open Sets

Definition 3.1 *A subset A of a space X is said to be semi-n-open if there exists a gn-open set U such that $U \subseteq A \subseteq nCl(U)$.*

Theorem 3.2 *Let A be a subset of a space X. Then A is semi-n-open if and only if $A \subseteq nCl(nInt(A))$.*

Proof. Assume $A \subseteq nCl(nInt(A))$. Then $nInt(A) \subseteq A \subseteq nCl(\text{nInt(A)})$ and by Theorem 2.12(d) $nInt(A)$ is gn-open. Therefore A is semi-n-open.

Assume A is semi-n-open and let U be a gn-open set such that $U \subseteq A \subseteq nCl(U)$. Therefore $U = nInt(U) \subseteq nInt(A)$ and hence $A \subseteq nCl(U) \subseteq nCl(nInt(A))$.

Theorem 3.3 *Let A be a subset of a space X. If A is gn-open, then A is semi-n-open.*

Proof. Since A is gn-open, by Theorem 2.12(b) $A = nInt(A)$. Then $A \subseteq nCl(A) = nCl(nInt(A))$. Thus A is semi-n-open.

Theorem 3.4 *Assume X is not discrete and let $A \subseteq X$. If A is semi-n-open, then A is gn-open.*

Proof. Assume A is semi-n-open. Then by Theorem 3.2 $A \subseteq nCl(nInt(A))$. It follows from Theorem 2.10(b) that $nInt(A) = A$ or $nInt(A) = \emptyset$. If $nInt(A) = A$, A is gn-open. If $nInt(A) = \emptyset$, then, since X is not discrete, $nCl(nInt(A)) = nCl(\emptyset) = \emptyset$. Since $A \subseteq nCl(nInt(A))$, $A = \emptyset$, which is gn-open. Therefore A is gn-open.

Corollary 3.5 *Assume X is not discrete and let $A \subseteq X$. Then A is semi-n-open if and only if A is gn-open.*

Remark 3.6 *If X is discrete, then $nCl(nInt(A)) = nCl(\emptyset) = X$ for every set $A \subseteq X$. Therefore every set is semi-n-open. However, $\emptyset$ is the only gn-open set. Thus, if X is discrete, gn-open $\Rightarrow$ semi-n-open, but semi-n-open $\not\Rightarrow$ gn-open.*

Definition 3.7 *A subset A of a space X is said to be pre-n-open if $A \subseteq nInt(nCl(A))$.*

Theorem 3.8 *Let A be a subset of a space X. If A is gn-open, then A is pre-n-open.*

Proof. Since A is gn-open, $A = nInt(A) \subseteq nInt(nCl(A))$. Thus A is pre-n-open.

Lemma 3.9 *A space X is discrete if and only if there exists a subset A of X such that $nInt(A) = \emptyset$ and $nCl(A) = X$.*

Proof. Let $A \subseteq X$ such that $nInt(A) = \emptyset$ and $nCl(A) = X$. Since $nInt(A) = \emptyset$, for every $x \in A$ $\{x\}$ is not n-open and hence must be clopen. Since $nCl(A) = X$, $nInt(X - A) = \emptyset$. Therefore for every $x \in X - A$ $\{x\}$ is not n-open and hence must be clopen. It follows that X is discrete.

If X is discrete, then every subset A of X satisfies the conditions that $\text{nInt(A)} = \emptyset$ and $nCl(A) = X$.

Theorem 3.10 *Let A be a subset of a space X. If A is pre-n-open, then A is gn-open.*

Proof. Assume A is pre-n-open. Then $A \subseteq nInt(nCl(A))$. If X is discrete, then $nInt(nCl(A)) = \emptyset$ and hence $A = \emptyset$, which is gn-open. Assume X is not discrete. By Theorem 2.10(a) $nCl(A) = A$ or $nCl(A) = X$. If $nCl(A) = A$, then $A \subseteq nInt(A)$ and hence $A = nInt(A)$. Therefore by Theorem 2.12(b) A is gn-open. If $nCl(A) = X$, then, since X is not discrete, it follows from Lemma 3.9 that $nInt(A) \neq \emptyset$. Thus $A = nInt(A)$ and hence A is gn-open. Therefore in either case A is gn-open.

Corollary 3.11 *Let A be a subset of a space X. Then A is pre-n-open if and only if A is gn-open.*

Corollary 3.12 *Assume X is not discrete and let $A \subseteq X$. Then A is pre-n-open if and only if A is semi-n-open.*

Remark 3.13 *If X is discrete, the only pre-n-open set is $\emptyset$ and every every set is semi-n-open. Therefore, if X is discrete, then pre-n-open $\Rightarrow$ semi-n-open, but semi-n-open $\not\Rightarrow$ pre-n-open.*

Theorem 3.14 *Assume X is not discrete. Let $A \subseteq X$. If A is not gn-closed, then A is gn-open.*

Proof. Assume A is not gn-closed. Then $nCl(A) \neq A$ and therefore $nCl(A) = X$. It follows from Lemma 3.9 that $nInt(A) \neq \emptyset$ and hence $nInt(A) = A$. Thus A is gn-open.

Since an n-open set is both gn-open and gn-closed, the converse of Theorem 3.14 does not hold. It follows from Theorem 3.14 that, if X is not discrete, then every subset of X is either gn-open or gn-closed.

Theorem 3.15 *If X is not discrete, then $nInt(nCl(A)) = nCl(nInt(A))$ for every $A \subseteq X$*

Proof. Let $A \subseteq X$. It follows from Theorem 2.10(a) and Theorem 2.10(b) that $nCl(A) = A$ or $nCl(A) = X$ and $nInt(A) = A$ or $nInt(A) = \emptyset$. It follows from Lemma 3.9 that it is not the case that $nCl(A) = X$ and $nInt(A) = \emptyset$. Therefore one of the following cases holds:

(1) $nCl(A) = A$ and $nInt(A) = A$.

(2) $nCl(A) = A$ and $nInt(A) = \emptyset$.

(3) $nCl(A) = X$ and $nInt(A) = A$.

If (1) holds, then $nCl(nInt(A)) = nCl(A) = A$ and $nInt(nCl(A)) = nInt(A) = A$. If we assume (2), then $nCl(nInt(A)) = nCl(\emptyset) = \emptyset$ and $nInt(nCl(A)) = nInt(A) = \emptyset$. Similarly, if (3) holds, then $nCl(nInt(A)) = nCl(A) = X$ and $nInt(nCl(A)) = nInt(X) = X$. In each case $nInt(nCl(A)) = nCl(nInt(A))$.

Remark 3.16 *We previously proved indirectly that, if X is not discrete and $A \subseteq X$, then A is semi-n-open if and only if A is pre-n-open by showing that both of the conditions are equivalent to gn-open. Obviously, Theorem 3.15 provides an immediate direct proof of this result.*

Definition 3.17 *A subset A of a space X is said to be regular n-open if $A = nInt(nCl(A))$ and regular n-closed if $A = nCl(nInt(A))$.*

Obviously regular n-open $\Rightarrow$ pre-n-open and regular n-closed $\Rightarrow$ semi-n-open. The following example shows that these implications cannot be reversed.

Example 3.18 *Let $X = \{a, b, c\}$ have the topology $\tau = \{X, \emptyset, \{a, b\}, \{c\}\}$. The n-open sets are $\{a\}, \{b\}, \{a, c\}$, and $\{b, c\}$. The set $A = \{a, b\}$ is neither regular n-open nor regular n-closed but is both semi-n-open and pre-n-open, since $nCl(nInt(A)) = nInt(nCl(A)) = X$.*

It follows immediately from Theorem 3.15 that, if X is not discrete, then a set A is regular n-open if and only if it is regular n-closed. However, if X is discrete, then, since $nCl(nInt(A)) = X$ and $nInt(nCl(A)) = \emptyset$ for every set $A \subseteq X$, X is the only regular-n-closed set and $\emptyset$ is the only regular n-open set.

The implications in the following diagrams have been established:

Assume X is not discrete.

regular n-open
$\Updownarrow$
regular n-closed
$\Downarrow$
gn-open $\Leftrightarrow$ semi-n-open $\Leftrightarrow$ pre-n-open
$\Uparrow$
not gn-closed

Assume X is discrete.

$$\begin{array}{ccccc} & & \text{regular } n\text{-open} & & \\ & & \Updownarrow & & \\ gn\text{-open} & \Leftrightarrow & \text{pre-}n\text{-open} & \Rightarrow & \text{semi-}n\text{-open} \\ & & & & \Uparrow \\ & & & & \text{regular } n\text{-closed} \\ & & & & \Updownarrow \\ & & & & gn\text{-closed} \end{array}$$

4 Algebraic Properties

A monoid is a semigroup with identity. In this section the n-closure and n-interior operators are considered to be elements of the monoid $\mathcal{O}(X) = \{F : \mathcal{P}(X) \to \mathcal{P}(X) : F \text{ is a function}\}$ under the operation of composition. The identity operator is denoted by I.

Definition 4.1 *A space X is said to be gn-discrete if every set $A \subseteq X$ is gn-open.*

Example 4.2 *If X is a connected space, then every proper nonempty subset of X is non clopen, hence n-open and therefore gn-open. Since X and $\emptyset$ are gn-open, X is gn-discrete.*

Lemma 4.3 *The operators I, nCl, $nInt$, and $nCl \circ nInt$ on a space X are all distinct if and only if X is not discrete and not gn-discrete.*

Proof. Assume X is not discrete and not gn-discrete. To begin with we show that I, nCl, and $nInt$ are distinct. Since X is not gn-discrete, there exists a set A that is not gn-open. Then by Theorem 2.12(b) $nInt(A) \neq A$. Therefore $nInt \neq I$. It follows from Theorem 3.14 that A is gn-closed and hence by Theorem 2.12(a) $nCl(A) = A$. Therefore $nInt \neq nCl$. Let B be the complement of A. Then B is not gn-closed and $nCl(B) \neq B$ and hence $nCl \neq I$.

Next it is established that $nCl \circ nInt \neq nInt$. Since $nCl(B) \neq B$, by Theorem 2.10(a) $nCl(B) = X$ Then, since X is not discrete, it follows from Lemma 3.9 that $nInt(B) \neq \emptyset$. Hence $nInt(B) = B$ and we have $nCl \circ nInt(B) = nCl(nInt(B)) = nCl(B) = X$. Since B is not gn-closed and X is gn-closed, $B \neq X$. Then $nInt(B) \neq X$ and hence $nCl \circ nInt \neq nInt$.

Finally we show that $nCl \circ nInt$ is distinct from nCl and I. Since $nInt(A) \neq A$, then $nInt(A) = \emptyset$. Hence we have $nCl(nInt(A)) = nCl(\emptyset) = \emptyset$. Since A is not gn-open and $\emptyset$ is gn-open, $A \neq \emptyset$. Thus $nCl \circ nInt \neq I$ and, since

$nCl(A) \neq \emptyset$, $nCl \circ nInt \neq nCl$. Therefore the operators I, nCl, $nInt$, and $nCl \circ nInt$ are all distinct.

Assume X is either discrete or gn-discrete. If X is discrete, then for every $A \subseteq X$ $nCl \circ nInt(A) = nCl(nInt(A)) = nCl(\emptyset) = X = nCl(A)$ and hence $nCl \circ nInt = nCl$. If X is gn-discrete, then every set $A \subseteq X$ is both gn-open and gn-closed. Therefore by Theorem 2.12(a), (b) for every set $A \subseteq X$ $nCl(A) = A$ and $nInt(A) = A$. It then follows that $nCl = nInt = nCl \circ nInt = I$. Thus in either case the operators I, nCl, $nInt$, and $nCl \circ nInt$ are not distinct.

The space (X, τ) in Example 3.18 is not discrete and also not gn-discrete, since the set $\{c\}$ is not gn-open.

Since a connected space is not discrete but is gn-discrete, not discrete $\not\Rightarrow$ not gn-discrete. Since a discrete space is also not gn-discrete, not gn-discrete $\not\Rightarrow$ not discrete. Thus not gn-discrete and not discrete are independent properties and hence Lemma 4.3 is a decomposition of the property that the operators I, nCl, $nInt$, and $nCl \circ nInt$ on a space X are all distinct.

Theorem 4.4 *If X is not discrete and not gn-discrete, then $M = \{I, nCl, nInt, nCl \circ nInt\}$ is a commutative monoid.*

Proof. By Lemma 4.3 the operators I, nCl, $nInt$, and $nCl \circ nInt$ are all distinct. The closure and commutativity follow from Theorem 3.15 ($nCl \circ nInt = nInt \circ nCl$), Theorem 2.10(c) ($nCl \circ nCl = nCl$), and Theorem 2.10(d) ($nInt \circ nInt = nInt$).

The next theorem is a consequence of Theorem 2.9.

Theorem 4.5 *If X is discrete, the set $M = \{I, nCl, nInt\}$ is a non commutative monoid.*

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