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On the Calderón-Zygmund Singular Integral Operators and Local Hardy-type Amalgam Spaces (Part 1)

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Abstract

In this paper, we prove that Calderón-Zygmund singular integral operators are bounded from local Hardy-type amalgam space $l^q(h^p)$ into $l^p(h^p)$ where $1 < q < \infty$ and $0 < p \leq q$.

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1 Introduction

Amalgam spaces are defined as follows:

$$l^q(L^p) \stackrel{\text{def}}{=} \left\{ f\colon \|f\|_{l^q(L^p)} = \left[\sum_{Q \in \mathcal{Q}} \|f\|_{L^p(Q)}^q \right]^{\frac{1}{q}} < \infty \right\},$$

where $0 < p, q < \infty$, $\mathcal{Q}$ is the collection of all cubes with uniform side length 1 and their vertices on integral lattices of $\mathbb{R}^n$. One of the main reason to study the amalgam spaces is that they allow us to separate the global behaviour from the local behaviour of a function. In [14] the author proved that pseudo-differential operators of order zero simultaneously preserve local and global behaviour of functions in the amalgam spaces in the sense that they are bounded from $l^q(L^p)$ into $l^q(L^p)$, where $0 < q \leq 1 < p < \infty$. In [15], the author proved that pseudo-differential operators of order zero are bounded from local Hardy-type amalgam space $l^q(h^p)$ into $l^q(h^p)$ where $0 < q \leq p \leq 1$. In [16], the author proved that Calderón-Zygmund singular integral operators are bounded from $l^1(L^p)$ into $weak - l^1(L^p)$. In this paper, we prove that Calderón-Zygmund singular integral operators are bounded from local Hardy-type amalgam space $l^q(h^p)$ into $l^p(h^p)$ where $1 < q < \infty$ and $0 < p \leq q$. The main results in this paper serve as a kind of preparation for proving the boundedness of Calderón-Zygmund singular integral operators from $l^1(h^p)$ into $weak - l^1(h^p)$, where $0 < p \leq 1$, which will appear in a separate paper [17].

2 Preliminary

Throughout this paper, we will adopt the following notations.

(a) If $1 \leqslant p \leqslant \infty$, then p' is denoted as the exponent conjugate to p. $Q(x,r)$ is the closed cube centered at x with side length r. $l(Q)$ is the side length of the cube Q. $B(x,r)$ is an open ball centered at x with radius r. If $t > 0, B$ is an open ball and Q is a closed cube in $\mathbb{R}^{\mathbf{n}}$, then tB, tQ have the same centers as B, Q respectively but whose radius and side length are expanded by the factor t. If E is a subset of $\mathbb{R}^{\mathbf{n}}$, then the complement of E in $\mathbb{R}^{\mathbf{n}}$ will be denoted as ${}^cE$. m_n is the usual Lebesgue measure on $\mathbb{R}^{\mathbf{n}}$.

(b) L^p will indicate the usual function spaces on $\mathbb{R}^{\mathbf{n}}$. L^p_c, C_c, C^∞_c are denoted as the spaces of all compactly supported L^p-functions, continuous functions and smooth functions respectively. $\mathcal{S}$ is the Schwartz class of all rapidly decreasing functions. Its dual space consists of all tempered distributions, denoted as $\mathcal{S}'$.

(c) Let $A^p_r f(x)$ be the average of f over the closed cube $Q(x,r)$ in p^{th}-mean, i.e. $A^p_r f(x) = \left[\frac{1}{r^n}\int_{Q(x,r)} |f(\omega)|^p \, d\omega\right]^{\frac{1}{p}}$ provided that $0 < p < \infty$. When $p = \infty$, we define $A^\infty_r f(x) = \|f\|_{L^\infty(Q(x,r))}$. When $r = 1$, we simply write $A^p f(x)$.

(d) The Hardy-Littlewood maximal function and its uncentered version are defined as

$$Mf(x) = \sup_{r>0} \frac{1}{c_n \, r^n} \int_{B(x,r)} |f| dm_n \ , \quad \widetilde{M}f(x) = \sup_{x \in B(y,r)} \frac{1}{c_n \, r^n} \int_{B(y,r)} |f| dm_n,$$

respectively, where c_n is the volume of the unit ball.

(e) $l^q(L^p) \stackrel{\text{def}}{=} \left\{ f\colon \|f\|_{l^q(L^p)} \stackrel{def}{=} \left[\sum_{Q\in\mathcal{Q}} \|f\|^q_{L^p(Q)}\right]^{\frac{1}{q}} < +\infty \right\}$, where $0 < p, q < \infty$, $\mathcal{Q}$ is the collection of closed cubes with uniform side length 1 and their vertices on integral lattices of $\mathbb{R}^{\mathbf{n}}$. If $q = \infty$, then $\|f\|_{l^\infty(L^p)} = \sup_{Q\in\mathcal{Q}} \|f\|_{L^p(Q)}$.

The following four lemmas will be frequently used throughout the thesis.

Lemma 2.1 *Let $f : \mathbb{R}^{\mathbf{n}} \to \mathbb{C}$ be a Borel measurable function, λ, $r > 0, k \in \mathbf{Z}^+, 0 < p < \infty$. Then*

$$|\{\ A^p_r f > \lambda\ \}| \leqslant c\ 2^{kn}\ |\{\ A^p_{r\cdot 2^{-k}} f > 2^{-n/p}\ \lambda\ \}|,$$

where c is a constant independent of f, r and λ.

Proof The reader may refer to [14].

Lemma 2.2 (Bertrandias, Datry, Dupuis [2]) *Suppose that $0 < p, q < \infty$. Then there exist two numbers c_1, c_2 dependent only on p, q, n such that for each $f\colon \mathbb{R}^{\mathbf{n}} \to \mathbb{C}$ be Borel measurable,*

$$c_1 \|A^p f\|_q \leqslant \left[\sum_{Q\in\mathcal{Q}} \left(\|f\|_{L^p(Q)}\right)^q\right]^{\frac{1}{q}} \leqslant c_2 \|A^p f\|_q.$$

Lemma 2.3 *Let $f : \mathbb{R}^{\mathbf{n}} \to \mathbb{C}$ be a Borel measurable function, $r > 0, k \in \mathbf{Z}^+, 0 < p, q < \infty$. Then*

$$\|A^p_r f\|_q \leqslant c(2^{kn})^{\frac{1}{q}} \|A^p_{r\cdot 2^{-k}} f\|_q,$$

where c is a constant independent of f and r but dependent on p, q.

Proof It is a consequence of Lemma 2.1 or 2.2. The proof is complete. ▮

Lemma 2.4 (Maximal Theorem) *The uncentered maximal operator $\widetilde{M}$ is bounded from L^1 into weak-L^1 and from L^p into L^p for each $1 < p < \infty$.*

Proof For the proof, the reader can refer to [21]. ▮

The class of operators which we will consider in this thesis is defined as follows. We call operators in this class to be the Calderón-Zygmund singular integral operators throughout the thesis.

Definition 2.5 (The Calderón-Zygmund Singular Integral Operators) *Let T be a bounded linear operator on L^2 that are translation-invariant. It is a well-known fact (see Stein [21]) that T is representable as*

$$\widehat{Tf}(\xi) = m(\xi) f(\xi) \quad \text{and} \quad |m(\xi)| \leqslant A.$$

for all $f \in L^2$. Now let W be the tempered distribution with $\hat{W} = m$. Then we have, for all $f \in \mathcal{S}$,

$$Tf = W * f.$$

We make the a priori assumption that the distribution W coincides, away from the origin, with a locally integrable function K. Then we have

$$Tf(x) = \int K(x-y)f(y)dy,$$

for all $f \in L^2_c$ and almost all $x \notin supp(f)$. Moreover we assume that

$$|K(x-y) - K(x-y')| \leqslant c\, \frac{|y-y'|^\delta}{|x-y|^{n+\delta}} \quad \text{for all } |x-y| \geqslant 2|y-y'|,$$

where $\delta \in (0,1]$ and $c > 0$ are fixed constants. Then we call the operator T to be a Calderón-Zygmund singular integral operator.

Such operator was known to be bounded on $L^p (1 < p < \infty)$ and from L^1 into *weak*-L^1 (see Stein [21]). The typical examples are the Hilbert transform on $\mathbb{R}^1$, whose kernel is $\frac{1}{x}$, and its n-dimensional analogue, the Riesz transforms on $\mathbb{R}^n$. The kernel of the j^{th} Riesz transform is $\frac{y_j}{|y|^{n+1}}$, where $1 \leqslant j \leqslant n$.

3 Local Hardy-Type Amalgam Spaces $l^q(h^p),\ 1 \leqslant q < \infty,\ 0 < p \leqslant q$

3.1 Definition of $l^q(h^p),\ 1 \leqslant q < \infty,\ 0 < p \leqslant q$

Definition 3.1.1 *Suppose $f \in \mathcal{S}'$, $x \in \mathbb{R}^n$.*

(a) *Let $F_N = \{\psi \in C_0^\infty(B(0,1)) : \|\partial^\beta \psi\|_\infty \leqslant 1,\ |\beta| \leqslant N\}$,*

(i) $M_1^{F_N} f(x) = \sup\limits_{0<t<1} \sup\limits_{\psi \in F_N} |f * \psi_t(x)|$,

(ii) $M^{F_N} f(x) = \sup\limits_{t>0} \sup\limits_{\psi \in F_N} |f * \psi_t(x)|$.

(b) *Let $\phi \in \mathcal{S}$ with $\int \phi \neq 0$.*

(i) $M_1^{\phi} f(x) = \sup\limits_{0<t<1} |f * \phi_t(x)|$,

(ii) $M^{\phi} f(x) = \sup\limits_{t>0} |f * \phi_t(x)|$.

Definition 3.1.2 *Let $1 \leqslant q < \infty,\ 0 < p \leqslant q$. We denote $l^q(h^p) = \{f$ is a distribution : $A^p(M_1^{F_N} f) \in L^q\}$, where F_N is the collection of smooth functions defined in Definition 3.1.1 and $N = max\{0, [n(\frac{1}{p} - 1)] + 1\}$, $[n(\frac{1}{p} - 1)]$ is the integral part of $n(\frac{1}{p} - 1)$.*

Throughout this section, N will be used to denote the number $\max\{0, [n(\frac{1}{p} - 1)] + 1\}$. We remark that with this choice of N, it is not difficult to verify that for each local $(2,p)$-atom a (see definition in Goldberg [12]), $\|M_1^{F_N} a\|_{L^p} \leqslant c$ where c is a constant independent of the choice of a. As a consequence, for $f \in h^p$, $\|M_1^{F_N} f\|_{L^p} \simeq \|f\|_{h^p}$. Besides, on this range of norm indices p, q, the basic assumption on f is that f is a distribution which, by definition, is a continuous linear functional on C_c^∞. We recall a characterization of distributions. A linear functional f on C_c^∞ is a distribution if and only if for each compact set $K \subseteq \mathbb{R}^{\mathbf{n}}$, there exist $C > 0$ and $m \in \mathbf{Z}^+$ such that

$$|f(\phi)| \leqslant C \sum_{|\alpha| \leqslant m} \|\partial^\alpha \phi\|_\infty \quad \text{whenever} \quad f \in C^\infty \quad \text{supported inside } K.$$

Lemma 3.1.3 *Suppose that* (i) $\phi \in C_0^\infty(B(0,1))$*, for each* $|\alpha| \leqslant N$, $\|\partial^\alpha \phi\|_\infty \leqslant 1$, (ii) $\eta \in C_0^\infty(Q_\eta)$ *where* Q_η *is a cube with* $l(Q_\eta) \geqslant 1$*, for each* $|\alpha| \leqslant N$, $\|\partial^\alpha \eta\|_\infty \leqslant 1$. *Then there exists a constant c (dependent only on N) such that*

$$M_1^\phi (f\eta)(x) \leqslant c \cdot \chi_{3Q_\eta}(x) M_1^{F_N} f(x), \qquad \textit{whenever } f \in \mathcal{S}', x \in \mathbb{R}^{\mathbf{n}},$$

where by definition, $(f\eta)(\varphi) := f(\eta\varphi)$ *for all* $\varphi \in \mathcal{S}$.

Proof Noticing that for all $x, \omega \in \mathbb{R}^{\mathbf{n}},\ t \in (0,1)$,

$$\begin{aligned} \eta(\omega)\phi_t(x-\omega) &= \eta_{x,t}(\frac{x-\omega}{t})\frac{1}{t^n}\phi(\frac{x-\omega}{t}) \qquad \text{where } \eta_{x,t}(\xi) = \eta(x - t\xi), \\ &= \frac{1}{t^n}(\eta_{x,t}\cdot\phi)(\frac{x-\omega}{t}). \end{aligned}$$

For all multi-index $|\beta| \leq N,\ \omega \in \mathbb{R}^{\mathbf{n}},\ t \in (0,1)$,

$$\begin{aligned} \partial_\omega^\beta(\eta_{x,t}\cdot\phi)(\omega) &= \sum_{\gamma \leqslant \beta} \binom{\beta}{\gamma} (\partial_\omega^\gamma \eta_{x,t})(\omega)(\partial_\omega^{\beta-\gamma}\phi)(\omega), \\ &= \sum_{\gamma \leqslant \beta} \binom{\beta}{\gamma} (-t)^{|\gamma|}(\partial_\omega^\gamma \eta)(x - t\omega)(\partial_\omega^{\beta-\gamma}\phi)(\omega). \end{aligned}$$

It follows that $\|\partial_\omega^\beta(\eta_{x,t}\cdot\phi)\|_\infty \leqslant \sum_{\gamma\leqslant\beta}\binom{\beta}{\gamma} \leqslant c_N$ because $\mathrm{supp}(\phi) \subseteq B(0,1),\ \forall |\gamma| \leqslant N,\ \|\partial^\gamma \eta\|_\infty,\ \|\partial^\gamma\phi\|_\infty \leqslant 1$ and $0 < t < 1$. So $\frac{1}{c_N}(\eta_{x,t}\cdot\phi) \in F_N$. Moreover $\mathrm{supp}(M_1^\phi(f\eta)) \subseteq 3Q_\eta$. Consequently, for all $x \in \mathbb{R}^{\mathbf{n}}$, $M_1^\phi(f\eta)(x) \leqslant c_N \cdot \chi_{3Q_\eta}(x)(M_1^{F_N} f)(x)$. The proof of the lemma is complete. ∎

We first start with a decomposition theorem for distributions in $l^q(h^p)$, in which the building blocks are required to be localized in some dyadic cubes.

As a consequence, we show that C_c^∞ is a dense subspace of $l^q(h^p)$, and each distribution in $l^q(h^p)$ is locally a h^p-distribution.

To state the lemma below, we introduce a smooth partition of unity as below: Let $\mathcal{Q}$ be a prescribed collection of closed cubes with uniform side length 1 whose interiors do not intersect with each other and union is equal to $\mathbb{R}^{\mathbf{n}}$. Denote $x_k =$ centre of Q_k, $Q_k \in \mathcal{Q}$, $k = 1, 2, 3, \ldots$ Take a C^∞-function ψ such that $0 \leqslant \psi \leqslant 1$ on $\mathbb{R}^{\mathbf{n}}$, $\psi \equiv 1$ on $\overline{Q(0,1)}$ and $\operatorname{supp}(\psi) \subseteq \overline{Q(0,2)}$. Define $\eta_k(x) = \frac{\psi(x-x_k)}{\sum_{j=1}^{\infty} \psi(x-x_j)}$ for each $x \in \mathbb{R}^{\mathbf{n}}$, $k = 1, 2, \ldots$. Then for each $x \in \mathbb{R}^{\mathbf{n}}$, $\sum_{j=1}^{\infty} \psi(x - x_j) \geqslant 1$. Moreover, for each α, k, $\|\partial^\alpha \eta_k\|_\infty \leqslant c_\alpha \|\partial^\alpha \psi\|_\infty$ where c_α is dependent on α and the dimension n only.

Lemma 3.1.4 *Let $1 \leqslant q < \infty$, $0 < p \leqslant q$. Suppose that $\{\eta_k\}_1^\infty$ is the smooth partition of unity stated in the previous paragraph. If f is a distribution, then the following three quantities are mutually comparable with bounds independent of f:*

(i) $\|A^p(M_1^{F_N} f)\|_{L^q}$,

(ii) $[\sum_{k=1}^\infty \|f\eta_k\|_{h^p}^q]^{\frac{1}{q}}$,

(iii) $\inf\{[\sum |\lambda_Q|^q]^{\frac{1}{q}} : f = \sum_{Q \in \mathcal{Q}} \lambda_Q a_Q,\ \|a_Q\|_{h^p} \leqslant 1,\ \operatorname{supp}(a_Q) \subseteq 4Q\}$, *where the convergence is taken in the sense of distribution.*

Proof Denote $N_1(f), N_2(f), N_3(f)$ to be the expressions in (i), (ii), (iii) respectively. For the proof of $N_2(f) \leqslant cN_1(f)$ where c being independent of the choice of f, it is already contained in the proof of Theorem 3.3.2 in [15]. We will not repeat it here. Besides, it is trivial to see that $N_3(f) \leqslant N_2(f)$. What we are going to show is that $N_1(f) \leqslant cN_3(f)$. Now let $f = \sum_{Q \in \mathcal{Q}} \lambda_Q a_Q$ where a_Q 's are satisfied the above mentioned properties. Then case 1: $1 \leqslant p \leqslant q$. We have

$$\begin{aligned}
\int_{\mathbb{R}^{\mathbf{n}}} |A^p(M_1^{F_N} f)(y)|^q dy &= \sum_{V \in \mathcal{Q}} \int_V |A^p(M_1^{F_N} f)(y)|^q dy, \\
&\leqslant \sum_{V \in \mathcal{Q}} \int_V |\sum_{Q \in \mathcal{Q}} A^p(M_1^{F_N}(\lambda_Q a_Q))(y)|^q dy, \\
&= \sum_{V \in \mathcal{Q}} \int_V |\sum_{4Q \cap 4V \neq \emptyset} A^p(M_1^{F_N}(\lambda_Q a_Q))(y)|^q dy, \\
&\leqslant \sum_{V \in \mathcal{Q}} (\sum_{4Q \cap 4V \neq \emptyset} \|\lambda_Q a_Q\|_{h^p})^q, \\
&\leqslant c_{n,q} \sum_{V \in \mathcal{Q}} |\lambda_V|^q,
\end{aligned}$$

where $c_{n,q}$ only depends on the dimension n and norm q. Case 2: $0 < p < 1$. We have

$$\begin{aligned}
\int_{\mathbb{R}^n} |A^p(M_1^{F_N} f)(y)|^q dy &= \sum_{V\in\mathcal{Q}} \int_V |A^p(M_1^{F_N} f)(y)|^q dy, \\
&\leqslant \sum_{V\in\mathcal{Q}} \int_V |\sum_{Q\in\mathcal{Q}} |\lambda_Q|^p \int_{Q(y,1)} |M_1^{F_N} a_Q|^p \, dm_n \,|^{\frac{q}{p}} \, dy, \\
&= \sum_{V\in\mathcal{Q}} \int_V |\sum_{4Q\cap 4V\neq\emptyset} |\lambda_Q|^p \int_{Q(y,1)} |M_1^{F_N} a_Q|^p \, dm_n \,|^{\frac{q}{p}} \, dy, \\
&\leqslant \sum_{V\in\mathcal{Q}} (\sum_{4Q\cap 4V\neq\emptyset} |\lambda_Q|^p)^{\frac{q}{p}}, \\
&\leqslant \sum_{V\in\mathcal{Q}} (m^{\frac{q}{p}-1} \sum_{4V\cap 4Q\neq\emptyset} |\lambda_Q|^q), \\
&\leqslant m^{\frac{q}{p}} \sum_{V\in\mathcal{Q}} |\lambda_V|^q,
\end{aligned}$$

where m is the number of all distinct elements in $\{Q \in \mathcal{Q} : 4Q \cap 4V \neq \emptyset\}$. Note that m is independent of the chioce of V. The second last inequality follows from the Hölder's inequality and the fact that $p \leqslant q$. This completes the proof of the lemma. ▮

As a consequence of the above lemma, each distribution in $l^q(h^p)$ $(1 \leqslant p \leqslant q < \infty)$ is locally an L^p-function because each $f\eta_k \in L^p$. As a result, it is easy to see that $l^q(h^p)$ is a Banach space. Moreover since $\|f\eta_k\|_{h^p} \simeq \|f\eta_k\|_{L^p}$ where $1 < p \leqslant q < \infty$, $\|A^p(M_1^{F_N} f)\|_q \simeq \|A^p f\|_q$. For $0 < p \leqslant 1 \leqslant q < \infty$, $(l^q(h^p), \|\cdot\|^p_{l^q(h^p)})$ defines a topological vector space with a complete translation invariant metric, $d(f,g) = \|f-g\|^p_{l^q(h^p)}$. Besides, we can show that C_c^∞ is a dense subspace of $l^q(h^p)$. Indeed, it is obvious that $\|f - \sum_{k=1}^N f\eta_k\|_{l^q(h^p)} \to 0$ as $N \to \infty$ by the previous lemma. Together with the fact that h^p space is stable under multiplication by $\mathcal{S}$ (see Goldberg [12]), each $f\eta_k$ can be approximated in h^p-norm by smooth functions which are compactly supported inside $4Q_k$ $(\mathrm{supp}(f\eta_k) \subset 2Q_k)$. Finally by the previous lemma, f can be approximated in $l^q(h^p)$-norm by compactly supported smooth functions.

In the rest of this section, we consider $A_r^p f(x)$ as the average of f over the open ball $B(x,r)$.

3.2 The Calderón-Zygmund Singular Integral Operators on $l^q(h^p)$, $1 < q < \infty$, $0 < p \leqslant q$

Proposition 3.2.1 *Let $0 < p < \infty$. Then there exists a constant $c > 0$ (dependent only on p) such that*

$$\sup_{\lambda>0} \lambda\,|\{A^p(M^{F_N}f) > \lambda\}| \leqslant c\,\|A^p(M_1^{F_N}f)\|_{L^1}, \qquad \textit{whenever} \quad f \in l^1(h^p).$$

Proof Let $f \in l^1(h^p)$. Denote $Gf = \sup_{\psi\in F_N} \sup_{t\geqslant 1} |f * \psi_t|$. Then $A^p(M^{F_N}f) \leqslant cA^p(M_1^{F_N}f) + cA^p(Gf)$. So

$$|\{A^p(M^{F_N}f) > \lambda\}| \leqslant |\{A^p(M_1^{F_N}f) > \frac{\lambda}{2c}\}| \; + \; |\{A^p(Gf) > \frac{\lambda}{2c}\}|.$$

Therefore, it is sufficient for us to show that the third quantity is less than $c'\,\lambda^{-1}\,\|f\|_{l^1(h^p)}$. To finish the proof, it suffices to show that

$$\|Gf\|_{C(B(y,1))} \leqslant c\,\widetilde{M}(A^p(M_1^{F_N}f))(y), \quad \text{for each} \quad y \in \mathbb{R}^{\mathbf{n}}, \tag{3.1}$$

where c is independent of y, f. Indeed, we fix $y_0 \in \mathbb{R}^{\mathbf{n}}, x_0 \in B(y_0, 1)$. Then take a function $\phi \in C_0^\infty(B(0, \frac{1}{4}))$ with $\int \phi = 1$. (The choice of ϕ is independent of f) Fix $t \geqslant 1$ and $\psi \in F_N$. Then

$$f * \psi_t(x_0) = \frac{1}{t^n} f(\psi(\frac{x_0 - \cdot}{t})) = \frac{1}{t^n} \int_{B(x_0,t+\frac{1}{2})} f(\psi(\frac{x_0 - \cdot}{t})\phi(w - \cdot))\,dw.$$

Now for each $w \in B(x_0, t + \frac{1}{2}), \beta \in B(w, \frac{1}{4})$, we define

$$g_{w,\beta}(y) = 2^{-n}\psi(\frac{x_0 - \beta + 2^{-1}y}{t})\phi(w - \beta + 2^{-1}y).$$

So $f(\psi(\frac{x_0-\cdot}{t})\phi(w - \cdot)) = f(2^n\; g_{w,\beta}(2\;(\beta - \cdot)))$. We claim that for such $g_{w,\beta}$ function, $\mathrm{supp}(g_{w,\beta}) \subseteq B(0, 1)$. It follows from the fact that $\mathrm{supp}(\phi) \subseteq B(0, \frac{1}{4})$ and $|w - \beta| < \frac{1}{4}$. Moreover by the Leibnitz' formula, there exists a positive number L (independent of x_0, β, w and t, but dependent on ϕ and N) such that for each $|\alpha| \leqslant N$, $\|\partial_y^\alpha(g_{w,\beta})\|_\infty \leqslant L$. So

$$|f(\psi(\frac{x_0 - \cdot}{t})\phi(w - \cdot))| \leqslant L \cdot (M_1^{F_N}f)(\beta), \quad \text{for each} \quad \beta \in B(w, \frac{1}{4}).$$

Integrating both sides in the p^{th}-mean over the ball $B(w, \frac{1}{4})$, we have

$$|f(\psi(\frac{x_0 - \cdot}{t})\phi(w - \cdot))| \leqslant cL \cdot A^p_{\frac{1}{4}}(M_1^{F_N}f)(w).$$

Since $t \geqslant 1$ and $y_0 \in B(x_0, t + 1)$, we finally have

$$|f * \psi_t(x_0)| \leqslant c\,\tilde{M}(A^p(M_1^{F_N}f))(y_0).$$

So (3.1) is established. The proof is therefore complete. ▮

Theorem 3.2.2 (Goldberg [12]) *Let $0 < p \leqslant 1, \psi \in \mathcal{S}$ with $\int \psi = 1$ and $\int x^\alpha \psi(x)dx = 0$ for each $\alpha \neq 0$. Then for each $f \in h^p, f - f * \psi \in H^p$ and $\|f - f*\psi\|_{H^p} \leqslant c\|f\|_{h^p}$ where c is independent of f. Moreover, $f*\psi = \sum \lambda_i b_i$ a.e. on $\mathbb{R}^n$ where b_i's are local $(2,p)$-atoms with their supporting cubes, $l(Q) \geqslant 1$, and $\sum |\lambda_i|^p \leqslant c\|f\|_{h^p}^p$.*

Theorem 3.2.3 (Stein [22]) *Let $0 < p \leqslant 1$, T be the Calderón-Zygmund singular integral operator with its kernel satisfying the following assumption*

$$|\partial^\beta K(x)| \leqslant c\,|x|^{-n-|\beta|} \qquad \textit{for} \quad |\beta| \leqslant [n(\frac{1}{p} - 1)],\ x \neq 0.$$

Then T is bounded from H^p into H^p.

Now we come to the main results in this paper.

Theorem 3.2.4 *Let $1 < q < \infty$, $0 < p \leqslant q$, T be the Calderón-Zygmund singular integral operator.* (i) *If $1 < p \leqslant q$, then there exists a constant c (dependent only on p, q, T) such that*

$$\|Tf\|_{l^q(h^p)} \leqslant c\,\|f\|_{l^q(h^p)} \quad \textit{whenever} \quad f \in C_c^\infty.$$

So T admits a unique bounded linear extension on $l^q(h^p)$. (ii) *If $0 < p \leqslant 1$ and K satisfies the following stronger assumption*

$$|\partial^\beta K(x)| \leqslant c\,|x|^{-n-|\beta|} \qquad \textit{for} \quad |\beta| \leqslant [n(\frac{1}{p} - 1)] + 2,\ x \neq 0,$$

then there exists a constant c' (dependent only on p, q, T) such that

$$\|Tf\|_{l^q(h^p)} \leqslant c'\,\|f\|_{l^q(h^p)} \quad \textit{whenever} \quad f \in C_c^\infty.$$

So T admits a unique bounded linear extension on $l^q(h^p)$.

Proof Take a smooth function ψ vanishing outside $B(0,1)$ and $\int \psi dx = 1$. Let $f \in C_c^\infty$. Then $T(f*\psi)$ is a well-defined function in L^q. We now write $Tf = T(f*\psi) + T(f - f*\psi)$. We claim that $\|T(f*\psi)\|_{l^q(h^p)} \leqslant c_q\|f\|_{l^q(h^p)}$. Indeed, from (3.1), we know $|f*\psi| \leqslant c\,\widetilde{M}(A^p(M_1^{F_N}f))$. So by the maximal theorem, the L^q-boundedness property of T and the Jensen inequality ($1 \leqslant \frac{q}{p}$), we have $\|T(f*\psi)\|_{l^q(h^p)} \leqslant c_q\,\|T(f*\psi)\|_{L^q} \leqslant c\,\|f*\psi\|_{L^q} \leqslant c\,\|\widetilde{M}(A^p(M_1^{F_N}f))\|_{L^q} \leqslant c\,\|f\|_{l^q(h^p)}$.

What remains to show is $\|T(f - f*\psi)\|_{l^q(h^p)} \leqslant c\,\|f\|_{l^q(h^p)}$. For convenience, we write $\widetilde{T}f = T(f - f*\psi)$. Take another smooth function $\eta \in C_0^\infty(B(0,5))$ such that $\eta \equiv 1$ on $B(0,4), 0 \leqslant \eta \leqslant 1$. Denote $\eta_x(\cdot) = \eta(x - \cdot)$ for each

$x \in \mathbb{R}^{\mathbf{n}}$. We are going to establish the inequality by dividing into three steps to show that for each $x \in \mathbb{R}^{\mathbf{n}}$,

$$A^p(M_1^{F_N}(\widetilde{T}(f\eta_x)))(x) \leqslant c\, A_{10}^p(M_1^{F_N}f)(x), \tag{3.2}$$

$$A^p(M_1^{F_N}(\widetilde{T}(f(1-\eta_x))))(x) \leqslant c\, M_1'f(x) \qquad \text{if } 1 < p \leqslant q, \tag{3.3}$$

$$A^p(M_1^{F_N}(\widetilde{T}(f(1-\eta_x))))(x) \leqslant c\, \|Gf\|_{C(B(x,1))} \qquad \text{if } 0 < p \leqslant 1, \tag{3.4}$$

where $M_1'f = \sup_{1<t} \frac{1}{c_n t^n} \int_{B(x,t)} |f| dm_n$ and Gf is defined as in Proposition 3.2.1. We fix a point $x_0 \in \mathbb{R}^{\mathbf{n}}$.

Step (i). We are going to establish (3.2). We argue that $f\eta_{x_0} \in h^p$. In fact, by Lemma 3.1.3, $\|f\eta_{x_0}\|_{h^p} \leqslant c\, A_{10}^p(M_1^{F_N}f)(x_0) \leqslant c\, A_{20}^p(M_1^{F_N}f)(\beta)$ for each $\beta \in B(x_0, 10)$. Then $f\eta_{x_0} \in h^p$ follows from Lemma 2.3. Case 1: $1 < p \leqslant q$. The L^p-boundedness property of T implies that $A^p(M_1^{F_N}(\widetilde{T}(f\eta_{x_0})))(x_0) \leqslant \|\widetilde{T}(f\eta_{x_0})\|_{L^p} \leqslant c\, \|f\eta_{x_0}\|_{L^p} \simeq c\, \|f\eta_{x_0}\|_{h^p} \leqslant c\, A_{10}^p(M_1^{F_N}f)(x_0)$. Case 2: $0 < p \leqslant 1$. By invoking Theorem 3.2.2, $f\eta_{x_0} = g_1 + g_2$ where $g_1 \in H^p$ with $\|g_1\|_{H^p} \leqslant c\|f\eta_{x_0}\|_{h^p}$ and $g_2 = \sum \lambda_i b_i$ a.e. on $\mathbb{R}^{\mathbf{n}}$ for some local $(2,p)$-atoms b_i's with their supporting cubes $l(Q) \geqslant 1$ and $\sum |\lambda_i|^p \leqslant c\|f\eta_{x_0}\|_{h^p}^p$. First we claim that for each i, $A^p(M_1^{F_N}(\widetilde{T}b_i))(x_0) \leqslant c$ where c is independent of i and f. Let Q_i be the smallest supporting cube of $b_i, l(Q_i) \geqslant 1$. Then by the Hölder's inequality,

$$\begin{aligned} |A^p(M_1^{F_N}(\widetilde{T}b_i))(x_0)|^p &\leqslant c\left[\int_{B(x_0,1)} |M_1^{F_N}(\widetilde{T}b_i)(y)|^{p\cdot\frac{2}{p}}\, dy\right]^{\frac{p}{2}}, \\ &\leqslant c\|M_1^{F_N}(\widetilde{T}b_i)\|_{L^2}^p, \\ &\leqslant c\|b_i\|_{L^2}^p, \\ &\leqslant c|Q_i|^{\frac{p}{2}-1}, \\ &\leqslant c. \end{aligned}$$

The claim is justified. Now we come back to establish (3.2). By Theorem 3.2.3,

$$\begin{aligned} |A^p(M_1^{F_N}(\widetilde{T}(f\eta_{x_0})))(x_0)|^p &\leqslant |A^p(M_1^{F_N}(\widetilde{T}g_1))(x_0)|^p + |A^p(M_1^{F_N}(\widetilde{T}(\sum \lambda_i b_i)))(x_0)|^p, \\ &\leqslant \|\widetilde{T}g_1\|_{H^p}^p + \sum |\lambda_i|^p\, |A^p(M_1^{F_N}(\widetilde{T}b_i))(x_0)|^p, \\ &\leqslant c\, \|g_1\|_{H^p}^p + c\, \|f\eta_{x_0}\|_{h^p}^p, \\ &\leqslant c\, \|f\eta_{x_0}\|_{h^p}^p. \end{aligned}$$

So (3.2) follows. (It is this point where we make use of Theorem 3.2.3.)

Step (ii). We are going to establish (3.3). Indeed, for each $\beta \in B(x_0, 2)$,

$$
\begin{aligned}
|\widetilde{T}(f(1-\eta_{x_0}))(\beta)| &= \left|\int_{\mathbb{R}^n} (K(\beta-y) - K*\psi(\beta-y)) \cdot f(y) \cdot (1-\eta_{x_0}(y)) dy\right|, \\
&\leqslant \int_{^cB(x_0,4)} |K(\beta-y) - K*\psi(\beta-y)||f(y)|dy, \\
&\leqslant \int_{^cB(x_0,4)} (\int_{\mathbb{R}^n} |K(\beta-y) - K(\beta-y-t)||\psi(t)|dt) \cdot |f(y)|dy, \\
&\leqslant c\|\psi\|_1 \cdot \int_{^cB(x_0,4)} \frac{|f(y)|}{|\beta-y|^{n+\delta}} dy \qquad (\text{since supp } (\psi) \subseteq B(0,1)), \\
&\leqslant c\|\psi\|_1 \cdot \int_{^cB(x_0,4)} \frac{|f(y)|}{|x_0-y|^{n+\delta}} dy \qquad (\text{since } \beta \in B(x_0,2)), \\
&\leqslant c\|\psi\|_1 \cdot \sum_{j=0}^{\infty} (\frac{1}{2})^{(n+\delta)(j+2)} \int_{2^{j+2} \leqslant |x_0-y| < 2^{j+3}} |f(y)|dy, \\
&\leqslant c\|\psi\|_1 \cdot (M_1^{'} f)(x_0).
\end{aligned}
$$

Step (iii). We are going to establish 3.4. For convenience, we write $\Theta(\cdot) \stackrel{def.}{=} \eta(\frac{\cdot}{2}) - \eta(\cdot)$. Noticing that for each $\beta \in B(x_0, 2)$,

$$
\begin{aligned}
\widetilde{T}(f(1-\eta_{x_0}))(\beta) &= \int_{\mathbb{R}^n} [K(\beta-y) - K*\psi(\beta-y)] \; f(y) \; (1-\eta_{x_0}(y)) \; dy, \\
&= \sum_{i=0}^{\infty} \int_{\mathbb{R}^n} [K(\beta-y) - K*\psi(\beta-y)] \; f(y) \; \Theta(\frac{x_0-y}{2^i}) \; dy, \\
&= \sum_{i=0}^{\infty} \int_{\mathbb{R}^n} [K_i(\beta-y) - K_i*\psi(\beta-y)] \; f(y) \; \Theta(\frac{x_0-y}{2^i}) \; dy,
\end{aligned}
$$

where $K_i(\omega) = K(\omega)\zeta(\frac{\omega}{2^i})$ and ζ is smooth function such that $\zeta(y) \equiv 1$ on $1 \leqslant |y| \leqslant 2^4$ and $\zeta(y) \equiv 0$ outside $2^{-1} \leqslant |y| \leqslant 2^5$. The last equality follows from the fact that $\Theta(\frac{x_0-\cdot}{2^i})$ vanishes outside $2^{i+2} \leqslant |x_0 - \cdot| \leqslant 5 \cdot 2^{i+1}$, on which we must have $2^i + 1 \leqslant |\beta - \cdot| \leqslant 2^{i+4} - 1$. Now let $\phi \in F_N, \omega \in B(x_0, 1), 0 < t \leqslant 1$. Define for each $y \in \mathbb{R}^n$,

$$
g_{i,\omega}(y) = 2^{n(i+4)} \; ((K_i - K_i*\psi)*\phi_t)(2^{i+4}y) \;\; [\eta(\frac{x_0-\omega}{2^{i+1}} + 8y) - \eta(\frac{x_0-\omega}{2^i} + 16y)].
$$

It follows that

$$
\phi_t * \widetilde{T}(f(1-\eta_{x_0}))(\omega) = \sum_{i=0}^{\infty} \int_{\mathbb{R}^n} (\frac{1}{2^{i+4}})^n \; g_{i,\omega}(\frac{\omega-y}{2^{i+4}}) \; f(y) \; dy.
$$

What we need to show is that there exists a positive number L (independent of i, ω, x_0, f, ϕ and t) such that for each $i = 0, 1, 2, \ldots,$ $2^i L \; g_{i,\omega} \in F_N$. Indeed, it is easy to see that $\text{supp}(g_{i,\omega}) \subseteq B(0,1)$. Moreover the smoothness of ϕ implies the function $g_{i,\omega}$ to be also smooth. Observe that for $|\gamma| \leqslant [n(\frac{1}{p}-1)]+2$, $\|\partial^\gamma K_i\|_\infty \leqslant C_\gamma\ 2^{-i(n+|\gamma|)}$ where C_γ depends only on ζ, K, N, but does not depend on $i, \omega, x_0, f, \phi, t$. Then $g_{i,\omega}$ satisfies the following differential inequality: for each $|\alpha| \leqslant N$, (let $c_{N,\eta} = \max\{2\cdot 16^N \|\partial^\beta \eta\|_\infty \colon |\beta| \leqslant N\}$)

$$\begin{aligned}
|\partial_y^\alpha (g_{i,\omega})(y)| &\leqslant 2^{n(i+4)}\cdot c_{N,\eta}\sum_{\gamma\leqslant\alpha}\binom{\alpha}{\gamma}2^{(i+4)|\gamma|}\ |\ [\partial_y^\gamma K_i - (\partial_y^\gamma K_i)*\psi]*\phi_t(2^{i+4}y)\ |,\\
&\leqslant c\ 2^{n(i+4)}\cdot c_{N,\eta}\sum_{\gamma\leqslant\alpha}\binom{\alpha}{\gamma}2^{(i+4)|\gamma|}\ \|\partial_y^\gamma K_i - (\partial_y^\gamma K_i)*\psi\|_\infty,\\
&\leqslant c\ 2^{n(i+4)}\cdot c_{N,\eta}\sum_{\gamma\leqslant\alpha}\binom{\alpha}{\gamma}2^{(i+4)|\gamma|}\ \sum_{|\beta|=1}\|\partial_y^{\gamma+\beta}K_i\|_\infty,\\
&\leqslant c\ 2^{n(i+4)}\cdot c_{N,\eta}\sum_{\gamma\leqslant\alpha}\binom{\alpha}{\gamma}2^{(i+4)|\gamma|}\ \sum_{|\beta|=1}C_{\gamma+\beta}\cdot 2^{-i(n+|\gamma|+1)},\\
&= c\ 2^{-i}\cdot 2^{4(n+N)}\cdot c_{N,\eta}\cdot\sum_{\gamma\leqslant\alpha}\binom{\alpha}{\gamma}\ \sum_{|\beta|=1}C_{\gamma+\beta},\\
&\leqslant 2^{-i}L^{-1},
\end{aligned}$$

where L depends only on N, η, K but does not depend on $i, \omega, x_0, f, \phi, t$. The claim is justified. So we have

$$M_1^{F_N}(\widetilde{T}(f(1-\eta_{x_0})))(\omega) \leqslant \sum_{i=0}^{\infty}\frac{1}{2^i L}Gf(\omega) \leqslant c\ \|Gf\|_{C(B(x_0,1))}.$$

Then (3.4) follows.

By invoking (3.1), (3.3) and (3.4) imply that $A^p(M_1^{F_N}(\widetilde{T}(f(1-\eta_{x_0}))))(x_0) \leqslant c\ \widetilde{M}(A^p(M_1^{F_N}f))(x_0)$. Finally we have

$$\|\widetilde{T}f\|_{l^q(h^p)} \leqslant c\ \|A_{10}^p(M_1^{F_N}f)\|_{L^q} + c\ \|\widetilde{M}(A^p(M_1^{F_N}f))\|_{L^q} \leqslant c\ \|f\|_{l^q(h^p)}.$$

The proof is therefore complete. ▮

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