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An Existence Result Arising from a Laplace-Neumann Problem on a Compact Manifold

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Abstract

A Laplace-Neumann problem is introduced and developed in a novel way. Important results are presented which are very important for studying this type of problem on a compact manifold. An existence theorem applicable to eigenvalue curves is also proved.

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Introduction and Preliminary Notes

1. Some properties of eigenvalues and eigenfunctions of the Laplace-Beltrami operator a compact Riemannian manifolds subjected to Neumann boundary conditions are investigated [1-3]. Difficulties can appear in dealing with g is allowed to vary through the space of metrics. Some Laplace-Neumann operator problems are formulated on compact manifolds. Every metric g determines a sequence $0 = \lambda_0(g) < \lambda_1(g) < \lambda_2(g) < \cdots < \lambda_k(g) < \cdots$ of eigenvalues of Δ_g counted with their multiplicities Each eigenvalue can be regarded as a function of $g \in \mathcal{M}$, the space of all C^k Riemannian metrics on M^n. The main result is to establish an existence result for the Laplace-Neumann operator. There exist analytic curves of eigenvalues for a Laplace-Neumann problem associated to analytic eigenfunction curves [4-5].

2. Some preliminary essential information which will be required here is established. Let M^n for $n \geq 2$ be a compact, oriented n-dimensional smooth manifold with boundary ∂M. Let $\mathcal{M}^k$ be the separable Banach space of all C^k Riemannian metrics on M^n for any $2 \leq k < \infty$ with C^k topology. The inner product is denoted $\langle T, S \rangle = \mathrm{Tr}(TS^*)$ induced by g acting on the space of $(0,2)$-tensors on M, where S^* denotes the adjoint of S. In local coordinates, we write $\langle T, S^* \rangle = g^{ik} g^{jl} T_{ij} S_{kl}$. For $f \in C^\infty(M)$ the Laplacian of f is $\Delta f = \langle \nabla^2 f, g \rangle$, where $\nabla^2 f = \nabla d f$ is the Hessian of f. As usual each $(0,2)$-tensor T on (M^n, g) can be associated to a unique $(1,1)$-tensor through the inner product $g(T(X), Y) = T(X, Y)$ for all vector fields $X, Y \in V(M^n)$, the set of vector fields on M^n. Writing this $(1,1)$ tensor as T, the $(0,1)$ tensor, the divergence is defined as

$$(\mathrm{div}\, T)(X)(p) = \mathrm{Tr}(Y \to (\nabla_Y T)(X)(p)), \tag{1}$$

with $p \in M^n$ and $X, Y \in V_p(M^n)$.

It may be recalled that if T is a symmetric $(0,2)$-tensor on a Riemannian manifold (M^n, g) and f a smooth function on M^n, then div satisfies

$$\mathrm{div}(T(fX)) = f \langle \mathrm{div}\, T, X \rangle + f \langle \nabla X, T \rangle + T(\nabla f, X \rangle \tag{2}$$

for each vector field X and the duality $(\mathrm{div}\, T)(X) = \langle \mathrm{div}\, T, X \rangle$ holds.

Let $t \to g(t)$ be a smooth variation of g such that $(M^n, g(t), d\mu_{g(t)})$ is a Riemannian manifold. Here $d\mu_{g(t)}$ is the volume form measure of $g(t)$. Let $d\sigma_{g(t)}$ be the volume element with respect to $g(t)$ restricted to ∂M. Denote by H a $(0,2)$-tensor defined by

$$H_{ij} = \frac{d}{dt}|_{t=0}\, g(t), \qquad h = \langle H, g \rangle. \tag{3}$$

Let $\tilde{h}$ denote the trace of the $(0,2)$-tensor $\tilde{H}$ induced by the derivative of $g(t)$ restricted to ∂M. The following derivatives will be required as well

$$\frac{d}{dt}\, d\mu_{g(t)} = \frac{1}{2} h\, d\mu_g, \qquad \frac{d}{dt}\, d\sigma_{g(t)} = \frac{1}{2} \tilde{h}\, d\sigma_g. \tag{4}$$

Vector fields $X, Y \in T(M^n)$ can be expanded with respect to the basis ∂_i such that $X = g^{ij} x_i(t) \partial_j$ and $Y = g^{kl} y_k(t) \partial_l$, where the coefficients in these expressions are given by $x_i(t) = \langle X, \partial_i \rangle$ and $y_j(t) = \langle Y, \partial_j \rangle$.

It is convenient to write $\dot{X} = g^{ij} \dot{x}_i(t) \partial_j$ and $\dot{Y} = g^{ij} \dot{y}_i(t) \partial_j$ such that $\dot{x}_i(t) = dx_i(t)/dt$ and $\dot{y}_j = dy_j(t)/dt$. We will use $t \to g(t)$ to denote a smooth variation of g.

Lemma 1. For every $X, Y \in V(M^n)$ and $f, g \in C^\infty(M^n)$, the following properties hold

$$(i) \qquad \frac{d}{dt} \langle X, Y \rangle = -H(X, Y) + \langle \dot{X}, Y \rangle + \langle X, \dot{Y} \rangle. \tag{5}$$

$$(ii) \qquad \frac{d}{dt}\langle \nabla_t f, \nabla_t g\rangle = -H(\nabla f, \nabla g). \tag{6}$$

$$(iii) \qquad \frac{d}{dt}\langle \nu_t, \nabla_t\, g(t)\rangle = -H(\nu, \nabla g) + \frac{1}{2}H(\nu,\nu)\langle \nu, \nabla g\rangle + \langle \nu, \nabla \dot{g}\rangle. \tag{7}$$

where

$$\nu_t = \frac{\nabla_t f}{|\nabla_t f|} \tag{8}$$

and ∇_t indicates the gradient with respect to $g(t)$.

Proof: The derivative of $g^{ij}(t)$, the inverse of $g_{ij}(t)$, is required. This can be obtained by using the fact that $g^{im}(t)g_{mj}(t) = \delta^i_j$, differentiating on both sides with respect to t and then solving

$$\frac{d}{dt}\langle X, Y\rangle = \frac{d}{dt}(g^{ij}(t)x_i(t)y_j(t)) = -g^{ik}H_{km}g^{mj}x_i(t)y_j(t) + g^{ij}\dot{x}_i(t)y_j(t) + g^{ij}(t)x_i(t)\dot{y}_j(t)$$

$$= -H(X,Y) + \langle \dot{X}, Y\rangle + \langle X, \dot{Y}\rangle. \tag{9}$$

To get (ii) let $X = \nabla f$ so that $x_i = \langle \nabla f, \partial_i\rangle = \partial_i f$ which is independent of t, and similarly for $Y = \nabla_t q$. Now substitute into (i).

For (iii) it suffices to note from the definition of ν_t,

$$\nu_i = \frac{\langle \nabla_t f, \partial_i\rangle}{|\nabla f|}.$$

Hence

$$\dot{\nu}_i = \frac{1}{2|\nabla f|}H(\nu,\nu)\partial_i f. \tag{10}$$

Take $X = \nu_t$ in (i) and use (10). □

Lemma 2: If ν is the exterior normal field on ∂M and $t \to g(t)$ a smooth variation of g, then

$$\frac{d}{dt}|_{t=0}\, \nu(t) = -H(\nu) + \frac{1}{2}H(\nu,\nu)\,\nu. \tag{11}$$

Proof: Let f be a smooth function on M such that $\nu(t)$ is given by (8). Then we have

$$\frac{d}{dt}\nabla_t f = -H^{ij}\partial_i f \partial_j = -g^{ik}g^{js}\,H(\partial_k,\partial_s)\partial_i f\,\partial_j = -g^{ik}H(g^{js}\partial_i f\,\partial_k, \partial_s)\partial_j = -g^{il}\langle H(\nabla_t f), \partial_l\rangle\,\partial_j$$

$$\equiv -H(\nabla_t\, f). \tag{12}$$

Using (6) in the form

$$\frac{d}{dt}\langle \nabla_t f, \nabla_t f\rangle = -H(\nabla_t f, \nabla_t f),$$

it follows that

$$\frac{d}{dt}\nu(t) = -\frac{1}{2|\nabla_t f|^3}\frac{d}{dt}\langle \nabla_t f, \nabla_t f\rangle \nabla_t f + \frac{1}{|\nabla_t f|}\frac{d}{dt}\nabla_t f$$

$$= \frac{1}{2|\nabla_t f|^3} H(\nabla_t f, \nabla_t f)\nabla_t f + \frac{1}{|\nabla_t f|} H(\nabla_t f) = \frac{1}{2|\nabla_t f|^3} H(\nabla_t f, \nabla_t f)\nabla_t f - \frac{1}{|\nabla_t f|} H(\nabla_t f). \tag{13}$$

Letting t go to zero on both sides of (13), result (11)

$$\frac{d}{dt}|_{t=0}\, \nu(t) = \frac{1}{2} H(\nu, \nu)\, \nu - H(\nu). \tag{14}$$

□

Theorem 1: The following integral formula holds for any two functions $f, g \in C^\infty(M^n)$,

$$\int_{M^n} \eta\, \Delta' \, f\, d\mu_g = \int_{M^n} \eta \Big(\frac{1}{2}\langle dh, df\rangle - \langle \operatorname{div} H, df\rangle - \langle H, \nabla^2 f\rangle \Big)\, d\mu_g, \tag{15}$$

where

$$\Delta' = \frac{d}{dt}|_{t=0}\, \Delta_{g(t)}. \tag{16}$$

Proof: By Stokes' Theorem it follows that

$$\int_{M^n} \eta \Delta_{g(t)} f\, d\mu_{g(t)} = -\int_{M^n} \langle df, d\eta\rangle\, d\mu_{g(t)} + \int_{\partial M} \eta \langle \nu_t, \nabla_t\, f\rangle\, d\sigma_{g(t)}. \tag{17}$$

By making use of (ii) and (iii), it follows from (17) at $t = 0$,

$$\int_{M^n} \eta \Delta' f\, d\mu_g + \int_{M^n} \frac{h}{2} \eta \Delta_g\, f\, d\mu_g = \int_{M^n} H(\nabla f, \nabla \eta)\, d\mu_g - \int_{M^n} \frac{h}{2}\, \langle df, d\eta\rangle\, d\mu_g$$

$$+ \int_{\partial M} \eta \langle -H(\nu, \nabla f) + \frac{1}{2} H(\nu, \nu)\, \frac{\partial f}{\partial \nu}\rangle\, d\sigma_g + \int_{\partial M} \frac{1}{2}\tilde{h}\eta \langle \nu, \nabla f\rangle \; d\sigma_g. \tag{18}$$

This result can be arranged so that it takes the following form,

$$\int_{M^n} \eta \Delta' f\, d\mu_g + \frac{1}{2}\int_{M^n} h\eta \Delta f\, d\mu_g = \int_{M^n} H(\nabla f, \nabla \eta) d\mu_g - \frac{1}{2}\int_{M^n} h\langle df, d\eta\rangle\, d\mu_g$$

$$+ \int_{M^n} \eta \big(-H(\nu, \nabla \nu) + \frac{1}{2} H(\nu, \nu)\langle \nu, \nabla f\rangle + \langle \nu, \nabla f\rangle \big)\, d\mu_g + \frac{1}{2}\int_{\partial M} \tilde{h}\, \langle , \nabla f, \rangle d\sigma_g$$

$$= \int_{M^n} H(\nabla f, \nabla \eta)\, d\mu_g - \frac{1}{2}\int_{M^n} h\langle df, d\eta\rangle\, d\mu_g - \int_{\partial M} \eta \big(H(\nu, \nabla \nu) \frac{\partial f}{\partial \nu}\, d\sigma_g + \frac{1}{2}\int_{\partial M} \tilde{h}\, \eta \langle \nu, \nabla f\rangle\, d\sigma_g. \tag{19}$$

It follows that

$$\int_{M^n} \eta \Delta' \, f\, d\mu_g = \int_{M^n} H(\nabla f, \nabla \eta) d\mu_g - \int_{\partial M} \eta H(\nu, \nabla \eta)\, d\sigma_g - \frac{1}{2}\int_{M^n} (h\langle df, d\eta\rangle + \eta h \Delta f)\, d\mu_g$$

$$+\frac{1}{2}\int_{\partial M}\eta(\tilde{h}+H(\nu,\nu))\frac{\partial f}{\partial\nu}\,d\sigma_g. \tag{20}$$

Using $\tilde{h}=h-H(\nu,\nu)$, (20) can be put in the form

$$\int_{M^n}\eta\Delta' f d\mu_g=\int_{M^n}H(\nabla f,\nabla\eta)d\mu_g-\int_{\partial M}\eta H(\nu,\nabla f)d\mu_g-\frac{1}{2}\int_{M^n}h\langle df,d\eta\rangle+\eta h\Delta f)d\mu_g$$

$$+\frac{1}{2}\int_{\partial M}\eta h\,\frac{\partial f}{\partial\nu}\,d\sigma_g. \tag{21}$$

Set $T=H$, $\varphi=\eta$ and $X=\nabla f$ in (2), it takes the form

$$\operatorname{div}(H(\eta\nabla f))=\eta\langle\operatorname{div}H\nabla f\rangle+\eta\langle\nabla^2 f,H\rangle+H\langle\nabla\eta,\nabla f\rangle. \tag{22}$$

Substituting (22) implies that the second term on the right of (21) is

$$\int_{\partial M}\eta H(\nu,\nabla f)\,d\mu_g=\int_{M^n}\operatorname{div}(H(\eta\nabla f))\,d\mu_g$$

$$=\int_{M^n}\eta\big(\langle\operatorname{div}H,\nabla^2 f\rangle+\langle H,\nabla^2 f\rangle\big)\,d\mu_g+\int_{M^n}H(\nabla f,\nabla\eta)\,d\mu_g. \tag{23}$$

In addition to the result (23), we have

$$\int_{M^n}\eta\Delta' f\,d\mu_g=\int_{M^n}(\eta h\Delta f+h\langle df,\eta\rangle)\,d\mu_g+\int_{M^n}\eta\langle df,dh\rangle\,d\mu_g. \tag{24}$$

Substitute (23), (24) into (21) and we arrive at

$$\int_{M^n}\eta\Delta' f\,d\mu_g=\int_{M^n}H(\nabla f,\nabla\eta)\,d\mu_g-\int_{M^n}\eta(\langle\operatorname{div}H,\nabla^2 f\rangle+\langle H,\nabla^2 f\rangle)\,d\mu_g-\int_{M^n}H(\nabla f,\nabla\eta)\,d\mu_g$$

$$-\frac{1}{2}\int_{M^n}(h\langle df,d\eta\rangle+h\eta\Delta f)\,d\mu_g+\frac{1}{2}\int_{M^n}(\eta h\Delta f+h\langle df,d\eta\rangle)\,d\mu_g+\frac{1}{2}\int_{M^n}h\langle df,dh\rangle\,d\mu_g, \tag{25}$$

for all functions $\eta\in C^\infty(M^n)$. Many of the terms in (25) cancel out and the desired result (15) remains. □

Theorem 2: Let $\{\varphi_i(t)\}\subset C^\infty(M^n)$ be a differentiable family of real functions such that $\langle\varphi_i(t),\varphi_j(t)\rangle_{L^2(M^n,d\mu_{g(t)})}=\delta_{ij}$ for all t, the following system holds:

$$-\Delta_{g(t)}\varphi_i(t)=\lambda(t)\varphi_i(t), \tag{26}$$

$$\frac{\partial}{\partial\nu_t}\varphi_i(t)=0 \tag{27}$$

where (26) holds on M^n and (27) on ∂M. Then

$$\lambda'(0)\delta_{ij}=\int_{M^n}\langle\frac{1}{4}\Delta(\varphi_i\varphi_j)g-d\varphi_i\otimes\varphi_j,H\rangle\,d\mu_g. \tag{28}$$

Proof: The derivative with respect to t is taken with respect to t and then set $t=0$ on both sides of the eigenvalue equation

$$-\Delta_{g(t)}\varphi(t) = \lambda(t)\varphi(t).$$

with the result,

$$-\Delta_g^{'}\varphi_i(t) - \Delta_g\dot{\varphi}(t) = \dot{\lambda}(t)\varphi_i(t) + \lambda(t)\dot{\varphi}_i(t). \tag{29}$$

Substitute for $\Delta_{g(t)}\varphi(t)$ from the eigenvalue equation (26), multiply by $\varphi_j(t)$ and integrate on both sides of (29) to get

$$-\int_M (\varphi_j\Delta_g^{'}\varphi_i + \varphi_j\Delta_g\dot{\varphi}_i)\,d\mu_g = \int_{M^n}(\dot{\lambda}\varphi_j\varphi_i - \dot{\varphi}_i\Delta_g\varphi_j)\,d\mu_g. \tag{30}$$

On the boundary, it is the case that

$$\langle\nu_t, \nabla_{g(t)}\,\varphi_i(t)\rangle = \frac{\partial}{\partial\nu_t}\varphi(t),$$

it follows that with $t=0$ and (27)

$$\langle\nu, \nabla\dot{\varphi}_i\rangle = H(\nu,\nabla\varphi_i) - \frac{1}{2}H(\nu,\nu)\langle\nu,\nabla\varphi_i\rangle = H(\nu,\nabla\varphi_i). \tag{31}$$

Integrating by parts in (30) and using (31), we obtain

$$\dot{\lambda}\delta_{ij} = -\int_{M^n}\varphi_j\Delta^{'}\varphi_i\,d\mu_g - \int_{\partial M}\varphi_j\frac{\partial}{\partial\nu}\dot{\varphi}_i\,d\mu_g = -\int_{M^n}\varphi_j\Delta_g^{'}\varphi_i\,d\mu_g - \int_{\partial M}\langle\nu,\nabla\dot{\varphi}_i\rangle\varphi_j\,d\mu_g$$

$$= -\int_{M^n}\varphi_j\Delta_g^{'}\varphi_i\,d\mu_g - \int_{\partial M}\varphi_j H(\nu,\nabla_g\varphi_i)\,d\sigma_g. \tag{32}$$

Consequently, it follows that

$$-2\dot{\lambda}\delta_{ij} = \int_{M^n}\varphi_j\Delta_g^{'}\varphi_i\,d\mu_g + \int_{M^n}\varphi_i\Delta^{'}\varphi_j d\mu_g + \int_{\partial M}\varphi_i H(\nu,\nabla\varphi_j)d\mu_g + \int_{\partial M}\varphi_j H(\nu,\nabla\,\varphi_i)d\mu_g. \tag{33}$$

Recall what has been developed already,

$$\int_{M^n}\varphi_j\Delta_g^{'}\varphi_i\,d\mu_g = \int_{M^n}\varphi_j\big(\frac{1}{2}\langle dh, d\varphi_i\rangle - \langle \mathrm{div}H, d\varphi_i\rangle - \langle H, \nabla^2\varphi_i\rangle\big)\,d\mu_g.$$

Hence (33) takes the form,

$$-2\dot{\lambda}\delta_{ij} = \int_{M^n}\langle\frac{1}{2}dh - \mathrm{div}H, \varphi_j d\varphi_i + \varphi_i d\varphi_j\rangle d\mu_g - \int_{M^n}\langle H, \varphi_j\nabla^2\varphi_i + \varphi_i\nabla^2\varphi_j\rangle\,d\mu_g$$

$$+\int_{\partial M}\varphi_i H(\nu,\nabla\varphi_j)d\sigma_g + \int_{\partial M}\varphi_j H(\nu,\nabla\varphi_i)\,d\sigma_g$$

$$= \int_{M^n} \langle \frac{1}{2} dh, d(\varphi_i \varphi_j) \rangle \, d\mu_g - \int_{M^n} \varphi_j (\langle \mathrm{div} H, d\varphi_i \rangle + \langle H, \nabla^2 \varphi_i \rangle) \, d\mu_g$$

$$+ \int_{\partial M} \varphi_j H(\nu, \nabla \varphi_j) \, d\sigma_g - \int_{M^n} \varphi_i \big(\langle \mathrm{div} H, d\varphi_j \rangle + \langle H, \nabla^2 \varphi_j \rangle \big) \, d\mu_g \tag{34}$$

$$+ \int_{\partial M} \varphi_i H(\nu, \nabla \varphi_j) \, d\mu_g.$$

Since

$$\int_{M^n} \mathrm{div}\, X \, d\mu_g = 0,$$

and using (2) adapted to this case

$$\mathrm{div}(H(\varphi_i d\varphi_j)) = \varphi_i \langle \mathrm{div} H, d\varphi_i \rangle + \varphi_i \langle \nabla^2 \varphi_j, H \rangle + H \langle \nabla \varphi_i, \nabla \varphi_j \rangle,$$

equation (34) drops into the following form,

$$-2\dot{\lambda}\delta_{ij} = \int_{M^n} \langle \frac{1}{2} dh, d(\varphi_i \varphi_j) \rangle \, d\mu_g - \mathrm{div}\,(H(\varphi_i d\varphi_j)) \, d\mu_g + H(\nabla \varphi_j, \nabla \varphi_i) \, d\mu_g + \int_{\partial M} \varphi_j H(\nu, \nabla \varphi_i) \, d\mu_g$$

$$- \int_{M^n} \mathrm{div}(H(\varphi_j d\varphi_i)) + H(\nabla \varphi_i, \nabla \varphi_j) \, d\mu_g + \int_{\partial M} \varphi_i H(\nu, d\varphi_j) \, d\mu_g$$

$$= - \int_{M^n} \frac{h}{2} \Delta(\varphi_i \varphi_j) \, d\mu_g + 2 \int_{M^n} H(\nabla \varphi_i, \nabla \varphi_j) \, d\mu_g.$$

Dividing out the factor of -2, the result in (28) is obtained. $\square$

1 An Existence Result

An existence result is established which makes use of the Lyapunov-Schmidt procedure. The problem to be considered is the following Neumann problem,

$$(\Delta_t + \lambda) u = 0,$$
$$\frac{\partial u}{\partial \nu_t} = 0. \tag{35}$$

The first equation in (35) holds on M^n and the second on ∂M. As usual (M^n, g) is an orientable, compact n-dimensional Remannian manifold with boundary ∂M and $\Delta_t = \Delta_{g(t)}$ so an analytic variation of g_0 is associated $t \to g(t)$ with $g(0) = g_0$. Also ν_t is a one-parameter family of unit exterior vectors along with $(\partial M, g(t))$.

Theorem 3: Let λ_0 be an eigenvalue of the Laplace-Neumann operator of multiplicity $m \geq 2$. For every $\epsilon > 0$, there exits a $\delta > 0$ such that for each

$|t| < \delta$, there exist exactly n to (35) eigenvalues including multiplicities in the interval $(\lambda_0 - \epsilon, \lambda_0 + \epsilon)$.

Proof: Let $\{\varphi_k\}_{k=1}^n$ be an orthonormal basis associated to eigenvalue λ_0, and define a projector P such that

$$P\,v = \sum_{j=1}^{n} \varphi_j \int_{M^n} \varphi_j \, v \, d\mu_g \tag{36}$$

is the projection on the corresponding eigenspace. As is well-known, P induces a splitting of L^2 such that

$$L^2(M^n, d\mu_g) = \mathcal{R}(P) \oplus \mathcal{N}(P).$$

Any function $v \in L^2(M^n, d\mu_g)$ can be broken up into a sum of two factors $\phi + \psi$ when $\phi \in \mathcal{R}(P) = \ker(\Delta + \lambda_0)$ and $\psi \in \mathcal{N}(P)$. Using this fact, Neumann problem (35) can be expressed equivalently as the following system of equations:

$$\begin{gathered} (I - P)(\Delta_t + \lambda)(\phi + \psi) = 0, \\ P(\Delta_t + \lambda)(\phi + \psi) = 0, \\ \frac{\partial}{\partial \nu_t}(\phi + \psi) = 0. \end{gathered} \tag{37}$$

The first two in (37) pertain to in M^n and the third on ∂M.

The Neumann problem can be decoupled and equivalently considered as a system of equations

$$(I - P)(\Delta_t + \lambda)(\phi + \psi) = 0, \tag{38}$$

$$P\,(\Delta_t + \lambda)(\phi + \psi) = 0, \tag{39}$$

$$\frac{\partial}{\partial \nu_t}\,(\phi + \psi) = 0, \tag{40}$$

where (38) and (39) apply on M^n and (40) on ∂M. Since $\phi_j \in \mathcal{R}(P)$ and ψ are orthogonal elements, the divergence theorem implies that

$$P(\Delta + \lambda)\psi = \sum_{j=1}^{m} \varphi_j \int_{M^n} \phi_j (\Delta + \lambda)\psi \, d\mu_{g_0} = \sum_{j=1}^{m} \varphi_j \int_{M^n} \varphi_j \frac{\partial \psi}{\partial \nu} \, d\mu_{g_0}, \tag{41}$$

and consequently,

$$(\Delta + \lambda)\psi = (I - P)\big((\Delta + \lambda)\psi\big) + \sum_{j=1}^{m} \varphi_j \int_{\partial M} \varphi_j \frac{\partial \psi}{\partial \nu} \, d\sigma_g. \tag{42}$$

It is therefore possible to state that

$$(\Delta + \lambda)\psi + (I - P)\big(\Delta_t + \Delta\big)(\phi + \psi) - \sum_{j=1}^{m} \varphi_j \int_{\partial M} \frac{\partial \psi}{\partial \nu} \, d\,\sigma_{g_0} = 0. \tag{43}$$

The part which is relevant to ∂M in equations (38)-(40) can be expressed as

$$\frac{\partial \psi}{\partial \nu} + (\frac{\partial}{\partial \nu_t} - \frac{\partial}{\partial \nu})(\phi + \psi) = 0. \tag{44}$$

Hence solving the first and third equations of (37) is equivalent to finding the zeros of the following function

$$\begin{aligned} F : \mathbb{R} \times \mathbb{R} \times \mathcal{R}(P) \times H^2(M^n) \cap \mathcal{N} \to \mathcal{N}(P) \times H^{3/2}(M^n), \\ (t, \lambda, \phi, \psi) \to (F_1(t, \lambda, , \phi, \psi), F_2(t, \lambda, \phi, \psi)), \end{aligned} \tag{45}$$

In (41), F_1 and F_2 are defined to be

$$F_1 = (\Delta + \lambda)\psi + (I - P)(\Delta_t - \Delta)(\phi + \psi) - \sum_{j=1}^{m} \varphi_j \int_{M^n} \varphi_j \frac{\partial \psi}{\partial \nu} \, d\sigma_{g_0}, \tag{46}$$

$$F_2 = \frac{\partial \psi}{\partial \nu} + \big(\frac{\partial}{\partial \nu_t} - \frac{\partial}{\partial \nu}\big)(\phi + \psi). \tag{47}$$

Clearly F depends differentially on the variables λ, t, ψ and ϕ. The idea is to use the implicit function theorem to show that $F(t, \lambda, \phi, \psi) = (0, 0)$ admits a solution which depends on λ, t and ϕ. To do so, observe that if $t = 0$, $\lambda = \lambda_0$ and $\psi = 0$,

$$\frac{\partial F}{\partial \psi}(0, \lambda_0, 0, 0)\dot{\psi} = \Big((\Delta + \lambda_0)\psi - \sum_{j=1}^{m} \varphi_j \int_{\partial M} \varphi_j \frac{\partial \dot{\psi}}{\partial \nu} \, d\sigma_0, \frac{\partial \dot{\psi}}{\partial \nu}\Big). \tag{48}$$

It is claimed that the map (48) is an isomorphism from $H^2(M^n) \cap \mathcal{N}(P)$ onto $\mathcal{N}(P) \times H^{3/2}(M^n)$.

The implicit function theorem requires that there exist two positive numbers δ, ϵ as well as a function $S(t, \lambda)\phi$ of class C^1 of the variables (t, λ) such that for every $|t| < \delta$ and $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$, it holds that $F(t, \lambda, \phi, S(t, \lambda)\phi) = (0, 0)$. Further $S(t, \lambda)\phi$ is analytic at λ and linear in ϕ. This solves (37) with respect to ψ.

Now for every $\phi \in \mathcal{R}(P)$, there exist real numbers $c_1, \ldots, c_m$ such that $\phi = \sum_{j=1}^{m} c_j \varphi_j$. The second equation in (37) can be regarded as a system of equations in variables $c_1, \ldots, c_m$

$$\sum_{j=1}^{m} c_j \int_{M^n} \varphi_k (\Delta_t + \lambda)(\varphi_j + S(t, \lambda)\varphi_j) \, d\mu_{g_0} = 0, \tag{49}$$

where $k = 0, \ldots, m$. Thus, λ is an eigenvalue of Δ_t if and only if $\det |A(t, \lambda)| = 0$, where the matrix elements of $A(t, \lambda)$ are obtained by calculating the integral

$$A_{kj}(t, \lambda) = \int_{M^n} \varphi_k (\Delta_t + \lambda)(\varphi_j + S(t, \lambda)\varphi_j) \, d\mu_g. \tag{50}$$

The associated eigenfunctions are given by

$$u(t,\lambda)=\sum_{j=1}^{m} c_j(\varphi_j+S(t,\lambda)\varphi_j). \tag{51}$$

This can be expressed in other words as $\mathbf{c}=(c_1,\ldots,c_m)$ must satisfy the system

$$A(t,\lambda)\mathbf{c}=\mathbf{0}.$$

By Rouché's Theorem we have that, for every $\epsilon>0$ there is a $\delta>0$ such that if $|t-t_0|<\delta$, there exists exactly m roots of

$$\det|A(t,\lambda)|=0$$

in the interval $(\lambda_0-\epsilon,\lambda_0+\epsilon)$. $\square$

Proposition 1: Let M^n $n\geq 2$ be a compact, oriented smooth manifold and let $g(t)$ be a real analytic 0ne-parameter family of Riemannian metrics on M^n with $g(0)=g_0$. Assume λ is an eigenvalue of multiplicity m for the Laplace-Neumann operator Δ_g. Then there exists an $\epsilon>0$ and a set of functions $\lambda_i(t)$ analytic in t and $\varphi_i(t)$, $i=1,\ldots,m$ such that

$$(\varphi_i(t),\varphi_j(t))_{L^2(M^n,d\mu_g)}=\delta_{ij}. \tag{52}$$

As well the following hold for every t in $|t|<\epsilon$:

$$(i)\quad \Delta_{g(t)}\,\varphi_i(t)=\lambda_i(t)\varphi_i(t), \qquad (ii)\quad \frac{\partial}{\partial\nu_t}\varphi_i(t)=0, \qquad (iii)\quad \lambda_i(0)=\lambda. \tag{53}$$

Moreover (i) holds in M^n and (ii) holds on ∂M in (53).

Proof: Suppose the same conditions as those of the previous result hold. It must be shown that there exist m analytic curves of eigenvalues $\lambda_j(t)$ for (35) associated with m-analytic eigenfunctions $\varphi_j(t)$. The idea is to reduce the problem to one that is finite-dimensional and then apply a Theorem of Kato often called the Selection theorem [6]. For this a slightly different construction shall be given from that used before.

Let $\{\varphi_j(t)\}$ be orthonormal eigenfunctions of the Laplace-Neumann system associated to λ_j. For each $k=1,\ldots,m$ consider the following problem

$$(\Delta+\lambda_0)u=0,$$

$$\frac{\partial}{\partial\nu_t}(\varphi_k+u)=0, \tag{54}$$

$$Pu=\sum_{j=1}^{m}\varphi_j\int_{M^n}\varphi_j u\,d\mu_{g_0}=0.$$

The first of these holds in M^n as well as the third, while the second holds on ∂M. Consider the orthogonal complement $\{\varphi_j\}^\perp$ of $\ker(\Delta+\lambda_0)$ in $L^2(M^n, d\mu_{g_0})$ and define the function

$$F : (-\delta, \delta) \times H^2(M^n, d\mu_{g_0}) \to \{\varphi_j\}^\perp \times \mathcal{R}(P) \times H^{3/2}(M^n, d\mu_{g_0}), \tag{55}$$

by

$$F(t, w) = \big((\Delta + \lambda_0)w, P\,w, \frac{\partial}{\partial\nu_t}(\varphi_k + w)\big). \tag{56}$$

Exactly as before $\partial F/\partial w\,(0,0)$ is an isomorphism. The Implicit Function Theorem asserts that there exists a $\delta > 0$ and functions $w_j(t, \lambda)$ such that for any t in $|t - t_0| < \delta$ and every λ in $|\lambda - \lambda_0| < \delta$, the equality $G_j(t, \lambda, w_j(t, \lambda)) = (0, 0, 0)$ holds. Now λ is an eigenvalue for (35) if and only if there exists a non-zero m-tuple $\mathbf{c} = (c_1, \ldots, c_m)$ of real numbers such that

$$A(t, \lambda)\mathbf{c} = 0,$$

where the matrix elements of A are calculated by means of the integrals

$$A_{ij}(t, \lambda) = \int_{M^n} \varphi_j(t)(\Delta_t + \lambda)(\varphi_j(t) + w_j(t, \lambda))\, d\mu_{g(t)}. \tag{57}$$

As before, λ is an eigenvalue of (35) if and only if $\det(A(t, \lambda)) = 0$.

By Rouché's Theorem there must exist m roots near λ_0 counting multiplicities for each t. Hence [7] there exist m analytic functions $t \to \lambda_j(t)$ which locally solve the equation $\det(A(t, \lambda)) = 0$. It can easily be seen that A is symmetric and so Kato's Theorem ensures an analytic curve $c^i(t) \in \mathbb{R}^m$ such that $A(t, \lambda_i(t)c^i(t)) = 0$ for each $i = 1, \ldots, m$. Consequently,

$$\psi_k(t) = \sum_{j=1}^{m} c_j^k(t)(\varphi_j + w_j(t, \lambda_k(t))) \tag{58}$$

is an analytic curve of eigenfunctions for (35) associated with $\lambda_j(t)$. Now with the same reasoning as Kato, m analytic curves of eigenfunctions $\{\varphi_i(t)\}_{i=1}^m$ can be obtained such that

$$\int_{M^n} \varphi_i(t)\varphi_j(t)\, d\mu_{g(t)} = \delta_{ij}. \tag{59}$$

□

In the particular case in which $m = m(\lambda_0) = 1$, the existence of a differentiable curve of eigenvalues through λ_0 follows from the Implicit Function Theorem applied to the mapping

$$F : S^k \times H^2(M^n, d\mu_{g_0}) \times \mathbb{R} \tag{60}$$

defined by

$$F(g,u,\lambda)=\big((\Delta_g+\lambda)\,u,\int_{M^n}u^2\,d\mu_{g_0}\big). \tag{61}$$

The corresponding formulas for the derivative $\dot{\lambda}(t)$ can be obtained by letting $i=j=1$ in (28).

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