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n-Open Sets and n-Continuous Functions

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Abstract

The collection of subsets of a topological space which satisfy the condition that their interior is not equal to their closure is investigated. The basic properties of this collection of sets, which we call n-open sets, are developed. These sets are used to define the concept of an n-continuous function. The basic properties are these functions are established.

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1 Introduction

Popa and Noiri developed the concept of a minimal structure [2] and used it to develop unified theories of continuity and various types of weak continuity [3]. In this paper we investigate the properties of the collection of subsets of a topological space which satisfy the condition that their interior is not equal to their closure. This collection of sets, which we call n-open sets, does not satisfy the conditions of a minimal structure but can be used as a generalization of a topology. The basic properties and relationships for n-open sets are investigated. For example, it is shown that the n-open sets are not closed under either union or intersection but are closed under complements. The collection of n-open sets is used to define the concept of n-continuity and the basic properties of these functions are established. Relationships between these functions and connected spaces are developed. In particular it is shown that connectedness can be characterized in terms of n-continuity. Also it is established that n-continuity is independent of continuity.

The symbols X and Y represent topological spaces with no separation properties assumed unless explicitly stated. All sets are considered to be subsets of topological spaces. The closure and interior of a set A are signified by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively.

Definition 1.1 *A function $f : X \to Y$ is said to be contra-continuous* [1] *if $f^{-1}(V)$ is closed for every open subset V of Y.*

2 n-open sets

Definition 2.1 *A subset A of a space X is said to be n-open if $Int(A) \neq Cl(A)$. A subset of X is called n-closed if its complement is n-open.*

Theorem 2.2 *A subset A of a space X is n-open if and only if A is not clopen.*

Proof. Let $A \subseteq X$. The set A is not n-open if and only if $\mathrm{Int}(A) = \mathrm{Cl}(A)$ if and only if $A = \mathrm{Int}(A)$ and $A = \mathrm{Cl}(A)$ if and only if A is open and A is closed if and only if A is clopen.

Corollary 2.3 *A subset A of a space X is n-open if and only if $X - A$ is n-open.*

Thus the n-open sets coincide with the n-closed sets.

Corollary 2.4 *A proper dense subset of a space X is n-open.*

Definition 2.5 *Let A be a subset of a space X. The n-interior of A is denoted by $nInt(A)$ and given by $nInt(A) = \cup\{U \subseteq X : U \subseteq A \text{ and } U \text{ is n-open}\}$. The n-closure of A is denoted by $nCl(A)$ and given by $nCl(A) = \cap\{F \subseteq X : A \subseteq F \text{ and } F \text{is n-closed}\}$.*

Example 2.6 *Let $X = \{a, b, c\}$ have the topology $\tau = \{X, \emptyset, \{a, b\}, \{c\}\}$. The n-open sets are $\{a\}, \{b\}, \{a, c\}$, and $\{b, c\}$. Then $nInt(\{a, b\}) = \{a\} \cup \{b\}$, which is not n-open and $nCl(\{c\}) = \{a, c\} \cap \{b, c\} = \{c\}$, which is not n-closed (hence also not n-open).*

It follows from Example (2.6) that for a set A, $\mathrm{nInt}(A)$ may not be n-open and $\mathrm{nCl}(A)$ may not be n-closed. Also the n-open sets are not closed under either union or intersection. It follows from the definitions that, if U is n-open, then $\mathrm{nInt}(U) = U$ and, if F is n-closed, then $\mathrm{nCl}(F) = F$.

Lemma 2.7 *A space X has an n-open subset if and only if it is not discrete.*

Proof. A space X is not discrete if and only if there exists a non-clopen set if and only if there exists an n-open set.

Remark 2.8 *Obviously a space is discrete if and only if there are no n-open sets.*

Theorem 2.9 *If X is not discrete, then for every $x \in X$ there exists an n-open set containing x.*

Proof. Let $x \in X$. Then by Lemma 2.7 there exists an n-open set U. Either $x \in U$ or $x \in X - U$, both of which are n-open.

Since $\emptyset$ and X are not n-open, the values of the operators nInt and nCl on the sets $\emptyset$ and X may depend on the space X.

Theorem 2.10 *If X is a space, then*

(a) $nCl(X) = X$.

(b) $nInt(\emptyset) = \emptyset$.

Proof. (a) $\text{nCl}(X) = \cap\{F \subseteq X : F \text{ is n-closed and } X \subseteq F\} = X$, since it is an intersection of an empty collection of sets.

(b) $\text{nInt}(\emptyset) = \cup\{U \subseteq X : U \text{ is n-open and } U \subseteq \emptyset\} = \emptyset$, since it is a union of an empty collection of sets.

Theorem 2.11 *If X is a discrete space,then*

(a) $nInt(A) = \emptyset$ *for every set* $A \subseteq X$.

(b) $nCl(A) = X$ *for every set* $A \subseteq X$.

Proof. (a) Let $A \subseteq X$. Then $\text{nInt}(A) = \cup\{U \subseteq X : U \subseteq A \text{ and } U \text{ is n-open}\} = \emptyset$, since the collection of sets is empty.

(b) Let $A \subseteq X$. Then $\text{nCl}(A) = \cap\{F \subseteq X : A \subseteq F \text{ and } F \text{ is n-closed}\} = X$, since the collection of sets is empty.

Theorem 2.12 *If X is an indiscrete space with at least two points, then*

(a) $nInt(A) = A$ *for every set* $A \subseteq X$.

(b) $nCl(A) = A$ *for every set* $A \subseteq X$.

Proof. By Theorem 2.10 $\mathrm{nInt}(\emptyset) = \emptyset$ and $\mathrm{nCl}(X) = X$. Since X has at least two points, the singleton sets are n-open (hence also n-closed) and thus $X = \cup\{\{x\} : x \in X\} \subseteq \mathrm{nInt}(X)$ and $\mathrm{nCl}(\emptyset) \subseteq \cap\{\{x\} : x \in X\} = \emptyset$. Therefore $\mathrm{nInt}(X) = X$ and $\mathrm{nCl}(\emptyset) = \emptyset$. Since every proper nonempty subset of X is non-clopen and therefore n-open and n-closed, $\mathrm{nInt}(A) = A$ and $\mathrm{nCl}(A) = A$ for every proper nonempty set $A \subseteq X$.

Theorem 2.13 *The following statements hold for every set $A \subseteq X$:*

(a) $nInt(X - A) = X - nCl(A)$.

(b) $nCl(X - A) = X - nInt(A)$.

(c) *$x \in nCl(A)$ if and only if $U \cap A \neq \emptyset$ for every n-open set U containing x.*

Proof. (a) Assume $x \in X$ and $A \subseteq X$. Then $x \in \mathrm{nInt}(X - A)$ if and only if there exists an n-open set $U \subseteq X$ such that $x \in U \subseteq X - A$ if and only if there exists an n-closed set $F \subseteq X$ such that $A \subseteq F$ and $x \notin F$ if and only if $x \notin \mathrm{nCl}(A)$ if and only if $x \in X - \mathrm{nCl}(A)$.

(b) Let $x \in X$ and let $A \subseteq X$. Then $x \in \mathrm{nCl}(X - A)$ if and only if for every n-open set U, whenever $X - A \subseteq X - U$, then $x \in X - U$ if and only if for every n-open set U, whenever $U \subseteq A$, then $x \notin U$ if and only if $x \notin \mathrm{nInt}(A)$ if and only if $x \in X - \mathrm{nInt}(A)$

(c) Let $x \in X$ and let $A \subseteq X$. Then $x \in \mathrm{nCl}(A)$ if and only if for every n-open set U, whenever $A \subseteq X - U$, then $x \notin U$ if and only if for every n-open set U, whenever $x \in U$, then $A \cap U \neq \emptyset$.

Definition 2.14 *A space X is said to be n-discrete if every proper nonempty set is n-open and n-indiscrete provided there are no n-open sets.*

Theorem 2.15 *A space X is connected if and only if it is n-discrete.*

Proof. A space X is connected if and only if there are no proper nonempty clopen sets if and only if every proper nonempty set is n-open.

Definition 2.16 *A space X is said to be n-disconnected if it is the union of two disjoint n-open sets. A space is called n-connected if it is not n-disconnected*

Corollary 2.17 *If a space X is connected and has at least two points, then X is n-disconnected*

Theorem 2.18 *A space X is n-connected if and only if X is n-indiscrete.*

Proof. Let X be a space. Then X is not n-indiscrete if and only if there exists an n-open set U if and only if there exists an n-open set U such that $X = U \cup (X - U)$ if and only if X is not n-connected.

The next result is a consequence of Remark 2.8.

Corollary 2.19 *A space X is n-connected if and only if X is discrete.*

Theorem 2.20 *Let $U \subseteq X$ and $V \subseteq Y$ be nonempty sets. Then $U \times V$ is n-open in $X \times Y$ if and only if U is n-open in X or V is n-open in Y.*

Proof. $U \times V$ is not n-open in $X \times Y$ if and only if $U \times V$ is clopen in $X \times Y$ if and only if U is clopen in X and V is clopen in Y if and only if U is not n-open in X and V is not n-open in Y. It then follows that $U \times V$ is n-open in $X \times Y$ if and only if U is n-open in X or V is n-open in Y.

Corollary 2.21 *If U and V are n-open sets in X and Y, respectively, then $U \times V$ is n-open in $X \times Y$.*

Definition 2.22 *A space X is said to be*

(a) *an nT_0-space if, whenever x and y are distinct points of X, there exists an n-open set containing one point but not the other.*

(b) *an nT_1-space if, whenever x and y are distinct points of X, each point is contained in an n-open set that does not contain the other.*

(c) *an nT_2-space if, whenever x and y are distinct points of X, there exist disjoint n-open sets U and V containing x and y, respectively.*

Lemma 2.23 *If U is an n-open set and $U = A \cup B$, then either A is n-open or B is n-open.*

Proof. Since $U = A \cup B$ and U is non-clopen, either A is non-clopen or B is non-clopen. Hence either A is n-open or B is n-open.

Theorem 2.24 *If X is not discrete, then X is an nT_0-space.*

Proof. Assume x and y are distinct points of X. By Theorem 2.9 there exists an n-open set U containing x. If $y \notin U$, then U is an n-open set containing x but not y. Assume $y \in U$. Then $U = (U - \{x\}) \cup (U - \{y\})$. Therefore by Lemma 2.23 either $U - \{x\}$ is an n-open set containing y but not x or $U - \{y\}$ is an n-open set containing x but not y. It follows that X is an nT_0-space.

Remark 2.25 *The process used in the proof of Theorem 2.24 can be used to construct a nested sequence of n-open subsets of an n-open set containing at least two points.*

As we see in the following example, $U - \{x\}$, where U is an n-open set containing x, is not necessarily n-open.

Example 2.26 *Let* (X, τ) *be the space in Example 2.6. The set* $U = \{a, c\}$ *is n-open, but* $U - \{a\} = \{c\}$ *is not n-open.*

Theorem 2.27 *If* X *is not discrete, then* X *is an* nT_2*-space.*

Proof Assume x and y are distinct points of X. By Theorem 2.24 X is an nT_0-space. Therefore there exists an n-open set containing x but not y or there exists an n-open set containing y but not x. If there exists an n-open set U containing x but not y, then U and $X - U$ are disjoint n-open sets containing x and y, respectively. Similarly, if there exists an n-open set V containing y but not x, then V and $X - V$ are disjoint n-open sets containing y and x, respectively. Therefore X is an nT_2-space.

Corollary 2.28 *The separation properties* nT_0*,* nT_1*, and* nT_2 *are equivalent.*

3 n-continuous functions

Definition 3.1 *A function* $f : X \to Y$ *is said to be n-continuous if* $f^{-1}(V)$ *is n-open in* X *for every proper nonempty open set* $V \subseteq Y$

Example 3.2 *Let* $X = \{a, b, c\}$ *have the topology* $\tau = \{X, \emptyset, \{a\}\}$ *and let* $f : (X, \tau) \to (X, \tau)$ *be given by* $f(a) = b, f(b) = a$*, and* $f(c) = a$*. Then, since* $f^{-1}(\{a\}) = \{b, c\}$ *which is not open and thus n-open,* f *is n-continuous.*

Example 3.3 *Let* $X = \{a, b, c\}$ *have the topologies* $\tau = \{X, \emptyset, \{a, b\}, \{c\}\}$ *and* $\sigma = \{X, \emptyset, \{c\}\}$ *The identity mapping* $f : (X, \tau) \to (X, \sigma)$ *is not n-continuous because* $f^{-1}(\{c\}) = \{c\}$*, which is clopen and therefor not n-open.*

These two examples together show that n-continuity is independent of continuity. The proof of the following theorem is straightforward.

Theorem 3.4 *A function* $f : X \to Y$ *is n-continuous if and only if* $f^{-1}(F)$ *is n-closed in* X *for every proper nonempty closed set* $F \subseteq Y$*.*

Remark 3.5 *If* Y *is not indiscrete and* X *is discrete, then there is no n-continuous function* $f : X \to Y$*.*

Theorem 3.6 *A space X is connected if and only if the identity mapping $f : X \to X$ is n-continuous.*

Proof. Let X be a space and let $f : X \to X$ be the identity mapping. Then X is connected if and only if every proper nonempty open subset V of X is non-clopen if and only if for every proper nonempty open subset V of X, $f^{-1}(V)$ is n-open if and only if f is n-continuous.

An analogous proof yields the next result.

Theorem 3.7 *A space X is connected if and only if for every space Y every function $f : X \to Y$ with the property that $f^{-1}(V)$ is a proper nonempty set for every proper nonempty open subset V of Y is n-continuous.*

Lemma 3.8 *If $f : X \to Y$ is surjective, then $f^{-1}(V)$ is a proper nonempty set for every proper nonempty subset V of Y.*

Corollary 3.9 *A space X is connected if and only if for every space Y every surjective function $f : X \to Y$ is n-continuous.*

Theorem 3.10 *If Y is not indiscrete and $f : X \to Y$ is n-continuous, then either f is not continuous or f is not contra-continuous.*

Proof. Since Y is not indiscrete, Y has a proper nonempty open set V. Because f is n-continuous, $f^{-1}(V)$ is n-open and hence not clopen. Therefore $f^{-1}(V)$ is either not open or not closed. Thus f either not continuous or not contra-continuous.

Example 3.11 *Let $X = \{a, b, c\}$ have the topologies $\tau = \{X, \emptyset, \{a\}, \{b, c\}\}$ and $\sigma = \{X, \emptyset, \{b\}, \{c\}, \{b, c\}\}$ The function $f : (X, \tau) \to (X, \sigma)$ given by $f(a) = c$, $f(b) = b$, and $f(c) = a$ is neither continuous nor contra-continuous, nor n-continuous. Note that $f^{-1}(\{b\})$ is neither open nor closed and $f^{-1}(\{c\})$ is not n-open.*

Thus the converse of Theorem 3.10 is does not hold.

Theorem 3.12 *If $f : X \to Y$ is n-continuous, then the following equivalent conditions hold:*

(a) *$nInt(f^{-1}(V)) = f^{-1}(V)$ for every proper open set $V \subseteq Y$.*

(b) *$nCl(f^{-1}(F)) = f^{-1}(F)$ for every nonempty closed set $F \subseteq Y$.*

Proof. Parts (a) and (b) follow from the definitions of the n-interior and the n-closure of a set and Theorem 2.10. The equivalence of (a) and (b) follows from Theorem 2.13.

As we see in the following example, neither (a) nor (b) of Theorem 3.12 implies n-continuity.

Example 3.13 *Let* $X = \{a, b, c\}$ *have the topologies* $\tau = \{X, \emptyset, \{a, b\}, \{c\}\}$ *and* $\sigma = \{X, \emptyset, \{a, b\}\}$ *The identity mapping* $f : (X, \tau) \to (X, \sigma)$ *satisfies both (a) and (b) of Theorem 3.12 but is not n-continuous, since* $f^{-1}(\{a, b\})$ *is not n-open.*

Recall that the graph of a function $f : X \to Y$ is given by $G(f) = \{(x, y) \in X \times Y : y = f(x)\}$.

Theorem 3.14 *If* Y *is* T_1 *and* $f : X \to Y$ *is n-continuous, then* $X \times Y - G(f)$ *is a union of n-open sets.*

Proof. Let $(x, y) \in X \times Y - G(f)$. Since $y \neq f(x)$, there exists an open set $V \subseteq Y$ such that $f(x) \in V$ and $y \notin V$. Then $(x, y) \in f^{-1}(V) \times (Y - V) \subseteq X \times Y - G(f)$. Since V is a proper nonempty open set and f is n-continuous, $f^{-1}(V)$ is n-open. It then follows from Theorem 2.20 that $f^{-1}(V) \times (Y - V)$ is n-open in $X \times Y$ and therefore $X \times Y - G(f)$ is a union of n-open sets.

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