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Generalized n-Closed Sets and Generalized n-Continuous Functions

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Abstract

The notion of a generalized n-closed set is introduced and the basic properties of these sets are established. A useful characterization of the generalized n-closed sets and a new property of the n-closure operator are proved. The concept of a generalized n-continuous function along with two related classes functions are developed.

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1 Introduction

The concept of an n-open set was introduced in [1]. In this note we continue this line of investigation by introducing generalized n-closed (briefly, gn-closed) sets. A useful characterization of these sets is proved. Specifically we show that a set A is gn-closed if and only if $\mathrm{nCl}(A) = A$. In general the gn-closed sets are better behaved and more useful than the n-open sets, although the sets do not necessarily form a minimal structure. Also a useful property of the n-closure operator is established. It is proved that for every subset A of a topological space X $\mathrm{nCl}(A) = A$ or $\mathrm{nCl}(A) = X$. The notion of a gn-continuous function is defined and the basic properties of these functions are developed. Conditions equivalent to gn-continuity are established. Also two classes of related functions, gn-closed functions and gn-irresolute functions, are introduced.

2 Preliminaries

The symbols X and Y represent topological spaces with no separation properties assumed unless explicitly stated. All sets are considered to be subsets of topological spaces. The closure and interior of a set A are signified by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively.

Definition 2.1 *Let X be a nonempty set and $\mathcal{P}(X)$ the power set of X. A subfamily m_X of $\mathcal{P}(X)$ is called a minimal structure (briefly an m-structure) on X* [2]*, if $\emptyset \in m_X$ and $X \in m_X$.*

Definition 2.2 *A subset A of a space X is said to be n-open* [1] *if $Int(A) \neq Cl(A)$. A subset of X is called n-closed if its complement is n-open.*

Theorem 2.3 [1] *A subset A of a space X is n-open if and only if A is not clopen.*

Corollary 2.4 [1] *A subset A of a space X is n-open if and only if $X - A$ is n-open.*

Thus the n-open sets coincide with the n-closed sets.

Definition 2.5 *Let A be a subset of a space X. The n-interior of A* [1] *is denoted by $nInt(A)$ and given by $nInt(A) = \cup\{U \subseteq X : U \subseteq A$ and U is n-open$\}$. The n-closure of A* [1] *is denoted by $nCl(A)$ and given by $nCl(A) = \cap\{F \subseteq X : A \subseteq F$ and F is n-closed$\}$.*

Theorem 2.6 [1] *The following statements hold for every set $A \subseteq X$:*

(a) $nInt(X - A) = X - nCl(A)$.

(b) $nCl(X - A) = X - nInt(A)$.

(c) $x \in nCl(A)$ *if and only if* $U \cap A \neq \emptyset$ *for every n-open set* U *containing* x.

Theorem 2.7 [1] *If X is a space, then*

(a) $nCl(X) = X$.

(b) $nInt(\emptyset) = \emptyset$.

Theorem 2.8 [1] *If X is a discrete space, then*

(a) *$nInt(A) = \emptyset$ for every set $A \subseteq X$.*

(b) *$nCl(A) = X$ for every set $A \subseteq X$.*

Theorem 2.9 [1] *If U is an n-open set and $U = A \cup B$, then either A is n-open or B is n-open.*

Definition 2.10 *A function $f : X \to Y$ is said to be n-continuous* [1] *if $f^{-1}(V)$ is n-open in X for every proper nonempty open set $V \subseteq Y$.*

See [1] for additional properties and notation concerning n-open sets.

3 Generalized n-Closed Sets

Definition 3.1 *A subset A of a space X is said to be generalized n-closed (briefly gn-closed) if whenever $A \subseteq U$ and U is open, then $nCl(A) \subseteq U$. A subset of X is called generalized n-open (briefly gn-open) if its complement is gn-closed.*

The collection of gn-closed sets may not form a minimal structure since it may not contain $\emptyset$.

Theorem 3.2 *Let A be a subset of a space X. Then A is gn-open if and only if $F \subseteq nInt(A)$ whenever $F \subseteq A$ and F is closed.*

Example 3.3 *Let $X = \{a, b, c\}$ have the topology $\tau = \{X, \emptyset, \{a, b\}, \{c\}\}$. The n-closed sets are $\{a\}, \{b\}, \{a, c\}$, and $\{b, c\}$. The gn-closed sets are $\{a\}, \{b\}$, $\{a, c\}$, $\{b, c\}, \{c\}$, X, and $\emptyset$. The set $\{c\}$ is gn-closed but not n-closed and the set $\{a, b\}$ is closed but not gn-closed.*

Obviously n-closed sets are gn-closed. Example 3.3 shows that in general the two collections are not equal. Also from Example 3.3 the gn-closed sets and the gn-open sets do not coincide and the gn-closed sets are not in general closed under union. The fact that n-open sets are not closed under either union or intersection is illustrated by Example 3.3.

Example 3.4 *Let $X = \{a, b, c\}$ have the topology $\tau = \{X, \emptyset, \{a\}\}$. All sets in X are gn-closed.*

Example 3.5 *If X is a discrete space, then X is the only gn-closed set.*

Example 3.6 *Let X denote the real numbers with the usual topology. Since there are no proper nonempty clopen sets, all proper nonempty sets are n-closed and hence all sets are gn-closed.*

Theorem 3.7 *Let A be a subset of a space X. Then A is gn-closed if and only if $nCl(A) = A$.*

Proof. For the sufficiency assume that A is gn-closed. If A is open, then $\mathrm{nCl}(A) \subseteq A$ and hence $\mathrm{nCl}(A) = A$. If A is not open, then A is not clopen and hence A is n-closed. Thus it follows from the definition of the n-closure operator that $\mathrm{nCl}(A) = A$.

The necessity follows immediately from the definition.

Corollary 3.8 *Let A be a subset of a space X. Then A is gn-open if and only if $nInt(A) = A$.*

Theorem 3.9 *The gn-closed sets in a space X are closed under arbitrary intersection.*

Proof. Let A_α be a gn-closed set for every $\alpha \in \mathcal{A}$. Then using Theorem 3.7 we obtain $\mathrm{nCl}(\cap_{\alpha\in\mathcal{A}} A_\alpha) \subseteq \cap_{\alpha\in\mathcal{A}} \mathrm{nCl}(A_\alpha) = \cap_{\alpha\in\mathcal{A}} A_\alpha$ and hence $\mathrm{nCl}(\cap_{\alpha\in\mathcal{A}} A_\alpha) = \cap_{\alpha\in\mathcal{A}} A_\alpha$. Thus $\cap_{\alpha\in\mathcal{A}} A_\alpha$ is gn-closed.

Remark 3.10 *If X is discrete, then X is the only gn-closed set in X and hence in the above proof $A_\alpha = X$ for every $\alpha \in \mathcal{A}$.*

Corollary 3.11 *The gn-open sets in a space X are closed under arbitrary union.*

Theorem 3.12 *Let A be a subset of a space X. Then $nCl(A) = A$ or $nCl(A) = X$.*

Proof. If A is n-closed, then by the definition of n-closure $\mathrm{nCl}(A) = A$. Assume A is not n-closed. If there is no n-closed set that contains A, then $\mathrm{nCl}(A) = X$. Assume F is an n-closed set such that $A \subseteq F$. Then $F = A \cup (F - A)$. Since F is n-closed and A is not n-closed, it follows from Theorem 2.9 that $F - A$ is n-closed. (Recall that a set is n-closed if and only if it is n-open.) Since it's complement $X - (F - A)$ is also n-closed and $A \subseteq F \cap (X - (F - A))$, it follows from the definition of the n-closure operator that $\mathrm{nCl}(A) \subseteq F \cap (X - (F - A))$. Since $F \cap (X - (F - A)) = A$, it follows that $\mathrm{nCl}(A) = A$.

Corollary 3.13 *Let A be a subset of a space X. Then $nCl(nCl(A)) = nCl(A)$.*

Corollary 3.14 *Let A be a subset of a space X. Then $nCl(A)$ is gn-closed.*

Corollary 3.15 *If A is a subset of a space X, then A is gn-closed if and only if $A = nCl(B)$ for some set $B \subseteq X$.*

Corollary 3.16 *If A is a proper subset of a space X, then A is gn-closed if and only if $nCl(A) \neq X$.*

Corollary 3.17 *The collection of all gn-closed sets of a space X is the set $\{A \subseteq X : nCl(A) \neq X\} \cup \{X\}$.*

Corollary 3.18 *Let A be a subset of a space X. If A is proper, gn-closed, and not n-closed, then A is the intersection of two n-closed sets.*

4 Generalized n-Continuous Functions

Definition 4.1 *A function $f : X \to Y$ is said to be generalized n-continuous (briefly gn-continuous) if $f^{-1}(F)$ is gn-closed in X for every closed set $F \subseteq Y$.*

Remark 4.2 *If X is a discrete space, then for every space Y there is no gn-continuous function $f : X \to Y$. Note that $f^{-1}(\emptyset) = \emptyset$, which is not gn-closed in X.*

Theorem 4.3 *The following conditions are equivalent for a function $f : X \to Y$:*

(a) *f is gn-continuous.*

(b) *$f^{-1}(V)$ is gn-open for every open set $V \subseteq Y$.*

(c) *$f^{-1}(Int(B)) \subseteq nInt(f^{-1}(B))$ for every set $B \subseteq Y$.*

(d) *$nCl(f^{-1}(B)) \subseteq f^{-1}(Cl(B))$ for every set $B \subseteq Y$.*

Proof. (a) $\Rightarrow$ (b) Let $V \subseteq Y$ be open. Then, using (a) and Teorem 3.7, we have $X - f^{-1}(V) = f^{-1}(Y - V) = \mathrm{nCl}(f^{-1}(Y - V)) = \mathrm{nCl}(X - f^{-1}(V)) = X - \mathrm{nInt}(f^{-1}(V))$. Thus $f^{-1}(V) = \mathrm{nInt}(f^{-1}(V))$ and by Corollary 3.8 $f^{-1}(V)$ is gn-open.

(b) $\Rightarrow$ (c) Let $B \subseteq Y$. By (b) $f^{-1}(\mathrm{Int}(B))$ is gn-open. Hence by Corollary 3.8 $f^{-1}(\mathrm{Int}(B)) = \mathrm{nInt}(f^{-1}(\mathrm{Int}(B))) \subseteq \mathrm{nInt}(f^{-1}(B))$.

(c) $\Rightarrow$ (d) Let $B \subseteq Y$. Then $X - f^{-1}(\mathrm{Cl}(B)) = f^{-1}(Y - \mathrm{Cl}(B)) = f^{-1}(\mathrm{Int}(Y - B)) \subseteq \mathrm{nInt}(f^{-1}(Y - B)) = \mathrm{nInt}(X - f^{-1}(B)) = X - \mathrm{nCl}(f^{-1}(B))$.

Therefore $\mathrm{nCl}(f^{-1}(B)) \subseteq f^{-1}(\mathrm{Cl}(B))$.

(d) $\Rightarrow$ (a) Let $F \subseteq Y$ be closed. It follows from (d) that $\mathrm{nCl}(f^{-1}(F)) \subseteq f^{-1}(\mathrm{Cl}(F)) = f^{-1}(F)$. By Theorem 3.7 $f^{-1}(F)$ is gn-closed and hence f is gn-continuous.

Theorem 4.4 *Assume X is not discrete. If $f : X \to Y$ is n-continuous, then f is gn-continuous.*

Remark 4.5 *If X is discrete and Y is indiscrete, then every function $f : X \to Y$ is n-continuous but not gn-continuous.*

Example 4.6 *Let $X = \{a, b, c\}$ have the topologies $\tau = \{X, \emptyset, \{a, b\}, \{c\}\}$ and $\sigma = \{X, \emptyset, \{a, b\}\}$. The identity function $f : (X, \tau) \to (X, \sigma)$ is gn-continuous but not n-continuous. Note that $f^{-1}(\{c\})$ is gn-closed but not n-closed.*

Definition 4.7 *A function $f : X \to Y$ is said to be generalized n-closed (briefly gn-closed) if $f(F)$ is gn-closed in Y for every gn-closed set $F \subseteq X$.*

Theorem 4.8 *The following conditions are equivalent for a function $f : X \to Y$:*

(a) *f is gn-closed.*

(b) *$f(nCl(A))$ is gn-closed for every set $A \subseteq X$.*

(c) *$nCl(f(A)) \subseteq f(nCl(A))$ for every set $A \subseteq X$.*

Proof. (a) $\Rightarrow$ (b) By Corollary 3.14 $\mathrm{nCl}(A)$ gn-closed for every set $A \subseteq X$.

(b) $\Rightarrow$ (c) Let $A \subseteq X$. Then $\mathrm{nCl}(f(A)) \subseteq \mathrm{nCl}(f(\mathrm{nCl}(A))) = f(\mathrm{nCl}(A))$.

(c) $\Rightarrow$ (a) $A \subseteq X$ be gn-closed. Then using (c) we obtain $\mathrm{nCl}(f(A)) \subseteq f(\mathrm{nCl}(A)) = f(A)$. Therefore $f(A) = \mathrm{nCl}(f(A))$ and hence $f(A)$ is gn-closed and f is gn-closed.

Definition 4.9 *A function $f : X \to Y$ is said to be generalized n-irresolute (briefly gn-irresolute) if $f^{-1}(F)$ is gn-closed in X for every gn-closed set $F \subseteq Y$.*

The proof of the following theorem is analogous to that of Theorem 4.8.

Theorem 4.10 *The following conditions are equivalent for a function $f : X \to Y$:*

(a) *f is gn-irresolute.*

(b) *$f^{-1}(nCl(A))$ is gn-closed for every set $A \subseteq Y$.*

(c) *$nCl(f^{-1}(A)) \subseteq f^{-1}(nCl(A))$ for every set $A \subseteq Y$.*

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