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Meet Countably Approximating Posets[1]

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Abstract

In this paper, the concept of meet countably approximating posets via the σ-Scott topology is introduced. Properties and characterizations of meet countably approximating posets are presented. The main results are: (1) a poset having countably directed joins is meet countably approximating iff its lattice of all σ-Scott-closed sets is a complete Heyting algebra; (2) a poset having countably directed joins is countably approximating iff it is meet countably approximating and generalized countably approximating.

Mathematical Subject Classifications: 06A11; 06B35; 54C35; 54D45

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1 Introduction

In 1972, Dana Scott introduced the notion of continuous lattices in order to provide models for the semantics of programming languages (see [10]). Later,

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a more general notion of continuous directed complete partially ordered sets (i.e., continuous dcpos or domains) was introduced and extensively studied (see [1]-[12]). It should be noted that a distinctive feature of the theory of continuous domains is that many of the considerations are closely interlinked with topological ideas. The Scott topology, as an order-theoretical topology, is of fundamental importance in domain theory. Lawson in [6] gave a remarkable characterization that a dcpo L is continuous iff the lattice $\sigma^*(L)$ of all Scott-closed subsets of L is completely distributive. Gierz, Lawson and Stralka in [2] introduced quasicontinuous domains, the most successful generalizations of continuous domains, and proved that quasicontinuous domains equipped with the Scott topologies are precisely spectra of hypercontinuous distributive lattices. A meet continuous lattice is a complete lattice in which the binary meet operation distributes over directed suprema (see [3]). This algebraic notion has a purely topological characterization that can be generalized to the setting of dcpos by the Scott topology in [3, 5] without involving the meet operations: A dcpo L is called *meet continuous* if for any $x \in L$ and any directed subset D with $\sup D \geqslant x$, one has $x \in cl_\sigma(\downarrow D \cap \downarrow x)$, where $cl_\sigma(\downarrow D \cap \downarrow x)$ is the Scott closure of the set $\downarrow D \cap \downarrow x$. It is well-known that a dcpo is continuous iff it is quasicontinuous and meet continuous.

On the other hand, Lee in [7] introduced the concept of countably approximating lattices, a generalization of continuous lattices. In [4], Han, Hong, Lee and Park further generalized the concept of countably approximating lattices to the concept of countably approximating posets and characterized countably approximating posets via the σ-Scott topology. Yang and Liu in [12] introduced the concept of generalized countably approximating posets and presented some properties of generalized countably approximating posets.

In this paper, we introduce the concept of meet countably approximating posets via the σ-Scott topology. Properties and characterizations of meet countably approximating posets are presented. With the obtained results, we are able to give some new characterizations of countably approximating posets.

2 Preliminaries

We quickly recall some basic notions and results (see, e.x., [3], [4] or [12]).

Let $(L, \leqslant)$ be a poset. Then L with the dual order is also a poset and denoted by L^{op}. A *principal ideal* (resp., *principal filter*) is a set of the form $\downarrow x = \{y \in L \mid y \leqslant x\}$ (resp., $\uparrow x = \{y \in L \mid x \leqslant y\}$). For $X \subseteq L$, we write $\downarrow X = \{y \in L \mid \exists\, x \in X, y \leqslant x\}$ and $\uparrow X = \{y \in L \mid \exists\, x \in X, x \leqslant y\}$. A subset X is a(n) *lower set* (resp., *upper set*) if $X = \downarrow X$ (resp., $X = \uparrow X$). The *supremum* of X is the least upper bound of X and denoted by $\vee X$ or $\sup X$. A subset D of L is *directed* if every finite subset of D has an upper bound in D. A subset D is *countably directed* if every countable subset of D has an upper bound in D. Clearly every countably directed set is directed but not vice versa. A poset L is a *directed complete partially ordered set* (dcpo, for short) if every directed subset of L has a supremum. A poset is said to have *countably directed joins* if every countably directed subset has a supremum.

It is clear that if D is countably directed and D is also countable, then D has a maximal element. By this observation, we see that every countable poset has countably directed joins and thus a poset having countably directed joins needn't be a dcpo.

Definition 2.1. (see [3, 11]) Let L be a poset and x, $y \in L$. We say that x is *way-below* y or x *approximates* y, written $x \ll y$ if whenever D is a directed set that has a supremum $\sup D \geqslant y$, then there is some $d \in D$ with $x \leqslant d$. For each $x \in L$, we write $\Downarrow x = \{y \in L \mid y \ll x\}$. A poset is said to be *continuous* if every element is the directed supremum of elements that approximate it. A continuous poset which is also a complete lattice is called a *continuous lattice*.

Definition 2.2. (see [4]) Let L be a poset and x, $y \in L$. We say that x is *countably way-below* y, written $x \ll_c y$ if for any countably directed subset D of L with $\sup D \geqslant y$, there is some $d \in D$ with $x \leqslant d$. For each $x \in L$, we write $\Downarrow_c x = \{y \in L \mid y \ll_c x\}$ and $\Uparrow_c x = \{y \in L \mid x \ll_c y\}$. A poset L having countably directed joins is called a *countably approximating poset* if for each $x \in L$, the set $\Downarrow_c x$ is countably directed and $x = \vee \Downarrow_c x$. A countably approximating poset which is also a complete lattice is called a *countably approximating lattice*.

In a poset L, since every singleton set is countably directed, it is clear that $x \ll_c y$ implies that $x \leqslant y$. Note that every countably directed set is directed, we have that $x \ll y$ implies $x \ll_c y$ for all x, $y \in L$. In other words, we have $\Downarrow y \subseteq \Downarrow_c y$ for each $y \in L$. However, the following example shows that the reverse implication need not be true.

Example 2.3. Let L be the unit interval $[0, 1]$. For all x, $y \in [0, 1]$, it is easy to check that $x \ll_c y \Leftrightarrow x \leqslant y$, and that $x \ll y \Leftrightarrow x = 0 = y$ or $x < y$.

It is clear that every countable poset is a countably approximating poset since every countably directed subset of a countable poset has a maximal element.

Proposition 2.4. *Let L be a poset and S a countable subset of L such that $\vee S$ exists. If $s \ll_c x$ for all $s \in S$, then $\vee S \ll_c x$.*

Proof. Let D be a countably directed subset of L with $\sup D \geqslant x$. For all $s \in S$, it follows from $s \ll_c x$ that there is $d_s \in D$ such that $s \leqslant d_s$. Since S is a countable subset of L, the set $\{d_s \mid s \in S\}$ is a countable subset of D. By the countable directedness of D, the set $\{d_s \mid s \in S\}$ has an upper bound d_0 in D. Thus $s \leqslant d_s \leqslant d_0$ for all $s \in S$. This shows that $\vee S \leqslant d_0$ and hence $\vee S \ll_c x$. □

By Proposition 2.4, in a complete lattice L, the set $\Downarrow_c x$ is automatically countably directed for each $x \in L$. So, a complete lattice L is countably approximating iff for each $x \in L$, $x = \vee \Downarrow_c x$. Thus every continuous lattice is a countably approximating lattice.

For a set X, we use $\mathcal{P}(X)$ to denote the power set of X and $\mathcal{P}_{fin}(X)$ to denote the set of all nonempty finite subsets of X. For a poset L, define a preorder $\leq$ (sometimes called *Smyth preorder*) on $\mathcal{P}(L)\setminus\{\emptyset\}$ by $G \leq H$ iff $\uparrow H \subseteq \uparrow G$ for all $G, H \subseteq L$. That is, $G \leq H$ iff for each $y \in H$ there is an element $x \in G$ with $x \leqslant y$. We say that a nonempty family $\mathcal{F}$ of subsets of L is (*countably*) *directed* if it is (countably) directed in the Smyth preorder. More precisely, $\mathcal{F}$ is directed if for all F_1, $F_2 \in \mathcal{F}$, there exists $F \in \mathcal{F}$ such that F_1, $F_2 \leq F$, i.e., $F \subseteq \uparrow F_1 \cap \uparrow F_2$.

Generalizing the relation $\ll_c$ on points of L to the nonempty subsets of L, one obtains the concept of generalized countably approximating posets.

Definition 2.5. (see [12]) Let L be a poset having countably directed joins. A binary relation $\ll_c$ on $\mathcal{P}(L)\setminus\{\emptyset\}$ is defined as follows: $A \ll_c B$ iff for any countably directed set $D \subseteq L$, $\vee D \in \uparrow B$ implies $D \cap \uparrow A \neq \emptyset$. We write $F \ll_c x$ for $F \ll_c \{x\}$ and $y \ll_c H$ for $\{y\} \ll_c H$. If for each $x \in L$, the family $\omega(x) = \{F \mid F \in \mathcal{P}_{fin}(L) \textit{ and } F \ll_c x\}$ is countably directed and $\uparrow x = \cap\{\uparrow F \mid F \in \omega(x)\}$, then L is called a *generalized countably approximating poset*.

By [12, Remark 2.12], every countably approximating poset is a generalized countably approximating poset.

Definition 2.6. (see [3, 11]) A subset U of a poset L is *Scott-open* if $\uparrow U = U$ and for any directed set $D \subseteq L$, $\sup D \in U$ implies $U \cap D \neq \emptyset$. All the Scott-open sets of L forms a topology, called the *Scott topology* and denoted by $\sigma(L)$. The complement of a Scott-open set is called a *Scott-closed* set. The collection of all Scott-closed sets of L is denoted by $\sigma^*(L)$. The topology generated by the complements of all principal filters $\uparrow x$ (resp., principal ideals $\downarrow x$) is called the *lower topology* (resp., *upper topology*) and denoted $\omega(L)$ (resp., $\nu(L)$). The common refinement $\sigma(L) \vee \omega(L)$ of the Scott topology and the lower topology is called the *Lawson topology*, denoted $\lambda(L)$.

Replacing directed sets with countably directed sets in Definition 2.6, we can get the concept of σ-Scott-open sets.

Definition 2.7. (see [4]) Let L be a poset. A subset U of L is called *σ-Scott-open* if $\uparrow U = U$ and for any countably directed set $D \subseteq L$, $\sup D \in U$ implies $U \cap D \neq \emptyset$. All the σ-Scott-open sets of L forms a topology, called the *σ-Scott topology* and denoted by $\sigma_c(L)$. The complement of a σ-Scott-open set is called a *σ-Scott-closed* set. The collection of all σ-Scott-closed sets of L is denoted by $\sigma_c^*(L)$.

Remark 2.8. (see [4, Remark 2.1]) (1) For a poset L, the σ-Scott topology $\sigma_c(L)$ is closed under countably intersections and the Scott topology $\sigma(L)$ is coarser than $\sigma_c(L)$, i.e., $\sigma(L) \subseteq \sigma_c(L)$.

(2) A subset F of a poset L is σ-Scott-closed if and only if F is a lower set and for any countably directed set $D \subseteq F$, $\sup D \in F$ whenever $\sup D$ exists.

Definition 2.9. Let L be a poset having countably directed joins. The common refinement $\sigma_c(L) \vee \omega(L)$ of the σ-Scott topology and the lower topology is called the *σ-Lawson topology*, denoted $\lambda_c(L)$.

Lemma 2.10. (see [12, Theorem 3.5]) Let L be a poset having countably directed joins. Then L is a countably approximating poset iff the lattice $\sigma_c(L)$ is a completely distributive lattice.

3 Meet countably approximating posets

In this section, in terms of the σ-Scott topology, the notion of meet countably approximating posets is introduced. Some properties and characterizations of meet countably approximating posets are presented. As one of main results, it is proved that a poset having countably directed joins is countably approximating iff it is meet countably approximating and generalized countably approximating.

Definition 3.1. Let L be a poset having countably directed joins. If for any $x \in L$ and any countably directed subset D with $\sup D \geqslant x$, one has $x \in cl_{\sigma_c}(\downarrow D \cap \downarrow x)$, where $cl_{\sigma_c}(\downarrow D \cap \downarrow x)$ is the σ-Scott closure of the set $\downarrow D \cap \downarrow x$, then L is called a *meet countably approximating poset*.

Proposition 3.2. *If L is a countably approximating poset, then L is a meet countably approximating poset.*

Proof. Let $x \in L$ and D a countably directed set with $\sup D \geqslant x$. It is clear that $\Downarrow_c x \subseteq \downarrow D$ and $\Downarrow_c x \subseteq \downarrow D \cap \downarrow x$. By the countably approximating property of L and the σ-Scott-closedness of $cl_{\sigma_c}(\downarrow D \cap \downarrow x)$, we have $x = \sup \Downarrow_c x \in cl_{\sigma_c}(\downarrow D \cap \downarrow x)$. This shows that L is a meet countably approximating poset. □

Theorem 3.3. *Let L be a poset having countably directed joins. Then the following statements are equivalent:*

(1) *L is a meet countably approximating poset;*
(2) *$\forall\, U \in \sigma_c(L)$, $\forall\, x \in L$, $\uparrow(U \cap \downarrow x) \in \sigma_c(L)$;*
(3) *$\forall\, U \in \sigma_c(L)$, for any lower set $C \subseteq L$, $\uparrow(U \cap C) \in \sigma_c(L)$.*

Proof. (1) $\Rightarrow$ (2): Let $x \in L$ and $U \in \sigma_c(L)$. Suppose that D is a countably directed subset with $\sup D \in \uparrow(U \cap \downarrow x)$. Then there is $y \in U \cap \downarrow x$ such that

$y \leqslant \sup D$. By the meet countably approximating property of L, we have $\downarrow D \cap \downarrow y \cap U \neq \emptyset$. So, $D \cap \uparrow(U \cap \downarrow x) \supseteq D \cap \uparrow(U \cap \downarrow y) \neq \emptyset$. This shows that $\uparrow(U \cap \downarrow x)$ is σ-Scott-open.

(2) $\Rightarrow$ (3): $\forall\, U \in \sigma_c(L)$, for any lower set $C \subseteq L$,

$$\uparrow(U \cap C) = \uparrow(U \cap (\bigcup_{x \in C} \downarrow x)) = \uparrow(\bigcup_{x \in C}(U \cap \downarrow x)) = \bigcup_{x \in C} \uparrow(U \cap \downarrow x).$$

By (2), one has $\uparrow(U \cap C) \in \sigma_c(L)$.

(3) $\Rightarrow$ (1): Let $x \in L$ and D a countably directed subset with $\sup D \geqslant x$. If x is not in $cl_{\sigma_c}(\downarrow D \cap \downarrow x)$, then there is $U \in \sigma_c(L)$ such that $x \in U$ and $U \cap \downarrow D \cap \downarrow x = \emptyset$. This implies $\uparrow (U \cap \downarrow x) \cap D = \emptyset$. It is clear that $\sup D \in \uparrow(U \cap \downarrow x)$. Then by (3), we see that $\uparrow(U \cap \downarrow x)$ is σ-Scott-open, there is $d \in \uparrow(U \cap \downarrow x) \cap D$, a contradiction. This shows that $x \in cl_{\sigma_c}(\downarrow D \cap \downarrow x)$ and hence L is meet countably approximating. □

We now arrive at a characterization of meet countably approximating posets via the lattice of σ-Scott-closed subsets.

Theorem 3.4. *Let L be a poset having countably directed joins. Then the following conditions are equivalent:*

(1) *L is a meet countably approximating poset;*

(2) *$\sigma_c^*(L)$ is a complete Heyting algebra.*

Proof. (1) $\Rightarrow$ (2): Clearly, $\sigma_c^*(L)$ is a complete lattice. So, it suffices to show the frame distributive law

$$F \wedge (\bigvee_{i \in I} F_i) = \bigvee_{i \in I}(F \wedge F_i)$$

holds for $\sigma_c^*(L)$, where $F, F_i \in \sigma_c^*(L)\,(i \in I)$. Clearly, $F \wedge (\bigvee_{i \in I} F_i) \supseteq \bigvee_{i \in I}(F \wedge F_i)$. To show $F \wedge (\bigvee_{i \in I} F_i) \subseteq \bigvee_{i \in I}(F \wedge F_i)$, let $x \in F \wedge (\bigvee_{i \in I} F_i) = F \cap (\bigvee_{i \in I} F_i) = F \cap cl_{\sigma_c}(\bigcup_{i \in I} F_i)$. Then for all $U \in \sigma_c(L)$ with $x \in U$, we have $x \in U \cap F$ and $x \in \uparrow (U \cap F) \in \sigma_c(L)$ by Theorem 3.3 (3). And then there is $i_0 \in I$ such that $\uparrow (U \cap F) \cap F_{i_0} \neq \emptyset$. So, $(U \cap F) \cap \downarrow F_{i_0} = U \cap (F \cap F_{i_0}) \neq \emptyset$. By the arbitrariness of $U \in \sigma(L)$, we have $x \in cl_{\sigma}(\bigcup_{i \in I}(F \cap F_i)) = \bigvee_{i \in I}(F \wedge F_i)$. The frame distributivity of $\sigma_c^*(L)$ is thus proved. Hence, $\sigma_c^*(L)$ is a complete Heyting algebra.

(2) $\Rightarrow$ (1): Let $x \in L$ and D a countably directed subset with $\sup D \geqslant x$. Then $\{\downarrow d \mid d \in D\}$ is a countably directed set of $\sigma_c^*(L)$. By Remark 2.8 (2), $\sup D \in cl_{\sigma_c}(\downarrow D) = \bigvee_{d \in D} \downarrow d$ and thus $\downarrow x \subseteq \bigvee_{d \in D} \downarrow d$. By (2),

$$x \in \downarrow x = \downarrow x \cap (\bigvee_{d \in D} \downarrow d) = \bigvee_{d \in D}(\downarrow d \cap \downarrow x) = cl_{\sigma}(\bigcup_{d \in D}(\downarrow d \cap \downarrow x)) = cl_{\sigma}(\downarrow D \cap \downarrow x).$$

Thus L is a meet countably approximating poset. □

Recall that a poset L is called a *hypercontinuous poset* (see [9]) if for all $x \in L$, the set $\{y \in L \mid y \prec_{\nu(L)} x\}$ is directed and $x = \sup\{y \in L \mid y \prec_{\nu(L)} x\}$, where $y \prec_{\nu(L)} x \Leftrightarrow x \in int_{\nu(L)} \uparrow y$. A hypercontinuous poset which is also a complete lattice is called a *hypercontinuous lattice*.

Lemma 3.5. (see [12, Theorem 3.4]) Let L be a poset having countably directed joins. Then L is a generalized countably approximating poset iff the lattice $\sigma_c(L)$ is a hypercontinuous lattice.

With the above results, we have the following characterization of countably approximating property of posets via meet countably approximating property.

Theorem 3.6. *Let L be a poset having countably directed joins. The following statements are equivalent:*

(1) *L is a countably approximating poset;*

(2) *L is a meet countably approximating and generalized countably approximating poset.*

Proof. (1) ⇒ (2): By [12, Remark 2.12] and Proposition 3.2.
(2) ⇒ (1): By Lemma 2.10, we need only to show the completely distributivity of $\sigma_c(L)$. By the generalized countably approximating property of L and Lemma 3.5, $\sigma_c(L)$ is a hypercontinuous lattice and thus $\sigma_c^*(L)$ is a generalized continuous lattice (see [1, Theorem 6.4]). Since L is also meet countably approximating, $\sigma_c^*(L)$ is a complete Heyting algebra by Theorem 3.4. Noticing that a complete Heyting algebra is precisely a meet continuous lattice, so by we see that $\sigma_c^*(L)$ is thus a continuous lattice. By [3, Theorem I-3.16], we see that $\sigma_c(L)$ is a completely distributive lattice. By Lemma 2.10, L is a countably approximating poset. □

References

[1] G. Gierz and J. D. Lawson, Generalized continuous and hypercontinuous lattices, *Rocky Mountain J. Math.*, **11** (1981), 271-296.
https://doi.org/10.1216/rmj-1981-11-2-271

[2] G. Gierz, J. D. Lawson, and A. Stralka, Quasicontinuous posets, *Houston Journal of Mathematics*, **9** (2) (1983), 191-208.

[3] G. Gierz, K. Hofmann, K. Keimel et al., *Continuous Lattices and Domains*, Cambridge University Press, Cambridge, 2003.
https://doi.org/10.1017/cbo9780511542725

[4] Y. H. Han, S. S. Hong, C. K. Lee, P. U. Park, A generalization of continuous posets, *Comm. Korean Math. Soc.*, **4** (1) (1989), 129-138.

[5] H. Kou, Y. M. Liu, M. K. Luo, On meet-continuous dcpos, in: G. Q. Zhang, J. Lawson, Y. M. Liu, M. K. Luo (Eds), *Domain Theory, Logic and Computation*, Kluwer Academic Publishers, 2003, 137-149. https://doi.org/10.1007/978-94-017-1291-0_6

[6] J. D. Lawson, The duality of continuous posets, *Houston J. of Mathematics*, **5** (1979), 357-394.

[7] S. O. Lee, On countably approximating lattices, *J. Korean Math. Soc.*, **25** (1), (1988), 11-23.

[8] Xuxin Mao, Luoshan Xu, Meet continuity properties of posets, *Theoretical Computer Science*, **410** (2009), 4234-4240. https://doi.org/10.1016/j.tcs.2009.06.017

[9] Xuxin Mao, Luoshan Xu, Quasicontinuity of posets via Scott topology and sobrification, *Order*, **23** (4) (2006), 359-369. https://doi.org/10.1007/s11083-007-9054-4

[10] D. S. Scott, *Continuous Lattice*, Lecture Notes in Mathematics, Vol. 274, Springer-Verlag, Berlin, 1972, 97-136. https://doi.org/10.1007/bfb0073967

[11] Luoshan Xu, Continuity of Posets via Scott Topology and Sobrification, *Topology and Its Applications*, **153** (2006), 1886-1894. https://doi.org/10.1016/j.topol.2004.02.024

[12] Jinbao Yang, Min Liu, On generalized countably approximating posets, *Journal of the Chungcheong Mathematical Society*, **25** (3) (2012), 415-424. https://doi.org/10.14403/jcms.2012.25.3.415

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