International Journal of Contemporary Mathematical Sciences
Vol. 15, 2020, no. 4, 215 - 224
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ijcms.2020.91460

Meet Countably Approximating Posets Revisited[1]

Xuxin Mao

College of Science, Nanjing University of Aeronautics and Astronautics
Nanjing 210016, China

Luoshan Xu [2]

Department of Mathematics, Yangzhou University
Yangzhou 225002, China

Abstract

In this paper, meet countably approximating posets are revisited. In terms of the σ-measurement topology and principal ideals, some topological and order-theoretical characterizations of meet countably approximating posets are presented. The main results are: (1) A poset L having countably directed joins is meet countably approximating iff for any open set U in the σ-measurement topology, $\uparrow U$ is σ-Scott-open; (2) Meet countably approximating posets are hereditary to σ-Scott-open and to σ-Scott-closed subsets; (3) A poset L having countably directed joins is meet countably approximating iff every principal ideal is meet countably approximating; (4) Lifts and retracts of meet countably approximating posets are still meet countably approximating.

Mathematics Subject Classifications: 06A11; 06B35; 54C35; 54D45

Keywords: σ-Scott topology; meet countably approximating poset; σ-measurement topology; principal ideal

---

[1]Supported by the NSF of China (11671008, 11101212, 61103018).
[2]Corresponding author

1 Introduction

In 1972, Dana Scott introduced the notion of continuous lattices in order to provide models for the semantics of programming languages [8]. Later, a more general notion of continuous directed complete partially ordered sets (i.e., continuous dcpos or domains) was introduced and extensively studied [1, 5]. It should be noted that a distinctive feature of domain theory is that many of the considerations are closely linked with topological ideas.

Lee in [4] introduced the concept of countably approximating lattices, a generalization of continuous lattices. Han, etc, in [2] generalized the concept of countably approximating lattices to the concept of countably approximating posets and characterized countably approximating posets via the σ-Scott topology. Yang and Liu in [10] introduced generalized countably approximating posets and presented some properties of them. Recently, Mao and Xu in [6] introduced the concept of meet countably approximating posets and showed that a poset having countably directed joins is countably approximating iff it is meet countably approximating and generalized countably approximating.

In this paper, in terms of the σ-measurement topology, some further topological characterizations of meet countably approximating posets are presented. Some operational properties of meet countably approximating posets are explored. We will see that meet countably approximating posets are hereditary to σ-Scott-open sets and to σ-Scott-closed subsets. A characterization theorem of meet countably approximating posets by principal ideals is also given. It is proved that meet countably approximating property is invariant under operations of coverings, liftings and retractions.

2 Preliminaries

We quickly recall some basic notions and results (see, e.x., [1], [2], [6]).

Let $(L, \leqslant)$ be a poset. Then L with the dual order is also a poset and denoted by L^{op}. A *principal ideal* (resp., *principal filter*) is a set of the form $\downarrow x = \{y \in L \mid y \leqslant x\}$ (resp., $\uparrow x = \{y \in L \mid x \leqslant y\}$). A *closed interval* $[x, y]$ is a set of the form $\uparrow x \cap \downarrow y$ for $x \leqslant y$. For $X \subseteq L$, we write $\downarrow X = \{y \in L \mid \exists\, x \in X, y \leqslant x\}$ and $\uparrow X = \{y \in L \mid \exists\, x \in X, x \leqslant y\}$. A subset X is a(n) *lower set* (resp., *upper set*) if $X = \downarrow X$ (resp., $X = \uparrow X$). The *supremum* of X is the least upper bound of X and denoted by $\vee X$ or $\sup X$. A subset D of L is *directed* if every finite subset of D has an upper bound in D. A subset D is *countably directed* if every countable subset of D has an upper bound in D. Clearly every countably directed set is directed but not vice versa. A poset L is a *directed complete partially ordered set* (dcpo, for short) if every directed subset of L has a supremum. A poset is said to have *countably directed joins* if every countably directed subset has a supremum.

It is clear that if D is countably directed and D is also countable, then D has a maximal element. By this observation, we see that every countable poset has countably directed joins and thus a poset having countably directed joins needn't be a dcpo.

Recall that the topology on a poset L whose open sets are upper/lower sets is called the *Alexandrov topology/dual Alexandrov topology* and denoted by $\alpha(L)/\alpha^*(L)$. The topology generated by the complements of all principal filters $\uparrow x$ (resp., principal ideals $\downarrow x$) is called the *lower topology* (resp., *upper topology*) and denoted by $\omega(L)$ (resp., $\nu(L)$).

Definition 2.1. (cf. [2, 6]) Let L be a poset. A subset U of L is called *σ-Scott-open* if $\uparrow U = U$ and for any countably directed set $D \subseteq L$, $\sup D \in U$ implies $U \cap D \neq \emptyset$. All the σ-Scott-open sets of L forms a topology, called the *σ-Scott topology* and denoted by $\sigma_c(L)$. The complement of a σ-Scott-open set is called a *σ-Scott-closed* set. The collection of all σ-Scott-closed sets of L is denoted by $\sigma_c^*(L)$. The common refinement $\sigma_c(L) \vee \omega(L)$ of the σ-Scott topology and the lower topology is called the *σ-Lawson topology*, denoted $\lambda_c(L)$.

Remark 2.2. (cf. [2, Remark 2.1]) A subset F of a poset L is σ-Scott-closed iff F is a lower set and $\sup D \in F$ for any countably directed set $D \subseteq F$.

Definition 2.3. (see [2]) Let P and Q be posets. A function $f : P \to Q$ is called *σ-Scott-continuous* if it is continuous with respect to the σ-Scott topologies on P and Q.

Proposition 2.4. (see [2, Remark 2.1]) Let L and M be posets having countably directed joins. A function $f : L \to M$ is σ-Scott-continuous iff it is order-preserving and $f(\sup D) = \sup f(D)$ whenever D is a countably directed set in L.

Definition 2.5. (see [6]) Let L be a poset having countably directed joins. If for any $x \in L$ and any countably directed subset D with $\sup D \geqslant x$, one has $x \in cl_{\sigma_c}(\downarrow D \cap \downarrow x)$, where $cl_{\sigma_c}(\downarrow D \cap \downarrow x)$ is the σ-Scott closure of the set $\downarrow D \cap \downarrow x$, then L is called a *meet countably approximating poset*.

Lemma 2.6. (see [6]) Let L be a poset having countably directed joins. Then the following conditions are equivalent:

(1) L is a meet countably approximating poset;

(2) $\sigma_c^*(L)$ is a complete Heyting algebra.

3 The σ-measurement topology

In this section, in terms of the σ-measurement topology, some further characterizations of meet countably approximating posets are presented.

Definition 3.1. Let L be a poset. The common refinement $\sigma_c(L) \vee \alpha^*(L)$ of the σ-Scott topology and the dual Alexandrov topology is called the *σ-measurement topology* and is denoted by $\mu_c(L)$.

Recall that for any topology τ on a set X, the collection $\{O \cap C \mid O, X \backslash C \in \tau\}$ forms a basis of a topology, the so-called *b-topology* for τ (see [9]).

Proposition 3.2. *Let L be a poset. Then the σ-measurement topology $\mu_c(L)$ is the b-topology for the σ-Scott topology $\sigma_c(L)$.*

Proof. Straightforward. □

Theorem 3.3. *Let L be a poset having countably directed joins. Then the following statements are equivalent:*

(1) *L is a meet countably approximating poset;*

(2) *$\forall\, U \in \sigma_c(L)$, $\forall\, x \in L$, $\uparrow(U\cap \downarrow x) \in \sigma_c(L)$;*

(3) *$\forall\, U \in \sigma_c(L)$, for any lower set $C \subseteq L$, $\uparrow(U \cap C) \in \sigma_c(L)$;*

(4) *$\forall\, U \in \mu_c(L)$, one has $\uparrow U \in \sigma_c(L)$, i.e., ${}^{\uparrow}\mu(L) = \{\uparrow U \mid U \in \mu_c(L)\} \subseteq \sigma_c(L)$.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3): Follows from [6, Theorem 3.3].

(3) $\Rightarrow$ (4): Suppose $U \in \mu_c(L)$. For all $t \in\uparrow U$, there are $V \in \sigma_c(L)$ and $C \in \alpha^*(L)$ such that $t \in\uparrow(V \cap C) \subseteq\uparrow U$. By (3), one has $\uparrow(V \cap C) \in \sigma_c(L)$. This shows that t is in the σ-Scott interior of $\uparrow U$. By the arbitrariness of $t \in\uparrow U$, one has $\uparrow U \in \sigma_c(L)$.

(4) $\Rightarrow$ (2): $\forall\, U \in \sigma_c(L)$, $\forall\, x \in L$, one has $U\cap \downarrow x \in \mu_c(L)$. By (4), $\uparrow(U\cap\downarrow x) \in \sigma_c(L)$. □

Corollary 3.4. *Let L be a meet countably approximating poset.*

(i) *If X is an upper set, then $int_{\sigma_c}X = int_{\lambda_c}X = int_{\mu_c}X$;*

(ii) *If X is a lower set, then $cl_{\sigma_c}X = cl_{\lambda_c}X = cl_{\mu_c}X$.*

Proof. (i) Suppose X is an upper set. Then $int_{\sigma_c}X \subseteq int_{\lambda_c}X \subseteq int_{\mu_c}X$ since $\sigma_c(L) \subseteq \lambda_c(L) \subseteq \mu_c(L)$. By Theorem 3.3(4), $int_{\mu_c}X \subseteq\uparrow int_{\mu_c}X = int_{\sigma_c}(\uparrow int_{\mu_c}X) \subseteq int_{\sigma_c}\uparrow X = int_{\sigma_c}X$. Hence, $int_{\sigma_c}X = int_{\lambda_c}X = int_{\mu_c}X$.

The equivalence of (i) and (ii) is straightforward. □

Recall that in a poset L, a nonempty subset F is *filtered* if for all $x, y \in F$, there is $z \in F$ such that $z \leqslant x$ and $z \leqslant y$. In this case, if $F =\uparrow F$, then F is called a *filter*. We say that a poset L with a topology has *small open filtered sets* iff each point has a neighborhood basis of open filtered sets (see [1, 3]).

Proposition 3.5. *Let L be a poset having countably directed joins. The following statements are equivalent:*

(1) *The σ-Scott topology $\sigma_c(L)$ has a basis of open filters;*

(2) *L is meet countably approximating and $\lambda_c(L)$ has small open filtered sets;*

(3) *L is meet countably approximating and $\mu_c(L)$ has small open filtered sets.*

Proof. (1) $\Rightarrow$ (2): Let $x \in L$ and D a countably directed set with $\sup D \geqslant x$. Then for all σ-Scott-open set $U \in \sigma_c(L)$ with $x \in U$, by (1) there is a σ-Scott-open filter V such that $x \in V \subseteq U$. By the σ-Scott-openness of V, we have that $D \cap V \neq \emptyset$. Pick $a \in D \cap V$. Since V is a filter, there is $b \in V$ such that $b \leqslant x$ and $b \leqslant a$. This shows that $b \in U \cap \downarrow x \cap \downarrow D \neq \emptyset$. Hence, $x \in cl_\sigma(\downarrow x \cap \downarrow D)$. By Definition 2.5, L is meet countably approximating. To show that $\lambda_c(L)$ has small open filtered sets, suppose that W is a σ-Lawson-open neighborhood of t. By (1), there are σ-Scott-open filter H and finite set F such that $t \in H \setminus \uparrow F \subseteq W$. Obviously, $H \setminus \uparrow F$ is σ-Lawson-open. For all v, $w \in H \setminus \uparrow F$, there is $h \in H$ such that $h \leqslant v$ and $h \leqslant w$ since H is a filter. It follows from $v, w \notin \uparrow F$ that $h \notin \uparrow F$ and $h \in H \setminus \uparrow F$. This shows that $H \setminus \uparrow F$ is filtered. Thus, $H \setminus \uparrow F$ is σ-Lawson-open filtered. By the arbitrariness of $t \in W$, $\lambda_c(L)$ has small open filtered sets.

(2) $\Rightarrow$ (1): Suppose $x \in U \in \sigma_c(L)$. It follows from $\sigma_c(L) \subseteq \lambda_c(L)$ and (2) that there is a σ-Lawson-open filtered set V such that $x \in V \subseteq U$. Hence, $x \in \uparrow V \subseteq \uparrow U = U$ and $\uparrow V$ is a filter. By the meet countably approximating property of L, $\lambda_c(L) \subseteq \mu_c(L)$ and Theorem 3.3(4), $\uparrow V$ is a σ-Scott-open filter. By the arbitrariness of $x \in U$, the σ-Scott topology $\sigma_c(L)$ has a basis of open filters.

(1) $\Rightarrow$ (3): Clearly, L is meet countably approximating by that (1) $\Leftrightarrow$ (2). Let $x \in W \in \mu_c(L)$. By Definition 3.1 and (1), there is a σ-Scott-open filter V such that $x \in V \cap \downarrow x \subseteq W$. Since V is a σ-Scott-open filter, it is easy to check that $V \cap \downarrow x$ is $\mu_c(L)$-open filtered. By the arbitrariness of $x \in W$, $\mu_c(L)$ has small open filtered sets.

(3) $\Rightarrow$ (1): Suppose $x \in U \in \sigma(L)$. It follows from $\sigma_c(L) \subseteq \mu_c(L)$ and (3) that there is a $\mu(L)$-open filtered set V such that $x \in V \subseteq U$. Hence, $x \in \uparrow V \subseteq \uparrow U = U$ and $\uparrow V$ is a filter. By the meet countably approximating property of L and Theorem 3.3(4), $\uparrow V$ is a σ-Scott-open filter. By the arbitrariness of $x \in U$, $\mu_c(L)$ has small open filtered sets. □

4 Heredity and Invariance

In this section, some operational properties of meet countably approximating posets are discussed. It will be established that a poset having countably directed joins is meet countably approximating iff every principal ideal is meet countably approximating.

Lemma 4.1. (see [5, Lemma 2.1]) Let L be a poset and $U \subseteq L$ an upper set. Then for all $\emptyset \neq A \subseteq U$, $\sup A = \sup_U A$ whenever one of them exists, where $\sup_U A$ denotes the supremum of A in U (with the relative order).

Lemma 4.2. *Let L be a poset and $U \subseteq L$ an upper set. Then $\sigma_c(L)|_U := \{W \cap U \mid W \in \sigma_c(L)\} \subseteq \sigma_c(U)$. If $U \in \sigma_c(L)$, then $\sigma_c(L)|_U = \sigma_c(U)$.*

Proof. Let $W \in \sigma_c(L)$. Trivially, $W \cap U$ is an upper set of U. For any countably directed $D \subseteq U$ with $\sup_U D \in W \cap U$, by Lemma 4.1, $\sup D = \sup_U D \in W \cap U$. By the σ-Scott openness of W, $D \cap W \neq \emptyset$ and $D \cap W \cap U \neq \emptyset$. This shows that $W \cap U \in \sigma_c(U)$. So, the relative σ-Scott topology on U is contained in the σ-Scott topology of the poset U.

If $U \in \sigma_c(L)$, then we show the converse containment is also true. Let $V \in \sigma_c(U)$. Trivially, V is an upper set of L. For any countably directed $D \subseteq L$ with $\sup D \in V$, by the σ-Scott-openness of U, $D \cap U \neq \emptyset$. Pick $d_0 \in D \cap U$. By Lemma 4.1 and the countably directedness of D, we have $\sup D = \sup(\uparrow d_0 \cap D) = \sup_U(\uparrow d_0 \cap D) \in V$. Since V is σ-Scott-open in U, $\uparrow d_0 \cap D \cap V \neq \emptyset$ and hence $D \cap V \neq \emptyset$. This shows that $V \in \sigma_c(L)$, as desired. □

Proposition 4.3. *Let L be a meet countably approximating poset and $U \subseteq L$ a σ-Scott-open set. Then U in the inherited order is meet countably approximating.*

Proof. Let L be a meet countably approximating poset. By Lemma 4.1, the subset U in the inherited order has countably directed joins. It follows from the meet countably approximating property of L and Lemma 2.6 that $\sigma_c^*(L)$ is a complete Heyting algebra. Thus it is straightforward to show that $\sigma_c^*(U)$ is a complete Heyting algebra by Lemma 4.2. Hence U in the inherited order is also meet countably approximating by Lemma 2.6. □

Lemma 4.4. *Let L be a poset having countably directed joins and $F \subseteq L$ a σ-Scott-closed set. Then $\sigma_c(L)|_F = \sigma_c(F)$.*

Proof. By Remark 2.2(2), it is straightforward to show that $\sigma_c^*(L)|_F = \sigma_c^*(F)$. Hence, $\sigma_c(L)|_F = \sigma_c(F)$. □

Proposition 4.5. *Let L be a meet countably approximating poset and $F \subseteq L$ a σ-Scott-closed set. Then F in the inherited order is also meet countably approximating. In particular, every principal ideal of L is a meet countably approximating poset.*

Proof. By Remark 2.2(2), the σ-Scott-closed set F in the inherited order has countably directed joins. It follows from the meet countably approximating property of L and Lemma 2.6 that $\sigma_c^*(L)$ is a complete Heyting algebra. Thus it is straightforward to show that $\sigma_c^*(F)$ is a complete Heyting algebra by Lemma 4.4. Hence F in the inherited order is also meet countably approximating by Lemma 2.6. □

Propositions 4.3 and 4.5 reveal that meet countably approximating property is hereditary to σ-Scott-open sets and to σ-Scott-closed sets.

Theorem 4.6. *Let L be a poset having countably directed joins. Then L is a meet countably approximating poset iff every principal ideal is a meet countably approximating poset.*

Proof. $\Rightarrow$: Follows from Proposition 4.5.

$\Leftarrow$: Assume each principal ideal of L is meet countably approximating. Let $x \in L$ and D a countably directed set with $\sup D := h \geqslant x$. Then $F = \downarrow h$ is a meet countably approximating poset and hence $x \in cl_{\sigma_c(F)}(\downarrow_F D \cap \downarrow_F x)$ by Definition 2.5. It follows from $\downarrow_F D = \downarrow_L D$, $\downarrow_F x = \downarrow_L x$ and Lemma 4.4 that $x \in cl_{\sigma_c(F)}(\downarrow_F D \cap \downarrow_F x) \subseteq cl_{\sigma_c(L)}(\downarrow_L D \cap \downarrow_L x)$. By Definition 2.5, L is a meet countably approximating poset. □

Corollary 4.7. *Let L be a poset having countably directed joins. Then L is meet countably approximating iff every σ-Scott-closed set in the inherited order is meet countably approximating.*

Proof. Apply Proposition 4.5 and Theorem 4.6. □

Corollary 4.8. *Let L be a poset having countably directed joins. Then every closed interval of L is meet countably approximating iff each principal filer $\uparrow x$ is meet countably approximating.*

Proof. Applying Theorem 4.6 to the principal filters of L. □

Let L be a poset. Adjoining an identity to L and forming poset $L^1 = L \cup \{1\}$ with $1 \notin L$ and $x < 1$ for all $x \in L$ is called the *covering* of L. The poset L^1 is called the *cover* of L. Adjoining a new bottom to L and forming poset $L_\perp = L \cup \{\perp\}$ with $x > \perp$ for all $x \in L$ is called the *lifting* of L. The poset $L_\perp$ is called the *lift* of L.

Proposition 4.9. *Let L be a poset having countably directed joins. Then L is meet countably approximating iff $L^1 = L \cup \{1\}$ is meet countably approximating.*

Proof. $\Leftarrow$: Let $L^1 = L \cup \{1\}$ is a meet countably approximating poset. Clearly, the singleton $\{1\} \in \sigma_c(L^1)$. Hence, L is a σ-Scott-closed subset of L^1. By Proposition 4.5, L is meet countably approximating.

$\Rightarrow$: Let L be a meet countably approximating poset. To show that $L^1 = L \cup \{1\}$ is meet countably approximating, by Definition 2.5 it suffices to show that for any $x \in L^1$ and any countably directed subset D with $\sup D \geqslant x$, one has $x \in cl_{\sigma_c(L^1)}(\downarrow D \cap \downarrow x)$. We divide the proof into three cases.

Case 1: $x = 1$. It follows from $x \leqslant \sup D$ that $1 \in D$. Then $\downarrow D \cap \downarrow x = L^1$ and hence $x \in cl_{\sigma_c(L^1)}(\downarrow D \cap \downarrow x)$.

Case 2: $x \in L$ and $1 \in D$. Then $\downarrow D \cap \downarrow x = \downarrow x$. Thus, $x \in cl_{\sigma_c(L^1)}(\downarrow D \cap \downarrow x)$.

Case 3: $x \in L$ and $1 \notin D \subset L$. By the meet countably approximating property of L, one has $x \in cl_{\sigma_c(L)}(\downarrow D \cap \downarrow x)$. Since L is a σ-Scott-closed subset of L^1, $x \in cl_{\sigma_c(L)}(\downarrow D \cap \downarrow x) = cl_{\sigma_c(L^1)}(\downarrow D \cap \downarrow x)$.

To sum up, in all cases, $x \in cl_{\sigma_c(L^1)}(\downarrow D \cap \downarrow x)$ and hence $L^1 = L \cup \{1\}$ is meet countably approximating. □

Lemma 4.10. *Let L be a poset having countably directed joins and $0 \in L$ a bottom. Then $\sigma_c(L) = \sigma_c(L \backslash \{0\}) \cup \{L\}$.*

Proof. Suppose $U \in \sigma_c(L)$. Then $U = L$ or $U \subset L$. If $U = L$, then $U \in \sigma_c(L \backslash \{0\}) \cup \{L\}$. If $U \subset L$, then $0 \notin U$. Since the singleton $\{0\}$ is σ-Scott-closed, one has $L \backslash \{0\} \in \sigma_c(L)$. By Lemma 4.2 and $U \in \sigma_c(L)$, we have $U = U \cap (L \backslash \{0\}) \in \sigma_c(L \backslash \{0\})$. So, $\sigma_c(L) \subseteq \sigma_c(L \backslash \{0\}) \cup \{L\}$.

Conversely, suppose $V \in \sigma_c(L \backslash \{0\}) \cup \{L\}$. If $V = L$, then $V \in \sigma_c(L)$. If $V \neq L$, then $V \in \sigma_c(L \backslash \{0\})$. By the σ-Scott-openness of $L \backslash \{0\}$ and Lemma 4.2, one has $V \in \sigma_c(L)$. This shows that $\sigma_c(L \backslash \{0\}) \cup \{L\} \subseteq \sigma_c(L)$. So, $\sigma_c(L) = \sigma_c(L \backslash \{0\}) \cup \{L\}$. □

Proposition 4.11. *Let L be a poset having countably directed joins and $0 \in L$ a bottom. Then L is meet countably approximating iff $L \backslash \{0\}$ is meet countably approximating.*

Proof. $\Rightarrow$: Suppose that L is a meet countably approximating poset. Since the singleton $\{0\}$ is σ-Scott-closed, one has $L \backslash \{0\} \in \sigma_c(L)$. By the meet countably approximating property of L and Proposition 4.3, $L \backslash \{0\}$ is meet countably approximating.

$\Leftarrow$: Suppose that $L \backslash \{0\}$ is meet countably approximating. To show that L is meet countably approximating, by Theorem 3.3(2) it suffices to show that $\forall U \in \sigma_c(L)$, $\forall\, x \in L$, one has $\uparrow_L (U \cap \downarrow_L x) \in \sigma_c(L)$. If $U = L$, then $\uparrow_L (U \cap \downarrow_L x) = \uparrow_L (\downarrow_L x) = L \in \sigma_c(L)$. If $U \neq L$ and $x = 0$, then $\uparrow_L (U \cap \downarrow_L x) = \emptyset \in \sigma_c(L)$. If $U \neq L$ and $x \neq 0$, then $U \in \sigma_c(L \backslash \{0\})$ by Lemma 4.10. It follows from the meet countably approximating property of $L \backslash \{0\}$ and $U \neq L$ that $\uparrow_L (U \cap \downarrow_L x) = \uparrow_L (U \cap \downarrow_{L \backslash \{0\}} x) = \uparrow_{L \backslash \{0\}} (U \cap \downarrow_{L \backslash \{0\}} x) \in \sigma_c(L \backslash \{0\} \subseteq \sigma_c(L)$. To sum up, in all cases, $\forall U \in \sigma_c(L)$, $\forall\, x \in L$, one has $\uparrow_L (U \cap \downarrow_L x) \in \sigma_c(L)$, as desired. □

Apply Proposition 4.11, we immediately have

Corollary 4.12. *Let L be a poset having countably directed joins. Then L is meet countably approximating iff the lift $L_\perp$ is meet countably approximating.*

Let L and M be posets having countably directed joins. M is called a *retract* of L if there exist σ-Scott-continuous functions $r : L \to M$ and $j : M \to L$ such that $rj = 1_M$ (see [2]). In this case, the function r is called a *retraction*.

Proposition 4.13. *Let L be a meet countably approximating poset. If M is a retract of L, then M is also meet countably approximating.*

Proof. To prove the meet countably approximating property of M by Definition 2.5, it suffices to show that for any $x \in M$ and any countably directed subset $D \subseteq M$ with $\sup_M D \geqslant x$, one has $x \in cl_{\sigma_c(M)}(\downarrow_M D \cap \downarrow_M x)$. To this end, suppose $U \in \sigma_c(M)$ with $x \in U$. Since M is a retract of L, there exist σ-Scott-continuous functions $r : L \to M$ and $j : M \to L$ such that $rj = 1_M$. By the σ-Scott-continuity of j and Proposition 2.4, $j(D) \subseteq L$ is countably directed and $j(x) \leqslant j(\sup D) = \sup j(D)$. Since $x = rj(x) \in U$ and r is σ-Scott-continuous, one has $j(x) \in r^{-1}(U)$ and $r^{-1}(U) \in \sigma_c(L)$. It follows from the meet countably approximating property of L and Definition 2.5 that $j(x) \in cl_{\sigma_c(L)}(\downarrow_L j(D) \cap \downarrow_L j(x))$. So, $r^{-1}(U) \cap (\downarrow_L j(D) \cap \downarrow_L j(x)) \neq \emptyset$. Pick $a \in r^{-1}(U) \cap (\downarrow_L j(D) \cap \downarrow_L j(x))$. Then $r(a) \in U \subseteq M$, $a \leqslant j(x)$ and there is $d \in D$ such that $a \leqslant j(d)$. Since r is σ-Scott-continuous, $r(a) \leqslant rj(x) = x$ and $r(a) \leqslant rj(d) = d \in D$. Hence, $r(a) \in U \cap (\downarrow_M D \cap \downarrow_M x) \neq \emptyset$. This shows that $x \in cl_{\sigma_c(M)}(\downarrow_M D \cap \downarrow_M x)$. □

Propositions 4.9, 4.13 and Corollary 4.12 reveal that meet countably approximating property is invariant under operations of coverings, liftings and retractions.

References

[1] G. Gierz, K. Hofmann, K. Keimel, et al., *Continuous Lattices and Domains*, Cambridge University Press, Cambridge, 2003.
https://doi.org/10.1017/cbo9780511542725

[2] Y. H. Han, S. S. Hong, C. K. Lee, P. U. Par,. A generalization of continuous posets, *Comm. Korean Math. Soc.*, **4** (1) (1989), 129-138.

[3] H. Kou, Y. M. Liu, M. K. Luo, On meet-continuous dcpos, in: G. Q. Zhang, J. Lawson, Y. M. Liu, M. K. Luo (Eds.), *Domain Theory, Logic and Computation*, Kluwer Academic Publishers, 2003, 137-149.

[4] S. O. Lee, On countably approximating lattices, *J. Korean Math. Soc.*, **25** (1) (1988), 11-23.

[5] Xuxin Mao, Luoshan Xu, Meet continuity properties of posets, *Theoretical Computer Science*, **410** (2009), 4234-4240. https://doi.org/10.1016/j.tcs.2009.06.017

[6] Xuxin Mao, Luoshan Xu, Meet Countably Approximating Posets, *Int. J. Contemp. Math. Science*, **15** (4) (2020), 191-198. https://doi.org/10.12988/ijcms.2020.91448

[7] K. Martin, *A Foundation for Computation*, Doctorial dissertation, Tulane University, 2000.

[8] D. S. Scott, *Continuous Lattice*, Lecture Notes in Mathematics 274, Springer-Verlag, Berlin, 1972, 97-136. https://doi.org/10.1007/bfb0073967

[9] P. Waszkiewiz, Distance and measurement in domain theory, *Electronic Notes in Computer Science*, **45** (2001), 448-462. https://doi.org/10.1016/s1571-0661(04)80975-7

[10] Jinbao Yang, Min Liu, On generalized countably approximating posets, *Journal of the Chungcheong Mathematical Society*, **25** (3) (2012), 415-424. https://doi.org/10.14403/jcms.2012.25.3.415

Received: July 19, 2020; Published: August 8, 2020