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The Pell Equations $x^2 - (k^2 - 1)y^2 = k^2, k \in \mathbb{N}, k \geq 2$

Recursion and Chebyshev Polynomials

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Dedicated to my colleague Prof. Dr. Dr. h.c. E. Ch. Wittmann

Abstract. Given $k \in \mathbb{N}, k \geq 2$ we give two recursion formulas for the elements in solution classes of our Pell equation $x^2 - (k^2 - 1)y^2 = k^2$ with parameter k. One of these recursions leads to a representation of the elements of each solution class by Chebshev polynomials and extends a result in [5] related only to the trivial solution class.

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1 Preliminaries

For the following theorems and definitions see [1] and [2].
Given parameter $k \in \mathbb{N}, k > 1$, then our subject are the Pell equations

$$x^2 - (k^2 - 1)y^2 = k^2 \qquad (1)$$

and to each its *related* Pell equation

$$x^2 - (k^2 - 1)y^2 = 1 \qquad (2)$$

which is also called Pell's *resolvent* [1].

If there are $x, y \in \mathbb{Z}$ with $x^2 - (k^2 - 1)y^2 = k^2$ then (x, y) is a solution to (1), for which we interchangeably write $x + y\sqrt{k^2 - 1}$. This can be traced back to the equation $x^2 - (k^2 - 1)y^2 = (x + y\sqrt{k^2 - 1})(x - y\sqrt{k^2 - 1})$. A solution (x, y) to (1) with $x, y > 0$ is called *positive solution*. If $gcd(x, y) = 1$ then (x, y) is called *primitive solution* to (1), otherwise *imprimitive solution*.

Our Pell equation (1) is always solvable by the trivial solution $(k, 0)$. The related Pell equation (2) is solvable with minimal positive primitive solution $(k, 1)$, and has infinitely many primitive solutions. It is well known that if (x, y) is any positive solution to (2) then there is $m \in \mathbb{N}$ such that

$$x + y\sqrt{k^2 - 1} = \left(k + 1\sqrt{k^2 - 1}\right)^m$$

Multiplication principle (MP). Let (a, b) be a solution to (2). If (x, y) is a solution to (1), then

$$x' + y'\sqrt{k^2 - 1} = \left(x + y\sqrt{k^2 - 1}\right)\left(a + b\sqrt{k^2 - 1}\right) =$$
$$(xa + yb(k^2 - 1) + (xb + ya)\sqrt{k^2 - 1},$$

which means

$$(x', y') = (xa + yb(k^2 - 1), xb + ya)$$

is also a solution to (1), which can be easily checked. Solution (x', y') to (1) is obtained by the *multiplication principle* MP, which shows furthermore, that (1) has infinitely many solutions.

Remark. If we look at the more general Pell equation $x^2 - dy^2 = k^2$, where $d \in \mathbb{N}$ is a non-square number and $k \in \mathbb{N}, k \geq 2$, then this equation is also always trivially solvable by $(k, 0)$. Let (u, v) be the fundamental solution to the related Pell equation $x^2 - dy^2 = 1$, then, by MP, (x_n, y_n), where $x_n + y_n\sqrt{d} = \left(k + 0\sqrt{d}\right)\left(u + v\sqrt{d}\right)^n, n \in \mathbb{N}$ is a solution to (1) and it is easily seen that $k|(x_n + y_n\sqrt{d})$ but $k^m \nmid (x_n + y_n\sqrt{d}), m > 1$.

Equivalent solutions ES. Let $k \in \mathbb{N}, k > 1$, the infinite set L_k of solutions to (1) can be partitioned into a finite number of classes in the following way. Solutions $x + y\sqrt{k^2 - 1}$, $x' + y'\sqrt{k^2 - 1}$ to (1) are *equivalent* if and only if there is a solution $a + b\sqrt{k^2 - 1}$ to (2) such that

$$x + y\sqrt{k^2 - 1} = \left(x' + y'\sqrt{k^2 - 1}\right)\left(a + b\sqrt{k^2 - 1}\right).$$

In this case we write $(x,y)\sim(x',y')$ or $x+y\sqrt{k^2-1}\sim x'+y'\sqrt{k^2-1}$. The relation $\sim$ is an equivalence relation in the set L_k and can be characterized in the following way, see [2].

Equivalent solutions criterion ESC (this criterion indeed holds for a much wider class of Pell equations).

Theorem 1.1 Let $(x,y),(x',y')$ be solutions to (1), then $(x,y)\sim(x',y')$ if and only if

$$xx' \equiv yy'(k^2-1) \; mod \; k^2 \text{ and } xy' \equiv x'y \; mod \; k^2$$

Proof. See [2].

Let (x,y) a solution to (1), then $(x,y)\sim(-x,-y),(x,-y)\sim(-x,y)$. The class which contains solution (x,y) of (1) is denoted by $K(x,y)$ and we call it *solution class*. The theorem above shows that, given (1), there are only a finite number of solution classes to (1).

In general $K(x,y)\neq K(-x,y)$, if not, $K(x,y)$ resp. $K(-x,y)$ is called *ambiguous class*.

Each solution class can be represented by any of its elements but it is comfortable to choose in each solution class as a representative the solution (x,y) with minimal nonnegative y, which is called *fundamental solution* of its class (see [2]). If $K(x,y)$ is ambiguous we furthermore prescribe $x>0$ for its fundamental solution.

Let (x_0,y_0) be the fundamental solution in the solution class $K(x_0,y_0)$ and (x,y) any solution to (1) in $K(x_0,y_0)$ then according to MP and ES there is $m\in\mathbb{N}$ such that

$$x+y\sqrt{k^2-1}=(x_0+y_0\sqrt{k^2-1})(k+1\sqrt{k^2-1})^m$$

The finitely many fundamental solutions to (1) can be found "in principle" in a bounded region.

Bounds on fundamental solutions BFS.

Theorem 1.3 Let $x+y\sqrt{k^2-1}$ be a fundamental solution to $x^2-(k^2-1)y^2=k^2$ and $k+\sqrt{k^2-1}$ the fundamental solution to the related Pell equation $x^2-(k^2-1)y^2=1$, then

$0<|x|\leq k\sqrt{\frac{1}{2}(k+1)}$ and $0\leq y\leq \frac{1}{\sqrt{2(k+1)}}k$

Proof. See [1], [2] for a more general assertion.

2 Recursion I

We use the multiplication principle in order to get an idea for the recursive computation of solutions to (1).

Let (x_n, y_n), resp. $x_n + y_n\sqrt{k^2-1}$ be any positive solution to (1), and $k + \sqrt{k^2-1}$ the fundamental solution to (2), then MP together with the definition of equivalent solutions tells us that

$$\left(x_n + y_n\sqrt{k^2-1}\right)\left(k + \sqrt{k^2-1}\right) =$$
$$= (kx_n + (k^2-1)y_n) + (x_n + ky_n)\sqrt{k^2-1} =: x_{n+1} + y_{n+1}\sqrt{k^2-1},$$

which means (x_{n+1}, y_{n+1}), is another solution to (1) in the same solution class in which (x_n, y_n) is.

From this observation we get the linear second order homogeneous recursion R

$$x_{n+1} = kx_n + y_n 1(k^2-1)$$
$$y_{n+1} = 1x_n + ky_n,\ n \geq 0$$

with initial values x_0, y_0, where $(x_0, y_0) \in L_k$.
Let $K(x,y)$ a solution class to (1). To get recursively the positive solutions in $K(x,y)$ it is obvious to choose the fundamental solution $(x_0, y_0) \in K(x,y)$ as initial values for recursion R.

Theorem 2.1 Let $k \in \mathbb{N}, k > 1$ and (x_0, y_0) be the fundamental solution in the solution class $K(x,y)$ of the Pell equation $x^2 - (k^2-1)y^2 = k^2$ (1) then:
1. The linear recursion R with initial values x_0, y_0 yields the solutions in $K(x,y)$:
$x_{n+1} = kx_n + y_n 1(k^2-1)$, initial values x_0, y_0, $n \geq 0$ $R1$
$y_{n+1} = x_n + ky_n$, initial values x_0, y_0, $n \geq 0$. $R2$

2. Recursion R can be represented with the unimodular matrix
$M = \begin{pmatrix} k & k^2-1 \\ 1 & k \end{pmatrix}$ by

$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} k & k^2-1 \\ 1 & k \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$, with initial value $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$.

Proof. 1. We use induction. Let (x_0, y_0) the fundamental solution in the solution class $K(x, y)$. According to R
$(x_1, y_1) = (kx_0 + (k^2 - 1)y_0, x_0 + ky_0)$.

On the other hand the multiplication principle, with fundamental solution
$k + \sqrt{(k^2 - 1)}$ to (2) yields

$$\left(x_0 + y_0\sqrt{(k^2 - 1)}\right)\left(k + \sqrt{(k^2 - 1)}\right) =$$
$$= (kx_0 + (k^2 - 1)y_0) + (x_0 + ky_0)\sqrt{(k^2 - 1)} = (x_1, y_1).$$

The definition of equivalent solutions to (1) shows that $(x_1, y_1) \sim (x_0, y_0)$, hence $(x_1, y_1) \in K(x_0, y_0)$.
Let $n \geq 1$, (x_n, y_n) given by R, and let $(x_n, y_n) \sim (x_0, y_0)$.
In order to show $(x_{n+1}, y_{n+1}) \sim (x_0, y_0)$ we first show $(x_{n+1}, y_{n+1}) \sim (x_n, y_n)$.
According to ESC it is sufficient to show
a) $x_{n+1}x_n - (k^2 - 1)y_{n+1}y_n \equiv 0 \; mod \; k^2$ and
b) $x_{n+1}y_n - y_{n+1}x_n \equiv 0 \; mod \; k^2$.
a) With R we get

$$x_{n+1}x_n - (k^2 - 1)y_{n+1}y_n = \left(kx_n + y_n(k^2 - 1)\right)x_n - (k^2 - 1)(x_n + ky_n)y_n =$$
$$\cdots = k(x_n^2 - (k^2 - 1)y_n^2) = kk^2 \equiv 0 \; mod \; k^2.$$

Since $(x_{n+1}, y_{n+1}) \sim (x_n, y_n)$ and $(x_n, y_n) \sim (x_0, y_0)$ we have $(x_{n+1}, y_{n+1}) \sim (x_0, y_0)$, hence $(x_{n+1}, y_{n+1}) \in K(x_0, y_0)$.

b) Similar to a).

2. The matrix representation of R is obvious. ■

Repeated application of R in matrix form brings powers M^n of $M = \begin{pmatrix} k & k^2 - 1 \\ 1 & k \end{pmatrix}$ into play.

According to the inductive definition of powers of M,

$M^1 := M, M^{n+1} := M^n M, n \in \mathbb{N}$ we get, for example,
$M^2 = \begin{pmatrix} 2k^2 - 1 & 2k(k^2 - 1) \\ 2k & 2k^2 - 1 \end{pmatrix}$, $M^3 = \begin{pmatrix} 4k^3 - 3k & (4k^2 - 1)(k^2 - 1) \\ 4k^2 - 1 & 4k^3 - 3k \end{pmatrix}$.

The polynomials in the cells of the matrices are Chebyshev polynomials $T_n(k)$ of the first kind and $U_n(k)$ of the second kind.

Chebyshev Polynomials of the first kind $T_n(k)$ resp. second kind $U_n(k)$ are defined recursively in the following way [3]:

$T_{n+1}(k) = 2kT_n(k) - T_{n-1}(k), \quad T_0(k) = 1, T_1(k) = k$
$U_{n+1}(k) = 2kU_n(k) - U_{n-1}(k), \quad U_0(k) = 1, U_1(k) = 2k$

Hence

$T_0(k) = 1, T_1(k) = k, T_2(k) = 2k^2 - 1, T_3(k) = 4k^3 - 3k, T_4(k) = 8k^4 - 8k^2 + 1, \ldots$

$U_0(k) = 1, U_1(k) = 2k, U_2(k) = 4k^2 - 1, U_3(k) = 8k^3 - 4k, U_4(k) = 16k^4 - 12k^2 + 1, \ldots$

In the following we shortly write e.g. T_n, U_n instead of $T_n(k)$ and $U_n(k)$.
If we compare, for example, matrix M^3 with Chebyshev polynomials then

$$M^3 = \begin{pmatrix} 4k^3 - 3k & (4k^2 - 1)(k^2 - 1) \\ 4k^2 - 1 & 4k^3 - 3k \end{pmatrix} = \begin{pmatrix} T_3 & U_2(k^2 - 1) \\ U_2 & T_3 \end{pmatrix}$$

This observation will be generalized in theorem 2.2.

First of all, we indicate the following lemma, whose assertions on Chebyshev polynomials will partially be used in proving theorem 2.2 but are also interesting, taken by itself.

Lemma 2.1 Let $k \in \mathbb{N}, k \geq 2$ then
a) $U_n = U_{n-1}k + T_n, n \in \mathbb{N}$,
b) $T_n = U_{n-1}k - U_{n-2}, n \geq 2$,
c) $T_{n+1} = T_nk + U_{n-1}(k^2 - 1), n \in \mathbb{N}$.
d) $2T_n = U_n - U_{n-2}, n \geq 2$,

Proof. We use induction to prove the three assertions. For c) we need a) and b).
a) Let $n = 1$ then $U_0k + T_1 = 1k + k = 2k = U_1$. Let $n \geq 1$ and a) be proven for each m with $1 \leq m \leq n$.

From this and the recursive definition of T_n follows

$$T_{n+1} = 2kT_n - T_{n-1} = 2k(U_n - U_{n-1}k) - (U_{n-1} - U_{n-2}k) = \\ 2kU_n - U_{n-1} - k(2kU_{n-1} - U_{n-2}) = U_{n+1} - kU_n \,.$$

Hence $U_{n+1} = kU_n + T_{n+1}$.
b) Let $n = 2$ then $U_1k - U_0 = 2k^2 - 1 = T_2$. Let $n \geq 2$ and b) be proven for each m with $2 \leq m \leq n$. From this and the recursive definition of U_n follows

$$T_{n+1} = 2kT_n - T_{n-1} = 2k(U_{n-1}k - U_{n-2}) - (U_{n-2}k - U_{n-3}) = \\ 2k^2U_{n-1} - 2kU_{n-2} - kU_{n-2} + U_{n-3} = \\ k(2kU_{n-1} - U_{n-2}) - (2kU_{n-2} - U_{n-3}) = kU_n - U_{n-1} \,.$$

Finally $T_{n+1} = kU_n - U_{n-1}$.

c) Let $n = 1$ then $T_1k + U_0(k^2 - 1) = kk + 1(k^2 - 1) = 2k^2 - 1 = T_2$.
From b) $T_{n+1} = U_nk - U_{n-1}$ and with a) we get

$$T_{n+1} = U_nk - U_{n-1} = (T_n + kU_{n-1})k - U_{n-1} = T_nk + U_{n-1}(k^2 - 1)$$

and finally $T_{n+1} = T_nk + U_{n-1}(k^2 - 1)$.

d) Let $n = 2$ then $U_2 - U_0 = (4k^2 - 1) - 1 = 4k^2 - 2 = 2T_2$.
Let $n \geq 2$ and d) be proven for each m with $2 \leq m \leq n$. From this and the recursive definition of T_n we get

$$2T_{n+1} = 2(2kT_n - T_{n-1}) = 4kT_n - 2T_{n-1} = 2k(U_n - U_{n-2}) - (U_{n-1} - \\ U_{n-3}) = (2kU_n - U_{n-1}) - (2kU_{n-2} - U_{n-3}) = U_{n+1} - U_{n-1}.$$

Hence $2T_{n+1} = U_{n+1} - U_{n-1}$. ■

Theorem 2.2 1. Let $k \in \mathbb{N}, k \geq 2$, and $M = \begin{pmatrix} k & k^2 - 1 \\ 1 & k \end{pmatrix} = \begin{pmatrix} T_1 & U_0(k^2 - 1) \\ U_0 & T_1 \end{pmatrix}$, then
$M^n = \begin{pmatrix} T_n & U_{n-1}(k^2 - 1) \\ U_{n-1} & T_n \end{pmatrix}$ for $n \in \mathbb{N}$.

2. Let $\binom{x_0}{y_0}$ be the fundamental solution in the solution class $K(x, y)$ of $x^2 - (k^2 - 1)y^2 = k^2$ then for the n-th solution (x_n, y_n) in $K(x, y), n \geq 1$

$$\binom{x_n}{y_n} = \begin{pmatrix} T_n & U_{n-1}(k^2-1) \\ U_{n-1} & T_n \end{pmatrix} \binom{x_0}{y_0}$$

3. With $\binom{x_0}{y_0} = \binom{k}{0}$ the trivial, imprimitive solutions in $K(k,0) \setminus \{(k,0)\}$ can be represented as

$$\binom{x_n}{y_n} = \begin{pmatrix} T_n & U_{n-1}(k^2-1) \\ U_{n-1} & T_n \end{pmatrix} \binom{k}{0} = \binom{kT_n}{kU_{n-1}}, n \geq 1.$$

Proof. 1. We use induction. Let $n = 1$ then the assertion is true.
Let the assertion be true for an $n \in \mathbb{N}$, then we have to show that

$$M^{n+1} = M^n M = \begin{pmatrix} T_n & U_{n-1}(k^2-1) \\ U_{n-1} & T_n \end{pmatrix} \begin{pmatrix} k & k^2-1 \\ 1 & k \end{pmatrix} = \begin{pmatrix} T_n k + U_{n-1}(k^2-1) & (U_{n-1}k + T_n)(k^2-1) \\ U_{n-1}k + T_n & T_n k + U_{n-1}(k^2-1) \end{pmatrix}.$$

In order to prove

$$\begin{pmatrix} T_{n+1} & U_n(k^2-1) \\ U_n & T_{n+1} \end{pmatrix} = \begin{pmatrix} T_n k + U_{n-1}(k^2-1) & (U_{n-1}k + T_n)(k^2-1) \\ U_{n-1}k + T_n & T_n k + U_{n-1}(k^2-1) \end{pmatrix}$$

we need that

i) $U_n = U_{n-1}k + T_n, n \in \mathbb{N}$
ii) $T_{n+1} = T_n k + U_{n-1}(k^2-1), n \in \mathbb{N}$,

which has been proven in Lemma 2.1 where i) corresponds to a) and ii) to c).

2. We know from theorem 2.1 that $\binom{x_{n+1}}{y_{n+1}} = \begin{pmatrix} k & k^2-1 \\ 1 & k \end{pmatrix} \binom{x_n}{y_n}$,

Let $\binom{x_0}{y_0}$ be the fundamental solution in the solution class $K(x,y)$ of $x^2 - (k^2-1)y^2 = k^2$ then a simple inductive argument, using 1., shows that for the n-th solution (x_n, y_n) in $K(x,y), n \geq 1$

$$\binom{x_n}{y_n} = \begin{pmatrix} T_n & U_{n-1}(k^2-1) \\ U_{n-1} & T_n \end{pmatrix} \binom{x_0}{y_0}.$$

3. With $\binom{x_0}{y_0} = \binom{k}{0}$ the trivial, imprimitive solutions in $K(k,0) \setminus \{(k,0)\}$ can be represented as

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} T_n & U_{n-1}(k^2-1) \\ U_{n-1} & T_n \end{pmatrix} \begin{pmatrix} k \\ 0 \end{pmatrix} = \begin{pmatrix} kT_n \\ kU_{n-1} \end{pmatrix}, n \geq 1$$

which is a direct consequence of 2. ■

3 Recursion II

By means of the multiplication principle we get another linear recursion for the solutions to the Pell equation $x^2 - (k^2-1)y^2 = k^2, k \in \mathbb{N}, k \geq 2$ (1).

Theorem 3.1. Let (x_0, y_0) the fundamental solution in the solution class $K(x_0, y_0)$ of (1) and $x_1 + y_1\sqrt{k^2-1} = \left(x_0 + y_0\sqrt{k^2-1}\right)(k + \sqrt{k^2-1})$, which means $(x_1, y_1) = (kx_0 + y_0(k^2-1), x_0 + ky_0)$.
The other solutions in $K(x_0, y_0)$ are supplied by the recursion S:
$x_{n+1} = 2kx_n - x_{n-1}$, $n \geq 1$, initial values x_0, x_1 $(S1)$
$y_{n+1} = 2ky_n - y_{n-1}$, $n \geq 1$, initial values y_0, y_1. $(S2)$
The initial values x_0, x_1 and y_0, y_1 arise from the solutions (x_0, y_0) und (x_1, y_1).

Proof. We use recursion I. According to R we have
$x_{n+1} = kx_n + y_n(k^2-1)$ $R1$
$y_{n+1} = x_n + ky_n,\ n \geq 0$ $R2$
The fundamental solution (x_0, y_0) of the solution class $K(x_0, y_0)$ yields the initial values for recursion R.
Substituting $R2$, $y_n = 1x_{n-1} + ky_{n-1}, n \geq 1$ into $R1$ we get

$$x_{n+1} = kx_n + y_n 1(k^2-1) = kx_n + (x_{n-1} + ky_{n-1})(k^2-1) =$$
$$kx_n + x_{n-1}(k^2-1) + k(x_{n-1} + kx_{n-1}) = 2kx_n + (x_{n-1}(k^2-1) - x_{n-1}k^2) =$$
$$2kx_n - x_{n-1}.$$

Hence $x_{n+1} = 2kx_n - x_{n-1}$ $(S1)$.
In order to get $(S2)$, $y_{n+1} = 2ky_n - y_{n-1}$ from (R) we proceed in a similar way. Substituting $(R1)$, $x_n = kx_{n-1} + y_{n-1}(k^2-1)$ into $(R2)$, $y_{n+1} = x_n + ky_n$ yields $y_{n+1} = x_n + ky_n = kx_{n-1} + y_{n-1}(k^2-1) + ky_n$.
From $(R2)$ we get $kx_{n-1} = ky_n - k^2y_{n-1}$. Hence

$$y_{n+1} = kx_{n-1} + y_{n-1}(k^2-1) + ky_n =$$
$$ky_n - k^2y_{n-1} + (y_{n-1}(k^2-1) + ky_n) = 2ky_n - y_{n-1}.$$

Hence $y_{n+1} = 2ky_n - y_{n-1}$ $(S2)$. ■

Recursion II can be used to get explicit formulas for (x_n, y_n). It is a standard way to go from a linear, homogenous recursion of second order to an explicit formula (see [4]).

Theorem 3.2 Let (x_0, y_0) the fundamental solution in the solution class $K(x_0, y_0)$ of (1) and $(x_1, y_1) = (kx_0 + y_0(k^2 - 1), x_0 + ky_0)$ as in theorem 3.1.
Recursion S:

$x_{n+1} = 2kx_n - x_{n-1}$, $n \geq 1$, initial values x_0, x_1 $S1$
$y_{n+1} = 2ky_n - y_{n-1}$, $n \geq 1$, initial values y_0, y_1, $n = 0,1, \ldots$ $S2$

leads to the following explicit solutions to S:

$$x_n = \frac{y_0\sqrt{k^2-1}+x_0}{2}(k + \sqrt{k^2 - 1})^n - \frac{y_0\sqrt{k^2-1}-x_0}{2}(k - \sqrt{k^2 - 1})^n,$$
$$y_n = \frac{y_0\sqrt{k^2-1}+x_0}{2\sqrt{k^2-1}}(k + \sqrt{k^2 - 1})^n + \frac{y_0\sqrt{k^2-1}-x_0}{2\sqrt{k^2-1}}(k - \sqrt{k^2 - 1})^n.$$

On the other hand: given these explicit formulas, where (x_0, y_0) is a solution to (1), then (x_n, y_n) is a solution to (1).

Proof. Recursion $(S1)$ $x_{n+1} = 2kx_n - x_{n-1}$, $n \geq 1$, initial values x_0, $x_1 = kx_0 + y_0(k^2 - 1)$ leads to the characteristic equation $t^2 - 2kt + 1 = 0$ with solutions $\alpha = k + \sqrt{k^2 - 1}$, $\beta = k - \sqrt{k^2 - 1}$. Since $\alpha \neq \beta,\ \alpha > \beta$ we have
$x_n = \mathrm{A}\alpha^n - B\beta^n$, where
$$\mathrm{A} = \frac{x_1 - \left(k - \sqrt{k^2 - 1}\right)x_0}{2\sqrt{k^2 - 1}}, \mathrm{B} = \frac{x_1 - \left(k + \sqrt{k^2 - 1}\right)x_0}{2\sqrt{k^2 - 1}}$$
Using $x_1 = kx_0 + y_0(k^2 - 1)$ we get
$$\mathrm{A} = \frac{y_0\sqrt{k^2 - 1} + x_0}{2}, \mathrm{B} = \frac{y_0\sqrt{k^2 - 1} - x_0}{2}$$
A similar procedure works for $(S2)$. We have
$y_n = \mathrm{A}'\alpha^n - B'\beta^n$, where
$$\mathrm{A}' = \frac{y_1 - \left(k - \sqrt{k^2 - 1}\right)y_0}{2\sqrt{k^2 - 1}}, \mathrm{B}' = \frac{y_1 - \left(k + \sqrt{k^2 - 1}\right)y_0}{2\sqrt{k^2 - 1}}$$
Using $y_1 = x_0 + ky_0$ we get
$$\mathrm{A}' = \frac{y_0\sqrt{k^2 - 1} + x_0}{2\sqrt{k^2 - 1}}, \mathrm{B}' = \frac{-y_0\sqrt{k^2 - 1} + x_0}{2\sqrt{k^2 - 1}}$$
Altogether we get the explicit solution formulas for solutions to S:

$x_n = \frac{y_0\sqrt{k^2-1}+x_0}{2}(k+\sqrt{k^2-1})^n - \frac{y_0\sqrt{k^2-1}-x_0}{2}(k-\sqrt{k^2-1})^n,$

$y_n = \frac{y_0\sqrt{k^2-1}+x_0}{2\sqrt{k^2-1}}(k+\sqrt{k^2-1})^n + \frac{y_0\sqrt{k^2-1}-x_0}{2\sqrt{k^2-1}}(k-\sqrt{k^2-1})^n, n = 0,1,\ldots$

Now we show, that given these explicit formulas, where (x_0, y_0) is a solution to (1), not necessarily a fundamental solution, then (x_n, y_n) is a solution to (1).

We use the following shortcuts

$$\mathrm{U} = \mathrm{A}\alpha^n, V = B\beta^n$$

then $x_n = U - V, y_n = \frac{1}{\sqrt{k^2-1}}(U+V)$

and it can easily be checked that $(x_n, y_n), n = 0,1, \ldots$ is a solution to (1).

Furthermore we show that (x_n, y_n) not only is a solution to (1) but is in the solution class $K(x_0, y_0)$, which means $(x_n, y_n) \sim (x_0, y_0), n = 0,1, \ldots$

The proof is by induction.

From the explicit formula we get

$$x_1 = \frac{y_0\sqrt{k^2-1}+x_0}{2}\left(k+\sqrt{k^2-1}\right)^1 - \frac{y_0\sqrt{k^2-1}-x_0}{2}\left(k-\sqrt{k^2-1}\right)^1 = x_0 k + y_0(k^2-1)$$

$$y_1 = \frac{y_0\sqrt{k^2-1}+x_0}{2\sqrt{k^2-1}}(k+\sqrt{k^2-1})^1 + \frac{y_0\sqrt{k^2-1}-x_0}{2\sqrt{k^2-1}}(k-\sqrt{k^2-1})^1 = x_0 + ky_0$$

Hence (see theorem 3.1) $(x_1, y_1) \sim (x_0, y_0)$, which means $(x_1, y_1) \in K(x_0, y_0)$. Suppose $(x_n, y_n) \sim (x_0, y_0)$ for $n \in \mathbb{N}$, then we show $(x_{n+1}, y_{n+1}) \sim (x_n, y_n)$, which finally yields $(x_{n+1}, y_{n+1}) \sim (x_0, y_0)$.

We show $(x_{n+1}, y_{n+1}) = (kx_n + y_n(k^2-1), x_n + ky_n)$. For that we use $x_n = \mathrm{A}\alpha^n - B\beta^n$ and $y_n = \frac{\mathrm{A}\alpha^n - B\beta^n}{\sqrt{k^2-1}}$ and get

$$\begin{aligned} kx_n + y_n(k^2-1) &= k\mathrm{A}\alpha^n - kB\beta^n + \frac{\mathrm{A}\alpha^n - B\beta^n}{\sqrt{k^2-1}}(k^2-1) \\ &= k\mathrm{A}\alpha^n - kB\beta^n + (\mathrm{A}\alpha^n - B\beta^n)\sqrt{k^2-1} \\ &= \mathrm{A}\alpha^n\left(k+\sqrt{k^2-1}\right) - B\beta^n\left(k-\sqrt{k^2-1}\right) = \mathrm{A}\alpha^{n+1} - B\beta^{n+1} \\ &= x_{n+1} \end{aligned}$$

Similarly, we have

$$\begin{aligned} x_n + ky_n &= \mathrm{A}\alpha^n - B\beta^n + \frac{\mathrm{kA}\alpha^n + kB\beta^n}{\sqrt{k^2-1}} \\ &= \mathrm{A}\alpha^n\left(1 + \frac{k}{\sqrt{k^2-1}}\right) + B\beta^n\left(-1 + \frac{k}{\sqrt{k^2-1}}\right) \\ &= \frac{\mathrm{A}\alpha^{n+1}}{\sqrt{k^2-1}} + \frac{B\beta^{n+1}}{\sqrt{k^2-1}} = y_{n+1} \end{aligned}$$

Altogether $(x_{n+1}, y_{n+1}) \sim (x_n, y_n)$, and with $(x_n, y_n) \sim (x_0, y_0)$ we get $(x_{n+1}, y_{n+1}) \sim (x_0, y_0)$, which shows that $(x_{n+1}, y_{n+1}) \in K(x_0, y_0)$. ■

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