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Generalized Derivations of *BM*-Algebras

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Abstract

In this paper, the notion of left-right (respectively right-left) derivation of *BM*-algebra are introduced and some of related properties are investigated. Also, the notions of generalized left-right (respectively right-left) derivations of *BM*-algebra and *B*-algebra are introduced and some of their properties are investigated. Then, we obtain some different properties of generalized derivation of *BM*-algebra and *B*-algebra.

Keywords: *BM*-algebra, *B*-algebra, (l,r)-derivation, (r,l)-derivation, generalized derivation

1 Introduction

Neggers and Kim [10] introduce the notion of *B*-algebra, which is a non-empty set X with a constant 0 and a binary operation "$*$" denoted by $(X;*,0)$, satisfying the following axioms: $(B1)$ $x*x=0$, $(B2)$ $x*0=x,$ and $(B3)$ $(x*y)*z=x*\big(z*(0*y)\big)$ for all $x,y,z\in X$. Then, Kim and Kim [6] introduce the notion of *BG*-algebra which is the generalization of *B*-algebra satisfying the following axioms: (*B*1), (*B*2), and (BG) $(x*y)*(0*y)=x$ for all $x,y\in X$. Kim and Kim [7] also introduce *BM*-algebra, which is a specialization of *B*-algebra, satisfying the following axioms: (*B*2) and $(A2)$ $(z*x)*(z*y)=y*x$

for all $x, y, z \in X$.

The first time, notion of derivation is discussed in ring and *near* ring. In the development of abstract algebra, the notion of derivation is also discussed in other algebraic structure. Kamaludin et al. [5] introduce the notion of derivations in *BG*-algebras. The results define a left-right or (l, r)-derivation, a right-left or (r, l)-derivation, a left derivation, and a regular in *BG*-algebra. Moreover, some of related properties are investigated. Aziz et al. [2] discuss the notion of (*f*, *g*)-derivation in *BG*-algebra, involving two endomorphisms *f*, *g* of *BG*-algebra, and the properties of *f*-derivation, and (*f*, *g*)-derivation in *BG*-algebra is obtained.

A new notion of derivation–generalized derivation–is introduced by some authors. Bawazeer et al. [3] introduce the generalized derivation in *BCC*-algebra defining the generalized (l, r)-derivation and generalized (r, l)-derivation in *BCC*-algebra, and some of related properties are investigated. Furthermore, the generalized of *f*-derivation where *f* is endomorphism is introduced by Jana et al. [4] in *KUS*-algebra and also Kim [8] in *BE*-algebra.

In this paper, the notion of (l, r)-derivation, (r, l)-derivation, and derivation in *BM*-algebra are introduced. Also, some of related properties are investigated. Then, we discuss the notion of the generalized (l, r)-derivation, generalized (r, l)-derivation, and generalized derivation in *BM*-algebra and some of their properties. The notion of the generalized (l, r)-derivation, generalized (r, l)-derivation, and generalized derivation in *B*-algebra are introduced and their properties are investigated. Finally, we obtain some different properties between the generalized derivations in *BM*-algebras and *B*-algebras.

2 Preliminaries

In this section, we recall the notion of *B*-algebra, *BM*-algebra, the generalized *BCC*-algebra, and review some properties that we need in the next section. Some definitions and theories related to the generalized of derivation in *BM*-algebra and *B*-algebra being discussed by several authors [1, 3, 7, 10] are also presented.

Definition 2.1. [10] A *B*-algebra is a non-empty set *X* with a constant 0 and a binary operation "$*$" satisfying the following axioms: (*B1*) $x * x = 0$, (*B2*) $x * 0 = x$, (*B3*) $(x * y) * z = x * \big(z * (0 * y)\big)$ for all $x, y, z \in X$.

Lemma 2.2. [10] If $(X; *, 0)$ is a *B*-algebra, then

(i) $0 * (0 * x) = x$,

(ii) $(x * y) * (0 * y) = x$,

(iii) $y * z = y * \big(0 * (0 * z)\big)$,

(iv) $x * (y * z) = \big(x * (0 * z)\big) * y$,

(v) $x * z = y * z$ implies $x = y$,

(vi) $x * y = 0$ implies $x = y$ for all $x, y, z \in X$.

Proof: Lemma 2.2 has been proved in [10]. ■

For a *B*-algebra $(X; *, 0)$, we denote $x \wedge y = y * (y * x)$ for all $x, y \in X$.

Definition 2.3. [1] Let $(X; *, 0)$ be a *B*-algebra. By an (l, r)-derivation of X, a self-map d of X satisfying the identity $d(x * y) = (d(x) * y) \wedge (x * d(y))$, for all $x, y \in X$. If X satisfies the identity $d(x * y) = (x * d(y)) \wedge (d(x) * y)$, for all $x, y \in X$, then we say that d is a (r, l)-derivation. Moreover, if d is both an (l, r)-derivation and an (r, l)-derivation, we say that d is a derivation of X.

Definition 2.4. [1] Let $(X; *, 0)$ be a *B*-algebra. A self-map d is said to be regular if $d(0) = 0$.

The notion of the generalized derivation in *BCC*-algebra is discussed by Bawazeer et al. [3].

Definition 2.5. [3] A *BCC*-algebra is a non-empty set *X* with a constant 0 and a binary operation "$*$" satisfying the following axioms:

(i) $\big((x * y)(z * y)\big) * (x * z) = 0$,

(ii) $0 * x = 0$,

(iii) $x * 0 = x$,

(iv) $x * x = 0$,

(v) $x * y = 0$ and $y * x = 0$ implies $x = y$ for all $x, y, z \in X$.

Definition 2.6. [3] Let X be a BCC-algebra. A mapping $D : X \to X$ is called a generalized (l, r)-derivation if there exists a (l, r)-derivation $d : X \to X$ such that $D(x * y) = (D(x) * y) \wedge \big(x * d(y)\big)$ for all $x, y \in X$, if there exists a (r, l)-derivation $d : X \to X$ such that $D(x * y) = \big(x * D(y)\big) \wedge (d(x) * y)$ for all $x, y \in X$, the mapping $D : X \to X$ is called a generalized (r, l)-derivation. Moreover, if *D* is both a generalized (l, r)-derivation and (r, l)-derivation, we say that *D* is a generalized derivation.

Definition 2.7. [7] A *BM*-algebra is a non-empty set *X* with a constant 0 and a binary operation "$*$" satisfying the following axioms (*A1*) $x * 0 = x$, (*A2*) $(z * x) * (z * y) = y * x$ for all $x, y, z \in X$.

Example 2.4. Let $X = \{0, 1, 2\}$ be a set with Cayley's table as seen in Table 2.2.

Table 2.2: Cayley's table for $(X; *, 0)$

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Then, $(X; *, 0)$ is a *BM*-algebra.

Lemma 2.8. [7] If $(X; *, 0)$ is a *BM*-algebra, then

(i) $x * x = 0$,
(ii) $0 * (0 * x) = x$,
(iii) $0 * (x * y) = y * x$,
(iv) $(x * z) * (y * z) = x * y$,
(v) $x * y = 0$ if and only if $y * x = 0$ for all $x, y, z \in X$.

Proof: Lemma 2.8 has been proved in [7]. ■

Theorem 2.9. [7] Every *BM*-algebra is a *B*-algebra.

Proof: Theorem 2.9 has been proved in [7]. ■

The converse of Theorem 2.9 does not hold in general.

Theorem 2.10. [7] If $(X; *, 0)$ is a *BM*-algebra, then $(x * y) * z = (x * z) * y$, for all $x, y, z \in X$.

Proof: Theorem 2.10 has been proved in [7]. ■

3 Derivations of *BM*-Algebras

In this section, an (l, r)-derivation, an (r, l)-derivation, and a derivation in *BM*-algebra are defined by a way similar to the construction of derivation in *B*-algebra by Al-Shehrie [1]. Then, some of related properties of (l, r)-derivation, (r, l)-derivation, and derivation in *BM*-algebra are obtained.

Let $(X; *, 0)$ be a *BM*-algebra, we denote $x \wedge y = y * (y * x)$ for all $x, y \in X$.

Definition 3.1. Let $(X; *, 0)$ be a *BM*-algebra. By an (l, r)-derivation of X, a self-map d of X satisfies the identity $d(x * y) = (d(x) * y) \wedge (x * d(y))$, for all $x, y \in X$. If X satisfies the identity $d(x * y) = (x * d(y)) \wedge (d(x) * y)$, for all $x, y \in X$, then we say that d is a (r, l)-derivation. Moreover, if d is both an (l, r)-derivation and an (r, l)-derivation, we say that d is a derivation of X.

Example 3.1. Let $(\mathbb{Z}; -, 0)$ be a set of integers $\mathbb{Z}$ with a subtraction operation and a constant 0. Then, it is easy to prove that $\mathbb{Z}$ is a *BM*-algebra. Let d is a self-map of $\mathbb{Z}$ by $d(x) = x - 2$ for all $x, y \in Z$. Then, d is an (l, r)-derivation in $\mathbb{Z}$. Since, $\big(3 - (1 - 2)\big) \wedge (3 - 2 - 1) = 4 \neq 0 = d(3 - 1)$. This shows that d is not an (r, l)-derivation in $\mathbb{Z}$.

Example 3.2. Let $X = \{0, 1, 2, 3\}$ be a set with Cayley's table as shown in Table 3.1.

Table 3.1: Cayley's table for $(X;*,0)$

$*$	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

Then, it is easy to show that *X* is a *BM*-algebra. Define a map $d : X \to X$ by

$$d(x) = \begin{cases} 2 & \text{if } x = 0, \\ 3 & \text{if } x = 1, \\ 0 & \text{if } x = 2, \\ 1 & \text{if } x = 3. \end{cases}$$

It can be shown that *d* is an (l,r)-derivation and an (r,l)-derivation of *X*, we say that *d* is a derivation of *X*.

Definition 3.2. Let $(X;*,0)$ be a *BM*-algebra. A self-map d is said to be regular if $d(0) = 0$.

Theorem 3.3. Let $(X;*,0)$ be a *BM*-algebra and *d* be an (l,r)-derivation in *X*, then

(i) $d(x*y) = d(x)*y$ for all $x,y \in X$,
(ii) $d(0) = d(x)*x$ for all $x \in X$,
(iii) $d(x*d(x)) = 0$ for all $x \in X$,
(iv) If *d* is a regular, then d is an identity function.

Proof: Let $(X;*,0)$ be a *BM*-algebra and *d* be an (l,r)-derivation in *X*.

(i) Since *d* is an (l,r)-derivation in *X*, then

$$\begin{aligned} d(x*y) &= (d(x)*y) \wedge (x*d(y)) \\ &= \big(x*d(y)\big) \wedge [\big(x*d(y)\big)*(d(x)*y)] \\ &= [(x*d(y))*0] \wedge [\big(x*d(y)\big)*(d(x)*y)] && \text{[by axiom } A1] \\ &= (d(x)*y)*0 && \text{[by axiom } A2] \\ d(x*y) &= d(x)*y. && \text{[by axiom } A1] \end{aligned}$$

Hence, this shows that $d(x*y) = d(x)*y$ for all $x,y \in X$.

(ii) By (i) It is obtained that $d(x*y) = d(x)*y$. By substitution $y = x$ then $d(x*x) = d(x)*x$, and by Lemma 2.8 (i) we get $d(0) = d(x)*x$ for all $x \in X$.

(iii) By (i) we have $d(x*d(x)) = d(x)*d(x) = 0$ for all $x \in X$.

(iv) Since d is a regular, then $d(0) = 0,$ and by (ii) it is obtained that $d(0) = d(x)*x = 0$.
Since every *BM*-algebra is a *B*-algebra, then *X* also satisfies Lemma 2.2 (vi), such that $d(x) = x$. ■

Theorem 3.4. Let $(X; *, 0)$ be a *BM*-algebra and d be an (r, l)-derivation in *X*, then

(i) $d(x * y) = x * d(y)$ for all $x, y \in X$,
(ii) $d(0) = x * d(x)$ for all $x \in X$,
(iii) $d(d(x) * x) = 0$ for all $x \in X$,
(iv) If d is a regular, then d is an identity function.

Proof: Let $(X; *, 0)$ be a *BM*-algebra and d be a (r, l)-derivation in *X*.

(i) Since d is an (r, l)-derivation in *X*, then

$$\begin{aligned} d(x * y) &= (x * d(y)) \wedge (d(x) * y) \\ &= (d(x) * y) \wedge [(d(x) * y) * (x * d(y))] \\ &= [(d(x) * y) * 0] \wedge [(d(x) * y) * \big(x * d(y)\big)] && \text{[by axiom } A1] \\ &= (x * d(y)) * 0 && \text{[by axiom } A2] \\ d(x * y) &= x * d(y). && .\text{[by axiom } A1] \end{aligned}$$

Thus, we have $d(x * y) = x * d(y)$ for all $x, y \in X$.

(ii) By (i) it is obtained that $d(x * y) = x * d(y)$. Substituting $y = x$ yields $d(x * x) = x * d(x)$, and by Lemma 2.8 (i) we get $d(0) = x * d(x)$ for all $x \in X$.

(iii) By (i) we have $d(d(x) * x) = d(x) * d(x) = 0$ for all $x \in X$.

(iv) Since d is a regular, then $d(0) = 0$, and by (ii) it is obtained that $d(0) = x * d(x) = 0$.
Since every *BM*-algebra is a *B*-algebra, then *X* also satisfies Lemma 2.2 (vi), such that $d(x) = x$. ■

Theorem 3.5. Let $(X; *, 0)$ be a *BM*-algebra and d be a derivation in *X*. d is a regular if and only if d is an identity function.

Proof: Let d is an (l, r)-derivation in *X*, then by Theorem 3.3 (iv) it shows that d is an identity function. If d is a (r, l)-derivation in *X*, then by Theorem 3.4 (iv) it shows that d is an identity function. Conversely, if d is an identity function, then clearly $d(0) = 0$. Hence, d is a regular. ■

4 Generalization Derivation of *BM*-Algebras

In this section, a generalized (l, r)-derivation, a generalized (r, l)-derivation, and a generalized derivation in *BM*-algebra are defined by a way similar to the construction of the generalized derivation in *BCC*-algebra by Bawazeer et al. [3]. Then, also we obtain some related properties.

Definition 4.1. Let X be a *BM*-algebra. A mapping $D : X \to X$ is called a generalized (l, r)-derivation if there exists a (l, r)-derivation $d : X \to X$ such that $D(x * y) = (D(x) * y) \wedge \big(x * d(y)\big)$ for all $x, y \in X$, if there exists a (r, l)-derivation $d : X \to X$ such that $D(x * y) = \big(x * D(y)\big) \wedge (d(x) * y)$ for all $x, y \in X$, the mapping $D : X \to X$ is called a generalized (r, l)-derivation. Moreover, if D

is both a generalized (l,r)-derivation and (r,l)-derivation, we say that D is a generalized derivation.

Example 4.1. Let $X = \{0, 1, 2, 3\}$ be a set with Cayley's table as shown in Table 4.1.

Table 4.1: Cayley's table for $(X; *, 0)$

$*$	0	1	2	3
0	0	2	1	3
1	1	0	3	2
2	2	3	0	1
3	3	1	2	0

Then, it is easy to show that X is a *BM*-algebra. Define a map $d : X \to X$ by $d(x) = x$ and $D : X \to X$ by

$$D(x) = \begin{cases} 3 & \text{if } x = 0, \\ 2 & \text{if } x = 1, \\ 1 & \text{if } x = 2, \\ 0 & \text{if } x = 3, \end{cases}$$

It can be shown that *d* is a derivation of *X* and *D* is a generalized derivation of *X*.

Definition 4.2. Let (*X*; $*$, 0) be a *BM*-algebra. A self-map *D* is said to be regular if $D(0) = 0$.

Theorem 4.3. Let $(X; *, 0)$ be a *BM*-algebra and *D* be a generalized (l,r)-derivation in *X*, then

(i) $D(x * y) = D(x) * y$ for all $x, y \in X$,
(ii) $D(0) = D(x) * x$ for all $x \in X$,
(iii) $D(x * d(x)) = D(x) * d(x)$ for all $x \in X$,
(iv) If *D* is a regular, then *D* is an identity function.

Proof: Let $(X; *, 0)$ be a *BM*-algebra and *D* be a generalized (l,r)-derivation in *X*.

(i) Since *D* is a generalized (l,r)-derivation in *X*, then

$$\begin{aligned} D(x * y) &= (D(x) * y) \wedge (x * d(y)) \\ &= \big(x * d(y)\big) \wedge [\big(x * d(y)\big) * (D(x) * y)] \\ &= [(x * d(y)) * 0] \wedge [\big(x * d(y)\big) * (D(x) * y)] && \text{[by axiom } A1] \\ &= (D(x) * y) * 0 && \text{[by axiom } A2] \\ D(x * y) &= D(x) * y. && \text{[by axiom } A1] \end{aligned}$$

Hence, it is obtained that $D(x * y) = D(x) * y$ for all $x, y \in X$.

(ii) By (i) we have $D(x * y) = D(x) * y$. Substitution of $y = x$ gives $D(x * x) = D(x) * x$, and by Lemma 2.8 (i) we get $D(0) = D(x) * x$ for all $x \in X$.

(iii) By (i) it is obtained that $D(x * d(x)) = D(x) * d(x)$ for all $x \in X$.

(iv) Since *D* is a regular, then $D(0) = 0$, and by (ii) we have $D(0) = D(x) * x = 0$.

Since every *BM*-algebra is a *B*-algebra, then *X* also satisfies Lemma 2.2 (vi), such that $D(x) = x$. ∎

Theorem 4.4. Let $(X;*,0)$ be a *BM*-algebra and *D* be a generalized (*r*, *l*)-derivation in *X*, then

(i) $D(x * y) = x * D(y)$ for all $x, y \in X$,

(ii) $D(0) = x * D(x)$ for all $x \in X$,

(iii) $D(d(x) * x) = d(x) * D(x)$ for all $x \in X$,

(iv) If *D* is a regular, then *D* is an identity function.

Proof: Let $(X; *,0)$ be a *BM*-algebra and *D* be a generalized (r, l)-derivation in *X*.

(i) Since *D* is a generalized (r, l)-derivation in *X*, then

$$
\begin{aligned}
D(x * y) &= (x * D(y)) \wedge (d(x) * y) \\
&= (d(x) * y) \wedge [(d(x) * y) * (x * D(y))] \\
&= [(d(x) * y) * 0] \wedge [(d(x) * y) * \big(x * D(y)\big)] && \text{[by axiom } A1\text{]} \\
&= (x * D(y)) * 0 && \text{[by axiom } A2\text{]} \\
D(x * y) &= x * D(y). && \text{[by axiom } A1\text{]}
\end{aligned}
$$

Hence, it is obtained that $D(x * y) = x * D(y)$ for all $x, y \in X$.

(ii) By (i) we have $D(x * y) = x * D(y)$. By substitution of $y = x$ then $D(x * x) = x * D(x)$, and by Lemma 2.8 (i) we get $D(0) = x * D(x)$ for all $x \in X$.

(iii) By (i) we have $D(d(x) * x) = d(x) * D(x)$ for all $x \in X$.

(iv) Since *D* is a regular, then $D(0) = 0$, and by (ii) it is obtained that $D(0) = x * D(x) = 0$.

Since every *BM*-algebra is a *B*-algebra, then *X* also satisfies Lemma 2.2 (vi), such that $D(x) = x$. ∎

Theorem 4.5. Let $(X; *,0)$ be a *BM*-algebra and *D* be a generalized derivation in *X*. *D* is a regular if and only if *D* is an identity function.

Proof: Let *D* be a generalized (l, r)-derivation in *X*, then by Theorem 4.3 (iv) it shows that *D* is an identity function. If *D* is a generalized (r, l)-derivation in *X*, then by Theorem 4.4 (iv) it shows that *D* is an identity function. Conversely, if *D* is an identity function, then clearly $D(0) = 0$. Hence, *D* is a regular. ∎

5 Conclusion

In this paper, the notions of left-right or (l, r)-derivation, right-left or (r, l)-derivation, and derivation in *BM*-algebra are introduced and some related properties are investigated. We obtain a property being similar to the (l, r)-derivation and (r, l)-derivation, that is if d is a regular then d is an identity function, it is meant that every derivation, which is a regular in *BM*-algebra, is an identity function. Furthermore, the notion of the generalized derivation in *BM*-algebra is equivalent to the generalized derivation in *BCC*-algebra, although there are some of their properties are different.

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