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Trees with the Same Distance Domination Number

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Abstract

The distance between two vertices u and v in a graph G equals the length of a shortest path from u to v. A set D of vertices is distance-k dominating if every vertex not belonging to D is at distance at most k of a vertex in D. The distance-k domination number of a graph G, denoted by $\gamma_k(G)$, is the minimum cardinality of a distance-k dominating set in G. Here we focus on the trees. For $n \geq 1$ and $k \geq 2$, let $\Gamma(k, n)$ be the set of trees satisfying $\gamma_k(T) = n$. In this paper, we provide a constructive characterization of $\Gamma(k, n)$ for all $n \geq 1$ and all $k \geq 2$.

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Keywords: tree, distance-k dominating set, distance-k domination number

1 Introduction

One of the fastest growing areas within graph theory is the study of domination and related subset problems. The decision problem of determining the domination number for arbitrary graphs is NP-complete [4]. The theory of distance dominating set was propsed by Slater [7] in 1976. The distance between two vertices u and v in a graph G equals the length of a shortest path from u to v. A set D of vertices is a *distance-k dominating set* (DkDS) if every vertex not belonging to D is at distance at most k of a vertex in D. The *distance-k*

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domination number of a graph G, denoted by $\gamma_k(G)$, is the minimum cardinality of a distance-k dominating set in G. Recently, it was then extensively studied the distance-k domination number $\gamma_k(G)$ for various classes of graphs G in the literature (see [2],[3],[5],[6],[7],[8],[9]).

For $n \geq 1$ and $k \geq 2$, let $\Gamma(k,n)$ be the set of trees T satisfying $\gamma_k(T) = n$. In this paper, we provide a constructive characterization of $\Gamma(k,n)$ for all $n \geq 1$ and all $k \geq 2$.

2 Notations and preliminary results

All graphs considered in this paper are finite, loopless, and without multiple edges. For a graph G, $V(G)$ and $E(G)$ denote the *vertex set* and the *edge set* of G, respectively. A *u-v path* $P : u = v_1, v_2, \ldots, v_k = v$ of G is a sequence of k vertices in G such that $v_i v_{i+1} \in E(G)$ for $i = 1, 2, \ldots, k-1$. Denote by P_n a n-path with n vertices. The length of P_n is n-1. For any two vertices u and v in G, the distance between u and v, denoted by $dist_G(u,v)$, is the minimum length of all u-v paths in G. For two different sets A and B, written $A - B$ is the set of all elements of A that are not elements of B.

The (open) *neighborhood* $N_G(v)$ of a vertex v is the set of vertices adjacent to v in G, and the *closed neighborhood* $N_G[v]$ is $N_G[v] = N_G(v) \cup \{v\}$. For $d \geq 0$, the closed neighborhood $N_G^d[v]$ of a vertex v is the set of the vertices u satisfying $dist_G(u,v) \leq d$ and the (open) neighborhood is $N_G^d(v) = N_G^d[v] - \{v\}$. We can see that $N_G^0[v] = \{v\}$, $N_G^1(v) = N_G(v)$ and $N_G^1[v] = N_G[v]$. For any subset A and $d \geq 0$, denote $N_G^d(A) = \bigcup_{v \in A} N_G^d(v)$ and $N_G^d[A] = \bigcup_{v \in A} N_G^d[v]$. The *degree* of v is the cardinality of $N_G(v)$, denoted by $\deg_G(v)$. A vertex v is an *isolated vertex* if $\deg_G(x) = 0$. A vertex x is said to be a *leaf* if $\deg_G(x) = 1$. A vertex of G is a *support vertex* if it is adjacent to a leaf in G. We denote by $L(G)$ and $U(G)$ the collections of the leaves and support vertices of G, respectively. A distance-k dominating set D of G is called a γ_k-set if $|D| = \gamma_k(G)$. For an edge $e \in E(G)$, the deletion $G - \{e\}$ of e from G is the graph $G - \{e\}$ obtained by removing the edge e. For a subset $A \subseteq V(G)$, the *deletion of A from G* is the graph $G - A$ obtained by removing all vertices in A and all edges incident to these vertices. The union of two disjoint graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with vertex set $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. The *diameter* of a graph G is the number $diam(G) = \max\{dist_G(u,v) : u, v \in V(G)\}$. A *forest* is a graph with no cycles, and a *tree* is a connected forest. For other undefined notions, the reader is referred to [1] for graph theory.

Let T be a tree and $diam(G) \geq 2k+1$. We introduce the end-tree $T^* = \widetilde{S}(z, A_1(z), A_2(z), \ldots, A_k(z))$, where $k \geq 2$, of T is a subtree with a center z such that the following all hold.

(i) z is lying on a longest path of T.
(ii) $dist_T(v,z) \leq k$ for every vertex v of T^*.
(iii) For $i = 1, \ldots, k$, $A_i(z) = \{v : dist_T(v,z) = k\}$.
(iv)$|A_k(z)| \geq 1$.
(v) The subgraph $T' = T - V(T^*)$ is a tree and $|T^*|$ is as large as possible.

The graph $\widetilde{S}(z, A_1(z), A_2(z), A_3(z))$ is shown Figure 1.

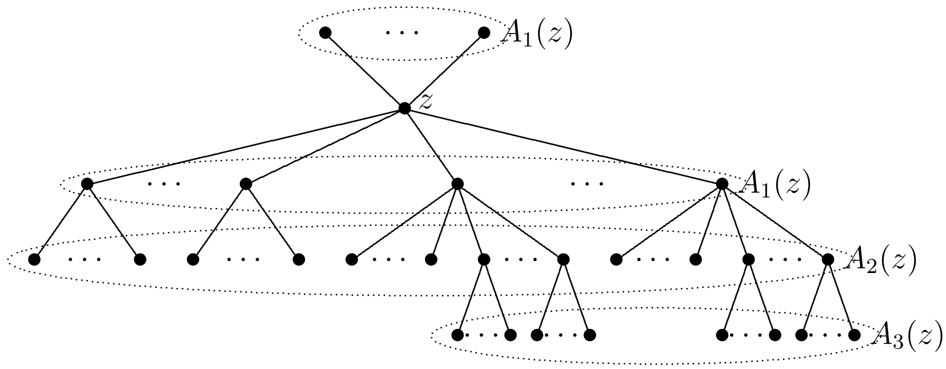

Figure 1: The tree $T^* = \widetilde{S}(z, A_1(z), A_2(z), A_3(z))$

The following are the useful lemmas.

Lemma 2.1. *Suppose $T^* = \widetilde{S}(z, A_1(z), A_2(z), \ldots, A_k(z))$ is an end-tree of a tree T, where $k \geq 2$ and $diam(T) \geq 2k+1$. Then there exists a γ_k-set D of T satisfying $D \cap V(T^*) = \{z\}$.*

Proof. Let D be a γ_k-set of T satisfying $z \in D$. If $D \cap V(T^*) = \{z\}$, then we are done. So we assume that $D \cap V(T^*) \neq \{z\}$. Let $w \in D$, where $w \neq z$, be a vertex of T^*. Suppose $T - uv = T^* \cup T'$, where $u \in V(T')$. If $N_T^k[w] \subseteq N_T^k[z]$, then $D - \{w\}$ is a DkDS of T with cardinality $\gamma_k(T) - 1$. This is a contradiction, so w is lying on the z-u path and $N_T^k[w] - N_T^k[z] \neq \emptyset$. Suppose, by contradiction, $u \in D$. Since $dist_T(z,u) \leq k+1$, this means that $dist_T(w,u) \leq k$. Then we can see that $D - \{w\}$ is a DkDS of T with cardinality $\gamma_k(T) - 1$. This is a contradiction, so $u \notin D$. We can see that $N_T^k[w] \subseteq N_T^k[z] \cup N_T^k[u]$. Thus $D' = (D - \{w\}) \cup \{u\}$ is a DkDS of T, so $\gamma_k(T) \leq |D'| = |D| = \gamma_k(T)$. Then D' is a γ_k-set of T satisfying $D \cap V(T^*) = \{z\}$, which completes the proof. □

Lemma 2.2. *Suppose $T^* = \widetilde{S}(z, A_1(z), A_2(z), \ldots, A_k(z))$, where $k \geq 2$, is an end-tree of a tree T and $T' = T - V(T^*)$. Then $\gamma_k(T') \leq \gamma_k(T) - 1$.*

Proof. By Lemma 1, assume that D is a γ_k-set of T satisfying $D \cap V(T^*) = \{z\}$. Let $D' = D - \{z\}$. If D' is a DkDS of T, then $\gamma_k(T') \leq |D'| = |D| - 1 = \gamma_k(T) - 1$. So we assume that D' is not a DkDS of T'. Let $T - uv = T^* \cup T'$, where $u \in V(T')$. Suppose $d = dist_T(z,u)$, then $1 \leq d \leq k+1$. If $d = k+1$, then $N_T^k[z] \cap V(T') = \emptyset$. Thus D' is a DkDS of T'. This is a contradiction, hence $1 \leq d \leq k$. Assume that $P : x_1, x_2, \ldots, x_k, z, \ldots, u, \ldots$ is a longest path of T. Let $T' - \{u\} = F \cup T''$, where F is a forest and T'' is a tree satisfying $V(P) \cap V(T'') \neq \emptyset$. Since $u \notin V(T^*)$ and $d \leq k$, we can see that $|F| \geq 1$.

Let $S = V(T') - N_{T'}^k[D']$. Since D' is not a DkDS of T', this implies that $S \neq \emptyset$. Then $S \subseteq N_{T'}^k[z]$ and $S \subseteq N_{T'}^{k-d}[u]$.

Claim 1. $V(F) - N_{T'}^{k-d}[u] \neq \emptyset$.

Suppose, by contradiction, $V(F) \subseteq N_{T'}^{k-d}[u]$. Then $V(F) \subseteq N_T^k[z]$, so T^* is not an end-tree of T. This is a contradiction.

Claim 2. $V(F) \cap D' \neq \emptyset$.

Suppose, by contradiction, $V(F) \cap D' = \emptyset$. By Claim 1, we have $V(F) \not\subseteq N_T^k[z]$. Then there exists a vertex $w \in D'$, where $w \in V(T'')$, satisfying $V(F) \subseteq N_T^k[w]$. Then D' is a DkDS of T', this is a contradiction.

By Claim 2, let $w \in D'$ be a vertex of F. Since P is a longest path of T, this implies that $dist_{T'}(u', u) \leq d+k$ for every vertex $u' \in V(F)$. Let w' be a vertex of F satisfying $dist_{T'}(w', u) = d$ and $N_{T'}^k[w] \subset N_{T'}^k[w']$. Then $S \subseteq N_T^k[w']$ and $D'' = (D - \{w\}) \cup \{w'\}$ is a DkDS of T'. So $\gamma_k(T') \leq |D''| = |D'| = \gamma_k(T) - 1$, which completes the proof. □

Lemma 2.3. *Suppose* $T^* = \widetilde{S}(z, A_1(z), A_2(z), \ldots, A_k(z))$*, where* $k \geq 2$*, is an end-tree of a tree* T *and* $T' = T - V(T^*)$*. Then* $\gamma_k(T') = \gamma_k(T) - 1$.

Proof. Suppose, by contradiction, $\gamma_k(T') \leq \gamma_k(T) - 2$. Let D' be a γ_k-set of T'. Then $D = D' \cup \{z\}$ is a DkDS of T , so $\gamma_k(T) \leq |D| = |D'| + 1 \leq (\gamma_k(T) - 2) + 1 = \gamma_k(T) - 1$. This is a contradiction, thus $\gamma_k(T') \geq \gamma_k(T) - 1$. By Lemma 2, we have that $\gamma_k(T) - 1 \leq \gamma_k(T') \leq \gamma_k(T) - 1$, which completes the proof. □

3 Characterization

In this section, we provide a constructive characterization of $\Gamma(k, n)$ for all $n \geq 1$ and all $k \geq 2$. Let T' be a tree. First we introduce some special subsets and some operations. Let T' be a tree.

(i) $A_0(z) = \{z\}$.

(ii) For $i = 0, 1, \ldots, k$,

$\overline{\mathcal{A}_i} = \{v : v \in A_i(z)$ for some z such that i is as small as possible$\}$.

(iii) For $d = 0, 1, \ldots, k-1$, $\mathcal{R}^d(T') = \{u : \gamma_k(T' - N_{T'}^d[u]) = \gamma_k(T')\}$.

Operation 1. Assume $u \in V(T')$. Add a new tree $T^* = \widetilde{S}(z, A_1(z), A_2(z), \ldots, A_k(z))$ and the edge uv, where $v \in \overline{\mathcal{A}_k}$, then we obtain the tree T and z is lying on a longest path of T.

Operation 2. Assume $u \in \mathcal{R}^0(T')$. Add a new tree $T^* = \widetilde{S}(z, A_1(z), A_2(z), \ldots, A_k(z))$ and the edge uv, where $v \in \overline{\mathcal{A}_{k-1}}$, then we obtain the tree T and z is lying on a longest path of T.

$\vdots$

Operation i. Assume $u \in \mathcal{R}^{i-2}(T')$. Add a new tree $T^* = \widetilde{S}(z, A_1(z), A_2(z), \ldots, A_k(z))$ and the edge uv, where $v \in \overline{\mathcal{A}_{k-i+1}}$, then we obtain the tree T and z is lying on a longest path of T.

$\vdots$

Operation k+1. Assume $u \in \mathcal{R}^{k-1}(T')$. Add a new tree $T^* = \widetilde{S}(z, A_1(z), A_2(z), \ldots, A_k(z))$ and the edge uv, where $v \in \overline{\mathcal{A}_0}$, then we obtain the tree T and z is lying on a longest path of T.

Let $\Psi(1)$ be the trees $T^* = \widetilde{S}(z, A_1(z), A_2(z), \ldots, A_k(z))$. Let Ψ be the collection of the trees T which are obtained from a sequence $T_1 \in \Psi(1), T_2, \ldots, T_m = T$ and, if $i = 1, \ldots, m-1$, T_{i+1} can be obtained recursively from T_i by one of the Operation 1 $\sim$ Operation k+1. Suppose $\Psi(k, n)$, where $k \geq 2$ and $n \geq 1$, is the collection of all trees $T \in \Psi$ satisfying $\gamma_k(T) = n$. We want to prove that $\Gamma(k, n) = \Psi(k, n)$ for for all $n \geq 1$ and all $k \geq 2$. The following theorem is the main theorem.

Theorem 3.1. *For $n \geq 1$ and $k \geq 2$, $\Gamma(k, n) = \Psi(k, n)$.*

For every tree $T \in \Psi(k, n)$, where $k \geq 2$ and $n \geq 1$, we can see that $\gamma_k(T) = n$. So $T \in \Gamma(k, n)$. Hence $\Psi(k, n) \subseteq \Gamma(k, n)$ for all $n \geq 1$ and all $k \geq 2$. On the other hand, we will prove $\Gamma(k, n) \subseteq \Psi(k, n)$ in the following lemma.

Lemma 3.2. *For $n \geq 1$ and $k \geq 2$, $\Gamma(k, n) \subseteq \Psi(k, n)$.*

Proof. We can see that $\Gamma(k, 1) = \Psi(k, 1)$. Suppose $T \in \Gamma(k, n)$ and $T \notin \Psi(k, n)$, where $n \geq 2$, such that n is as small as possible. Suppose $T^* = \widetilde{S}(z, A_1(z), A_2(z), \ldots, A_k(z))$ is an end-tree of T and $T' = T - V(T^*)$. By Lemma 2.3, $\gamma_k(T') = \gamma_k(T) - 1 = n - 1$. Thus $T' \in \Gamma(k, n-1)$, by the hypothesis, $T' \in \Psi(k, n-1)$. Assume that $T - \{uv\} = T^* \cup T'$, where $u \in V(T')$. We consider two cases.
Case 1. $v \in \overline{\mathcal{A}_k}$. Then T can be obtained from $T' \in \Psi(k, n-1)$ by the Operation 1. So $T \in \Psi(k, n)$. This is a contradiction.
Case 2. $v \in \overline{\mathcal{A}_i}$ for some $i = 0, 1, \ldots, k-1$. Then $dist_T(z, v) = i$ and $dist_T(z, u) = i+1$, where $v \in \mathcal{A}_i(z)$. Suppose, by contradiction, $u \notin \mathcal{R}^{k-i-1}(T')$. Then $\gamma_k(T' - N_{T'}^{k-i-1}[u]) \leq \gamma_k(T') - 1 = n - 2$. Let $H = T' - N_{T'}^{k-i-1}[u]$ and D' be a γ_k-set of H. Suppose $D'' = D' \cup \{z\}$. Since $N_{T'}^{k-i-1}[u] \subseteq N_T^k[z]$, this means that D'' is a DkDS of T. Thus $n = \gamma_k(T) \leq |D''| = |D'| + 1 = \gamma_k(H) + 1 \leq (n-2) + 1 = n - 1$, this is a contradiction. So $u \in \mathcal{R}^{k-i-1}(T')$. Then T can

be obtained from $T' \in \Psi(k, n-1)$ by the Operation $k-i+1$. So $T \in \Psi(k, n)$. This is a contradiction.

By Case 1 and Case 2, we have that $\Gamma(k, n) \subseteq \Psi(k, n)$ for all $n \geq 1$ and all $k \geq 2$. □

As an immediate consequence of Lemma 3.2, we obtain the Theorem 3.1. Hence we provide a constructive characterization $\Psi(k, n)$ of $\Gamma(k, n)$ for all $n \geq 1$ and all $k \geq 2$.

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