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On the 2-Independence Number of Connected Graphs

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Abstract

The distance between two vertices u and v in a graph G equals the length of a shortest path from u to v. A set S of vertices is a 2-independent set if the distance between any two elements in S is greater than two in G. The 2-independence number of a graph G, denoted by $\alpha_2(G)$, is the maximum size of a 2-independent set in G. In this paper, we determine a sharp upper bound for the 2-independence number in a connected graph and provide a characterization of the connected graphs achieving this sharp upper bound.

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Keywords: connected graph, 2-independent set, 2-independence number

1 Introduction

All graphs considered in this paper are finite, loopless, and without multiple edges. For a graph G, $V(G)$ and $E(G)$ denote the *vertex set* and the *edge set* of G, respectively. A *u-v path* $P : u = v_1, v_2, \ldots, v_k = v$ of G is a sequence of k vertices in G such that $v_iv_{i+1} \in E(G)$ for $i = 1, 2, \ldots, k-1$. Denote by P_n a *n-path* with n vertices. The *length* of P_n is n-1. For any two vertices u and v in G, the *distance between u and v*, denoted by $dist_G(u, v)$, is the minimum length of all u-v paths in G. A set S of vertices is a *k-independent set* if the distance between any two elements in S is greater than k in G. The *k-independence number* of a graph G, denoted by $\alpha_k(G)$, is the maximum size

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of a k-independent set in G. The study of the number of independent sets in a graph has a rich history. Finding a k-independent set of a graph is NP-hard (see [6], [7]). A. Abiad, G. Coutinho and M.A. Fiol [1] found the spectral bounds on the k-independence number of a graph. For some cases, they also showed that the bounds are sharp. Jou [3] determined the k-th largest number of 2-independent sets among all extra-free forest of order $n \geq 2$, where $k = 1, 2$ and 3. Extremal graphs achieving these values are also given. Min-Jen Jou and Jenq-Jong Lin [4] considered the problem of determining the small numbers of maximal 2-independent sets among all trees of order n. Extremal graphs achieving these values are also given. Min-Jen Jou, Jenq-Jong Lin and Qian-Yu Lin [5] determined a sharp upper bound for the 2-independence number in a tree. We also provided a constructive characterization of the extremal trees achieving this sharp upper bound. In this paper, we determine a sharp upper bound for the 2-independence number in a connected graph and provide a characterization of the connected graphs achieving this sharp upper bound.

2 A sharp upper bound

In this section, we determine a sharp upper bound for the 2-independence number in a connected graph. First, we introduce some notations.

The (open) *neighborhood* $N_G(v)$ of a vertex v is the set of vertices adjacent to v in G, and the *closed neighborhood* $N_G[v]$ is $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of v is the cardinality of $N_G(v)$, denoted by $\deg_G(v)$. A vertex x is said to be a *leaf* if $\deg_G(x) = 1$. A vertex of G is a *support vertex* if it is adjacent to a leaf in G. We denote by $L(G)$ and $U(G)$ the collections of the leaves and support vertices of G, respectively. Two leaves x and x' are called *duplicated leaves* in a graph G if they are adjacent to the same support vertex. The *closed 2-neighborhood* $N^2_G[v]$ of a vertex v is the set of the vertices u satisfying $dist_G(u,v) \leq 2$ and the (*open*) *2-neighborhood* is $N^2_G(v) = N^2_G[v] - \{v\}$. For any subset A, denote $N^2_G[A] = \bigcup_{v \in A} N^2_G[v]$ and $N^2_G(A) = \bigcup_{v \in A} N^2_G(v)$. Let $L^*(G) = \{x : x \in L(G), |N^2_G[x] \cap L(G)| = 1\}$. A connected graph is said to be *fresh* if $L(G) = L^*(G)$. A 2-independent set S of G is called a α_2*-set* if $|S| = \alpha_2(G)$. For a subset $A \subseteq V(G)$, the *induced subgraph* $\prec A \succ_G$ of a graph G is a subgraph $G' = (A, E(G'))$, where $E(G') = \{uv : u, v \in A, uv \in E(G)\}$. For a subset $A \subseteq V(G)$, the *deletion of* A *from* G is the graph $G - A$ obtained by removing all vertices in A and all edges incident to these vertices. For a subset $B \subseteq E(G)$, the *edge-deletion of* B *from* G is the graph $G - B$ obtained by removing all edges of B. For two different sets A and B, written $A - B$, is the set of all elements of A that are not elements of B. A *forest* is a graph with no cycles, and a *tree* is a connected forest. For other undefined notations, the reader is referred to [2] for graph theory.

The following are the useful lemmas.

Lemma 2.1. *Let H be a deletion of a connected graph G. If H is connected, then $\alpha_2(H) \leq \alpha_2(G)$.*

Proof. Let S be a α_2-set of H. Thus S is a 2-independent set of G. Hence $\alpha_2(H) = |S| \leq \alpha_2(G)$, which completes the proof. □

Lemma 2.2. *Let G' be an edge-deletion of a connected graph G. If G' is connected, then $\alpha_2(G) \leq \alpha_2(G')$.*

Proof. Let S be a α_2-set of G. For two distinct vertices u and v of S, $dist_{G'}(u, v) \geq dist_G(u, v) \geq 3$. Thus S is a 2-independent set of G'. Hence $\alpha_2(G) = |S| \leq \alpha_2(G')$, which completes the proof. □

Lemma 2.3. *Let G be a connected graph. Then there exists a α_2-set S satisfying $L^*(G) \subseteq S$.*

Proof. Let S be a α_2-set of G. If $L^*(G) \subseteq S$, then we are done. So we assume that $L^*(G) - S = \{x_1, \ldots, x_t\}$, where $t \geq 1$. Let u_i be a vertex of S such that $dist_G(u_i, x_i)$ is as small as possible. Then $dist_G(x_i, u_i) \leq 2$ for all i. Let $S^* = (S - \{u_1, \ldots, u_t\}) \cup \{x_1, \ldots, x_t\}$. So $|S^*| \geq |S| = \alpha_2(G)$. We can see that $dist_G(x_i, x_j) \geq 3$ for all $i \neq j$. For $w \in S^*$ and $w \neq x_i$, $dist_G(x_i, w) \geq dist_G(u_i, w) \geq 3$ Hence S^* is a α_2-set of G satisfying $L^*(G) \subseteq S^*$. □

Lemma 2.4. *Let G be a connected graph of order $|G| \geq 3$. If x and x' are duplicated leaves of G and $N_G(x) = N_G(x')$, then $\alpha_2(G - \{x'\}) = \alpha_2(G)$.*

Proof. Let $x \in L^*(G)$. By Lemma 2.3, there exists a α_2-set S of G satisfying $x \in S$. So S is a α_2-set of $G - \{x'\}$. Hence $\alpha_2(G - \{x'\}) = |S| = \alpha_2(G)$, which completes the proof. □

Theorem 2.5. [5] *Let T be a tree of order $|T| \geq 2$. Then $\alpha_2(T) \leq \lfloor \frac{|T|}{2} \rfloor$ and the upper bound is sharp.*

$|T(n)| = 2m$ is even $\qquad$ $|T(n)| = 2m + 1$ is odd

Figure 1: The tree $T(n)$, where $\lfloor \frac{n}{2} \rfloor = m \geq 1$.

Let $T(n)$ be as in Figure 1, where $\lfloor \frac{n}{2} \rfloor = m$. We can see that $\alpha_2(T(n)) = m$, as a result, the upper bound in Theorem 2.6 is sharp.

Theorem 2.6. *Let G be a graph of order $|G| \geq 2$. Then $\alpha_2(G) \leq \lfloor \frac{|G|}{2} \rfloor$ and the upper bound in is sharp.*

Proof. Let T be a spanning tree of G. Then $|T| = |G|$. By Lemma 2.2 and Theorem 2.5, $\alpha_2(G) \leq \alpha_2(T) \leq \lfloor \frac{|T|}{2} \rfloor = \lfloor \frac{|G|}{2} \rfloor$, which completes the proof. □

3 Characterization

In this section, we provide a characterization of the connected graphs achieving this sharp upper bound in Theorem 2.6. For the convenience of the characterization, let $\mathscr{G}(m)$ be the set of connected graph G satisfying $\alpha_2(G) = \lfloor \frac{|G|}{2} \rfloor = m$, where $m \geq 1$. Then $|G| = 2m$ or $2m+1$ for every $G \in \mathscr{G}(m)$. We want to construction $\mathscr{G}(m)$ for all $m \geq 1$. Due to the construction, we first mention two sets $\mathcal{H}_1(m)$ and $\mathcal{H}_2(m)$. They are collections of some connected graphs G.

(I) $\mathcal{H}_1(m) = \{G : G$ hold the following properties a and $b\}$.
(a) $|L^*(G)| = m$.
(b)$\prec V(G) - L^*(G) \succ_G$ is connected.

(II) $\mathcal{H}_2(m) = \{G : G$ hold the following properties c, d and $e\}$.
(c) $|L^*(G)| = m - 1$.
(d)$\prec V(G) - L^*(G) \succ_G$ is connected.
(e) $C = \{w_1, w_2, w_3\} = V(G) - N_G[L^*(G)]$ and $N_G(w_1) = \{w_2, w_3\}$.

The following Theorem is the main theorem.

Theorem 3.1. *For $m \geq 1$, $\mathscr{G}(m) = \bigcup_{i=1}^{2} \mathcal{H}_i(m)$.*

Suppose that $G \in \mathcal{H}_i(m)$ for some i, where $m \geq 1$, we can see that G is a connected graph satisfying $\alpha_2(G) = \lfloor \frac{|G|}{2} \rfloor = m$. Hence we obtain that $\mathcal{H}_i(m) \subseteq \mathscr{G}(m)$ for $i = 1, 2$. On the other hand, we will prove $\mathscr{G}(m) \subseteq \bigcup_{i=1}^{2} \mathcal{H}_i(m)$ and we prove it through a sequence of lemmas.

Lemma 3.2. *For $n \geq 3$, $\alpha_2(C_n) = \lfloor \frac{n}{3} \rfloor$.*

Proof. Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $\lfloor \frac{n}{3} \rfloor = k$. Then $S = \{v_3, \ldots, v_{3k}\}$ is a 2-independent set of C_n, so $\alpha_2(C_n) \geq |S| = k$. Suppose, by contradiction, $\alpha_2(C_n) \geq k+1$ and $S^* = \{v_{s_1}, \ldots, v_{s_{k+1}}\}$ is a 2-independent set of C_n, where $s_1 < s_2 < \cdots < s_{k+1}$. Then $dist_{C_n}(v_{s_i}, v_{s_{i+1}}) \geq 3$ for $i = 1, \ldots, k$ and $dist_{C_n}(v_{s_1}, v_{s_{k+1}}) \geq 3$. Then $|C_n| \geq 3k + 3 > n$. This is a contradiction, so $\alpha_2(C_n) \leq k$. Thus $\lfloor \frac{n}{3} \rfloor = k \leq \alpha_2(C_n) \leq k = \lfloor \frac{n}{3} \rfloor$, which completes the proof. □

Lemma 3.3. *Let $G \in \mathscr{G}(m)$ and $|G| = 2m$, where $m \geq 1$. Then the following hold.*
(i) $|L(G)| = |L^*(G)| = m$.
(ii) *The induced subgraph $\prec V(G) - L(G) \succ_G$ is connected.*

Proof. If $L(G) = \emptyset$, by Lemma 2.2 and Lemma 3.2, $m \leq \alpha_2(G) \leq \alpha_2(C_{2m}) = \lfloor \frac{2m}{3} \rfloor < m$. This is a contradiction, so $L(G) \neq \emptyset$.

Claim 1. $L(G) = L^*(G)$. Suppose, by contradiction, x and x' are duplicated leaves of G and $N_G(x) = N_G(x')$. By Lemma 2.4 and Theorem 2.6, $m = \alpha_2(G) = \alpha_2(G - \{x'\}) \leq \lfloor \frac{2m-1}{2} \rfloor = m - 1$. This is a contradiction, so $L(G) = L^*(G)$.

(i) We prove it by induction on m. We can see that $\mathscr{G}(1) = \{P_2, P_3, C_3\}$. If $|G| = 2$ is even, this means that $G = P_2$. So it's true for $m = 1$. Assume that it's true for $m-1$, where $m \geq 2$. Let $G \in \mathscr{G}(m)$ and $|G| = 2m$. Let $u_0 \in L(G)$ and $u_0' \in N_G(u_0)$. Suppose $G' = G - \{u_0, u_0'\}$, by Claim 1, then G' is a connected graph of order $|G'| = 2(m-1)$. By Theorem 2.6, $m-1 \leq \alpha_2(G') \leq \lfloor \frac{|G'|}{2} \rfloor = \lfloor \frac{|G|}{2} \rfloor - 1 = m-1$. The equalities hold, so $\alpha_2(G') = \lfloor \frac{|G'|}{2} \rfloor = m-1$. Hence $G' \in \mathscr{G}(m-1)$, by the induction hypothesis, $|L(G')| = |L^*(G')| = m-1$. Let $L_1 = L^*(G') = L(G') = \{u_1, \ldots, u_{m-1}\}$ and $U_1 = \{u_1', \ldots, u_{m-1}'\}$, where $u_i' \in N_{G'}(u_i)$ for $i = 1, \ldots, m-1$. So $V(G') = L_1 \cup U_1$.

Claim 2. $u_0' \notin N_G(L_1)$. Suppose, by contradiction, $A = N_G(u_0') \cap L_1 \neq \emptyset$ and $B = L_1 - A$. Let $A' = N_{G'}(A)$ and $B' = N_{G'}(B)$, where $|A| = |A'| = a$ and $|B| = |B'| = b$. By Lemma 2.3, there exists a α_2-set S of G satisfying $L(G) \subset S$. Then $u_0 \in S$ and $A \cap S = \emptyset$. Since G is connected, $N^2_{G'}(A') \cap B \neq \emptyset$. Then $|A' \cap S| \leq a-1$ and $m = \alpha_2(G) = |\{u_0\}| + |A' \cap S| + |B| \leq 1 + (a-1) + b = a + b = m-1$. This is a contradiction, so $u_0' \notin N_G(L_1)$.

By Claim 2, we have that $L(G) = L^*(G) = \{u_0, u_1, \ldots, u_{m-1}\}$ and $|L(G)| = |L^*(G)| = m$.

(ii) Since G is connected, this implies that $\prec V(G) - L(G) \succ_G$ is connected. □

Lemma 3.4. *Let $G \in \mathscr{G}(m)$ and $|G| = 2m+1$, where $m \geq 1$. If G have duplicated leaves, then $|L^*(G)| = |L(G)| - 1 = m$.*

Proof. By lemma 2.4, suppose G' is a fresh graph, where G' is a subgraph of G and $|G'| = |G| - a$, such that $\alpha_2(G') = \alpha_2(G) = m$. By Theorem 2.6, $m = \alpha_2(G') \leq \lfloor \frac{|G'|}{2} \rfloor = \lfloor \frac{|G|-a}{2} \rfloor \leq \lfloor \frac{2m}{2} \rfloor = m$. The equalities hold, so $a = 1$ and $G' \in \mathscr{G}(m)$. By Lemma 3.3, $|L(G')| = |L^*(G')| = m$. Hence $|L^*(G)| = |L(G)| - 1 = m$. □

Lemma 3.5. *If $G \in \mathscr{G}(m)$ and $|G| = 2m+1$, where $m \geq 1$, then $|L^*(G)| = m$ or $m-1$.*

Proof. We prove it by induction on m. We can see that $\mathscr{G}(1) = \{P_2, P_3, C_3\}$. If $|G| = 3$ is odd, this means that $G = P_3$ or C_3. So it's true for $m = 1$. Assume that it's true for $m-1$, where $m \geq 2$. Let $G \in \mathscr{G}(m)$ and $|G| = 2m+1$. If $L(G) = \emptyset$, by Lemma 2.2 and lemma 3.2, then $m = \alpha_2(G) \leq \alpha_2(C_{2m+1}) = \lfloor \frac{2m+1}{3} \rfloor < m$, where $m \geq 2$. This is a contradiction, so $L(G) \neq \emptyset$.

If $L^*(G) \neq L(G)$. By Lemma 3.4, $|L^*(G)| = |L(G)| - 1 = m$. So we assume that $L^*(G) = L(G)$. Let $u_0 \in L(G)$ and $u_0' \in N_G(u_0)$. Suppose $G' = G - \{u_0, u_0'\}$, then G' is a connected graph of order $|G'| = 2(m-1)+1$. By Theorem 2.6, $m-1 \leq \alpha_2(G') \leq \lfloor \frac{|G'|}{2} \rfloor = \lfloor \frac{|G|}{2} \rfloor - 1 = m-1$. The equalities hold, so $\alpha_2(G') = \lfloor \frac{|G'|}{2} \rfloor = m-1$. Thus $G' \in \mathscr{G}(m-1)$, by the induction hypothesis, $|L^*(G')| = m-1$ or $m-2$. Let $L_1 = L^*(G') = \{u_1, \ldots, u_k\}$, where $m-2 \leq k \leq m-1$, and $U_1 = N_{G'}(L_1) \cap U(T) = \{u_1', \ldots, u_k'\}$. If $N_G(u_0') \cap L_1 = \emptyset$, then $|L^*(G)| = k+1$ and $|L^*(G)| = m$ or $m-1$. So we assume that $A = N_G(u_0') \cap L_1 \neq \emptyset$ and $B = L_1 - A$. Let $A' = N_{G'}(A)$ and $B' = N_{G'}(B)$, where $|A| = |A'| = a$ and $|B| = |B'| = b$. By Lemma 2.3, there exists a α_2-set S of G satisfying $L^*(G) \subset S$. Then $u_0 \in S$ and $A \cap S = \emptyset$. Since $a + b = k \leq m-1$, $C = V(G') - (A \cup A' \cup B \cup B')$ and $|C| = 1$ or 3.

Claim. $|S \cap A'| \leq 1$. Suppose, by contradiction, $|S \cap A'| \geq 2$. Since G is connected, $\prec A' \succ_{G'}$ is connected or $N^2_{G'}(A') \cap (C \cup B') \neq \emptyset$. Then we have that $2 \leq |S \cap A'| \leq a-1$ and $m = \alpha_2(G) = |\{u_0\}| + |A' \cap S| + |B| \leq 1 + (a-1) + b = a + b = m-1$. This is a contradiction, so $|S \cap A'| \leq 1$.

Since $\alpha_2(G) = \alpha_2(G') + 1$, by Claim, $|A| = a = 1$, say $A = \{u_1\}$. If $k = m-2$, then $C = \{w_1, w_2, w_3\}$, where $N_{G'}(w_1) = \{w_2, w_3\}$, and $C \cap L^*(G) = \emptyset$. Since G' is connected, $u_1' \in N_{G'}(\{w_2, w_3\} \cup B')$. Thus $dist_{G'}(u_1', w_1) = 2$ or $dist_{G'}(u_1', u_i) = 2$ for some $u_i \in B$. Then $u_1' \notin S$ and $m = |S| = |\{u_0, w_1\}| + |B| = 2 + (m-2-1) = m-1$. This is a contradiction, so $k = m-1$. So $m \geq |L^*(G)| \geq |B| + 1 = m-2+1 = m-1$. □

Lemma 3.6. *For $m \geq 1$, $\mathscr{G}(m) \subseteq \bigcup_{i=1}^{2} \mathcal{H}_i(m)$.*

Proof. Let $G \in \mathscr{G}(m)$. Then $|G| = 2m$ or $2m+1$. If $|G| = 2m$, by Lemma 3.3, $|L(G)| = |L^*(G)| = m$ and $|G - L^*(G)| = m$, where $V(G) = L^*(G) \cup U(G)$. By Lemma 2.3, $L^*(G)$ is a α_2-set of G. Since G is connected, this implies that $G' = G - L^*(G)$ is connected. Hence $G \in \mathcal{H}_1(m)$. So we assume that $|G| = 2m+1$. By Lemma 3.4, $|L^*(G)| = m$ or $m-1$.

Case 1. $|L^*(G)| = m$. Then $L^*(G)$ is a α_2-set of G. Since G is connected, this implies that $G' = G - L^*(G)$ is connected. Hence $G \in \mathcal{H}_1(m)$.

Case 2. $|L^*(G)| = m-1$. Let $C = V(G) - N_G[L^*(G)]$. Then $|C| = 3$, say $C = \{w_1, w_2, w_3\}$ and $C \cap L(G) = \emptyset$. This means that $N_G(w_1) = \{w_2, w_2\}$. Hence $G \in \mathcal{H}_2(m)$.

By Cases 1 and 2, $G \in \mathcal{H}_1(m)$ or $G \in \mathcal{H}_2(m)$. □

As an immediate consequence of Lemma 3.6, we obtain the Theorem 3.1. Hence we provide a characterization of $\mathscr{G}(m)$ for all $m \geq 1$.

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