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On the 2-Independence Number of Trees

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Abstract

The distance between two vertices u and v in a graph G equals the length of a shortest path from u to v. A set S of vertices is a 2-independent set if the distance between any two elements in S is greater than two in G. The 2-independence number of a graph G, denoted by $\alpha_2(G)$, is the maximum size of a 2-independent set in G. Here we focus on the trees. In this paper, we determine a sharp upper bound for the 2-independence number in a tree. We also provide a constructive characterization of the extremal trees achieving this sharp upper bound.

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1 Introduction

All graphs considered in this paper are finite, loopless, and without multiple edges. For a graph G, $V(G)$ and $E(G)$ denote the *vertex set* and the *edge set* of G, respectively. A *u-v path* $P : u = v_1, v_2, \ldots, v_k = v$ of G is a sequence of k vertices in G such that $v_iv_{i+1} \in E(G)$ for $i = 1, 2, \ldots, k-1$. Denote by P_n a *n-path* with n vertices. The *length* of P_n is n-1. For any two vertices u and v in G, the *distance between u and v*, denoted by $dist_G(u, v)$, is the minimum length of all u-v paths in G. A set S of vertices is a *k-independent set* if the distance between any two elements in S is greater than k in G. The *k-independence number* of a graph G, denoted by $\alpha_k(G)$, is the maximum size

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of a k-independent set in G. The study of the number of independent sets in a graph has a rich history. Finding a k-independent set of a graph is NP-hard (see [5], [6]). A. Abiad, G. Coutinho and M.A. Fiol [1] found the spectral bounds on the k-independence number of a graph. For some cases, they also showed that the bounds are sharp. Jou [3] determined the k-th largest number of 2-independent sets among all extra-free forest of order $n \geq 2$, where $k = 1$, 2 and 3. Extremal graphs achieving these values are also given. Min-Jen Jou and Jenq-Jong Lin [4] considered the problem of determining the small numbers of maximal 2-independent sets among all trees of order n. Extremal graphs achieving these values are also given. In this paper, we determine a sharp upper bound for the 2-independence number in a tree. We also provide a constructive characterization of the extremal trees achieving this sharp upper bound.

2 A sharp upper bound

In this section, we determine a sharp upper bound for the 2-independence number in a tree. First, we introduce some notations.

The (open) *neighborhood* $N_G(v)$ of a vertex v is the set of vertices adjacent to v in G, and the *closed neighborhood* $N_G[v]$ is $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of v is the cardinality of $N_G(v)$, denoted by $\deg_G(v)$. A vertex x is said to be a *leaf* if $\deg_G(x) = 1$. A vertex of G is a *support vertex* if it is adjacent to a leaf in G. Two leaves x and x' are called *duplicated leaves* in a graph G if they are adjacent to the same support vertex. A k-independent set S of G is called a *α_k-set* if $|S| = \alpha_k(G)$. For a subset $A \subseteq V(G)$, the *deletion of A from G* is the graph $G - A$ obtained by removing all vertices in A and all edges incident to these vertices. The *diameter* of a graph G is the number $diam(G) = \max\{dist_G(u, v) : u, v \in V(G)\}$. For two different sets A and B, written $A - B$ is the set of all elements of A that are not elements of B. A *forest* is a graph with no cycles, and a *tree* is a connected forest. For other undefined notions, the reader is referred to [2] for graph theory.

The following are the useful lemmas.

Lemma 2.1. *Let T be a tree of order $|T| \geq 2$ and T' be a subtree of T. Then $\alpha_2(T') \leq \alpha_2(T)$.*

Proof. Let S' be a α_2-set of T'. Then S' is a 2-independent set of T. So $\alpha_2(T) \geq |S'| = \alpha_2(T')$, which completes the proof. □

Lemma 2.2. *Suppose T is a tree of order $|T| \geq 2$ and x is a leaf of T. Then there exists a α_2-set S of T satisfying $x \in S$.*

Proof. Let S be a α_2-set of T. If $x \in S$, then we are done. So we assume that $x \notin S$. Let v be a vertex in S such that $dist_T(x, v)$ is as small as possible. Then

$dist_T(x,v) \leq 2$ and $dist_T(x,u) \geq 3$ for every vertex $u \in S$, where $u \neq v$. Let $S' = (S - \{v\}) \cup \{x\}$. For every pair of vertices u and w in S', $dist_T(u,w) \geq 3$. Thus S' is a 2-independent set of T, so $\alpha_2(T) = |S| = |S'| \leq \alpha_2(T)$. Hence the equalities hold, the set S' is a α_2-set of T satisfying $x \in S'$. This completes the proof. □

Lemma 2.3. *Let T be a tree of order $|T| \geq 3$. Suppose x and x' are duplicated leaves of T, then $T' = T - \{x'\}$ is a tree and $\alpha_2(T') = \alpha_2(T)$.*

Proof. Since x' is a leaf of T, this implies that T' is a tree. By Lemma 2.2, there exists a α_2-set S of T satisfying $x \in S$. Then $x' \notin S$, so S is a 2-independent set of T'. By Lemma 2.1, $\alpha_2(T') \leq \alpha_2(T)$ and $\alpha_2(T) = |S| \leq \alpha_2(T') \leq \alpha_2(T)$. Then the equalities hold, hence $\alpha_2(T') = \alpha_2(T)$. □

Lemma 2.4. *Suppose T is a tree of order $|T| \geq 2$ and $\alpha_2(T) = m$, where $m \geq 1$. Then $|T| \geq 2m$.*

Proof. Suppose $S = \{w_1, \ldots, w_m\}$, where $m \geq 1$, is a α_2-set of T. For $i = 1, \ldots, m$, let $W(i)$ be the set of vertices $v \in N_T(w_i)$ which are not adjacent to another w_j, where $j \neq i$. Since $dist_T(w_i, w_j) \geq 3$ for all $i \neq j$, we have that $W(i) \neq \emptyset$ and $W(i) \cap W(j) = \emptyset$. So $|T| \geq |S| + \sum_{i=1}^{m} |W(i)| \geq m + m = 2m$. □

Now we determine an upper bound for the 2-independence number in a tree.

Theorem 2.5. *Let T be a tree of order $|T| \geq 2$. Then $\alpha_2(T) \leq \lfloor \frac{|T|}{2} \rfloor$.*

Proof. Let $|T| = n$. If n is even, by Lemma 2.4, $\alpha_2(T) \leq \frac{|T|}{2} = \lfloor \frac{|T|}{2} \rfloor$. So it's true for all even $n \geq 2$. Now we assume that $|T| = n = 2k+1$ is odd, where $k \geq 1$. Suppose, by contradiction, $\alpha_2(T) \geq \lfloor \frac{|T|}{2} \rfloor + 1$, then $\alpha_2(T) \geq k+1$. By Lemma 2.4, $2k+1 = |T| \geq 2 \cdot \alpha_2(T) \geq 2(k+1) = 2k+2$. This is a contradiction, which completes the proof. □

$|T(n)| = 2m$ is even $\quad\quad$ $|T(n)| = 2m+1$ is odd

Figure 1: The tree $T(n)$, where $\lfloor \frac{n}{2} \rfloor = m \geq 1$.

Let $T(n)$ be as in Figure 1, where $\lfloor \frac{n}{2} \rfloor = m$. We can see that $\alpha_2(T(n)) = m$, as a result, the upper bound in Theorem 2.5 is sharp.

3 Characterization

In this section, we provide a constructive characterization of the extremal trees achieving the sharp upper bound in Theorem 2.5. For the convenience of the characterization, let $\mathscr{T}(m)$ be the set of trees T satisfying $\alpha_2(T) = \lfloor \frac{|T|}{2} \rfloor = m$, where $m \geq 1$. Due to the construction, we first mention the following lemma.

Lemma 3.1. *Let $T \in \mathscr{T}(m)$, where $m \geq 2$. Suppose $T' \in \mathscr{T}(m-1)$ is a subtree of T, then $|T'| + 1 \leq |T| \leq |T'| + 3$.*

Proof. We can see that $|T| \geq |T'| + 1$. By Theorem 2.5, $|T'| \geq 2(m-1) = 2m-2$ and $|T| \leq 2m+1$. So $|T| - |T'| \leq (2m+1) - (2m-2) = 3$. Hence $|T'| + 1 \leq |T| \leq |T'| + 3$. □

Let T' be a tree of order $n' \geq 2$ and S' be a α_2-set of T'. We introduce a special subset $\mathcal{R}(T')$ and four operations. Let $\mathcal{R}(T') = \{v : v \in V(T'), N_{T'}[v] \cap S' = \emptyset\}$.

Operation O1. Assume $n' \geq 5$ is odd and $u \in \mathcal{R}(T')$, add a new vertex x_1 and the edge ux_1.
Operation O2. Assume $u \notin S'$, add a new path $P_2 : x_1, x_2$ and the edge ux_2.
Operation O3. Assume $n' \geq 2$ is even and $u \in V(T')$, add a new path $P_3 : x_1, x_2, x_3$ and the edge ux_3.
Operation O4. Assume $n' \geq 2$ is even and $u \in V(T')$, add a new path $P_3 : x_1, x_2, x_3$ and the edge ux_2.

Let $\mathcal{A}(1) = \{P_2, P_3\}$. Suppose $\mathcal{A}$ is the collection of the trees T which are obtained from a sequence $T_1 \in \mathcal{A}(1), T_2, \ldots, T_k = T$ and, if $i = 1, \ldots, k-1$, T_{i+1} can be obtained recursively from T_i by one of the operations O1-O4. Suppose $\mathcal{A}(m)$, where $m \geq 2$, is the collection of all trees $T \in \mathcal{A}$ satisfying $\alpha_2(T) = m$. We want to prove that $\mathscr{T}(m) = \mathcal{A}(m)$ for for all $m \geq 1$. The following Theorem is the main theorem.

Theorem 3.2. *For $m \geq 1$, $\mathscr{T}(m) = \mathcal{A}(m)$.*

For every tree $T \in \mathcal{A}(m)$, where $m \geq 1$, we can see that T is a tree satisfying $\alpha_2(T) = \lfloor \frac{|T|}{2} \rfloor = m$. Hence we obtain that $\mathcal{A}(m) \subseteq \mathscr{T}(m)$. On the other hand, we will prove $\mathscr{T}(m) \subseteq \mathcal{A}(m)$ in the following lemma.

Lemma 3.3. *For $m \geq 1$, $\mathscr{T}(m) \subseteq \mathcal{A}(m)$.*

Proof. We can see that $\mathscr{T}(1) = \{P_2, P_3\} = \mathcal{A}(1)$, so it's true for $m = 1$. Suppose, by contradiction, $T \in \mathscr{T}(m)$ and $T \notin \mathcal{A}(m)$ such that m is as small as possible. Then $2m \leq |T| \leq 2m+1$, where $m \geq 2$, and $diam(T) \geq 4$.

First, we assume that T has no duplicated leaf. Let $P : x_1, x_2, x_3, x_4, \ldots$ be a longest path of T. By Lemma 2.2, there exists a α_2-set S of T satisfying $x_1 \in S$. Let $S' = S - \{x_1\}$ and $T' = T - \{x_1, x_2\}$. Then T' is a tree of order $|T'| = |T| - 2$ and S' is a 2-independent set of T'. By Theorem 2.5, $m - 1 = |S| - 1 = |S'| \leq \alpha_2(T') \leq \lfloor \frac{|T'|}{2} \rfloor = \lfloor \frac{|T|-2}{2} \rfloor = \lfloor \frac{|T|}{2} \rfloor - 1 = m - 1$. Thus the equalities hold, $\alpha_2(T') = m-1$ and S' is a α_2-set of T'. So $T' \in \mathscr{T}(m-1)$, by the hypothesis, $T' \in \mathcal{A}(m-1)$. Since $x_3 \notin S'$, T can be obtained from $T' \in \mathcal{A}(m-1)$ by the Operation O2. So $T \in \mathcal{A}(m)$. This is a contradiction, hence T have duplicated leaves.

By Lemma 2.3, let T^* be a subtree of T such that T^* has no duplicated leaf and $\alpha_2(T^*) = \alpha_2(T)$. By Theorem 2.5, then $m = \alpha_2(T) = \alpha_2(T^*) \leq \lfloor \frac{|T^*|}{2} \rfloor \leq \lfloor \frac{|T|-1}{2} \rfloor \leq \lfloor \frac{|T|}{2} \rfloor = m$. Thus the equalities hold, so $|T^*| = |T| - 1 = 2m$. Then T have only two duplicated leaves, say u and u'. Let $N_T(y) = \{u, u', z\}$. Suppose S^* is a α_2-set of T satisfying $u \in S^*$. Then $S^{**} = S^* - \{u\}$ is a 2-independent set of $T^{**} = T - \{u, u', y\}$, where T^{**} is a tree of order $|T^{**}| = |T| - 3 = 2(m-1)$. By Theorem 2.5, $m - 1 = |S^*| - 1 = |S^{**}| \leq \alpha_2(T^{**}) \leq \lfloor \frac{|T^{**}|}{2} \rfloor = m - 1$. Thus the equalities hold, $\alpha_2(T^{**}) = m - 1$ and S^{**} is a α_2-set of T^{**}. So $T^{**} \in \mathscr{T}(m-1)$, by the hypothesis, $T^{**} \in \mathcal{A}(m-1)$. Since $z \notin S^{**}$ and $|T^{**}| = 2(m-1)$, T can be obtained from $T^{**} \in \mathcal{A}(m-1)$ by the Operation O4. So $T \in \mathcal{A}(m)$. This is a contradiction again, which completes the proof. □

As an immediate consequence of Lemma 3.3, we obtain the Theorem 3.2. Hence we provide a constructive characterization $\mathcal{A}(m)$ of $\mathscr{T}(m)$ for all $m \geq 1$.

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