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Ranks in Elliptic Curves $y^2 = x^3 \pm Ax$ with Varied Primes

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Abstract

Take E_{-p} as elliptic curve $y^2 = x^3 - px$ with prime p then, we shall numerate rank of this curve. We appoint that $E_{\mp pq}$ are elliptic curves $y^2 = x^3 \mp pqx$ then, we will calculate ranks of these curves according to the assumption of distinct odd primes p and q. In addition, the ranks of $E_{-2p}: y^2 = x^3 - 2px$ will be compared with that of in $E_{\mp pq}$

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1 Introduction

Generally, the result in systematized rank of elliptic curve is from 0 to 3. Even though, there is a case of rank 4 or at least 5 it is rare. And for this in certain cases it needs some conjecture or other supposition. Thus, on the basis of no hypothesis above range of calculation is general consequence. In [1], Antoniewicz showed that rank of elliptic curve $C_m: y^2 = x^3 - m^2x + 1$ with $m \in Z_+$ is rank$C_m \geq 2$ for $m \geq 2$ and rank $C_m \geq 3$ for $m \geq 4$, $m \equiv 0(mod\ 4)$. In [6], the authors showed that rank of $E_{q,p,m}: y^2 = x(x - 2^m)(x + q - 2^m)$ is $r_{q,p,4} \leq 1$ and in addition if $p \equiv 3$, $7(mod\ 8)$ then, there comes $r_{q,p,2} = 0$. In [22], Yoshida calculated that rank of elliptic curve $y^2 = (x + m)(x^2 + m^2)$ is 1 if prime p is the form $p \equiv -1(mod\ 8)$ with $m = p, \pm 2p$. In [16], Knapp presented that rank of $y^2 = x^3 - p^2x$ where an odd prime p is the form $p \equiv 3(mod\ 8)$ is 0. In [21], Walsh enumerated that if there exist two positive rational points $P_1 =(x_1, y_1)$, $P_2 =(x_2, y_2)$ on E_{-2p}: $y^2 = x^3 - 2px$ and a positive rational point $P_3 =(x_3, y_3)$ on E_{8p}: $y^2 = x^3 + 8px$ with $x_i = d_i u_i{}^2(1 \le i \le 3)$ where each d_i is squarefree and $u_i \in Q$ and in addition $(d_1, d_2) \in \{(-1, 2), (-1, -2), (-1, p), (2, -2), (2, p)\}$ and $d_3 \in \{2, p\}$ then, the rank of E_{-2p} is 3. In [20], Walsh treated rank of elliptic curve $y^2 = x^3 - px$ with prime p. In this article, we shall numerate the ranks of forms of elliptic curves $y^2 = x^3 \pm Ax$.

In section 2, we will consider the results of ranks in several curves $y^2 = x^3 \pm p_1p_2x$($p_1$and p_2 are different primes) and other curves.

In section 3, we shall treat the rank of curve $E_{-p}: y^2 = x^3 - px$.

In section 4, we will enumerate the ranks of elliptic curves $E_{\mp pq}: y^2 = x^3 \mp pqx$.

In section 5, the rank of curve $E_{-2p}: y^2 = x^3 - 2px$ with prime p will be researched and compared the results with that of consequences in $E_{\mp pq}$

In section 6, we shall submit the examples of previous calculations.

First, we ought to treat the notations in [17].

Denote E as an elliptic curve of the form $y^2 = x^3 + ax^2 + bx$ and take Γ as the set of rational points on E. Then, because of $Mordell's$ Theorem the set Γ became a finitely generated abelian group. Besides, the structure $\Gamma \cong E(Q)_{tors} \oplus Z^r$ is derived with torsion subgroup $E(Q)_{tors}$ and Mordell-Weil rank r. Moreover, define $Q^\times$ as multiplicative group that is composed of non-zero rational numbers. In addition, let $Q^{\times 2}$ be the subgroup of squares of elements of $Q^\times$.

Define α as a homomorphism $\alpha: \Gamma \to Q^\times/Q^{\times 2}$ where

$$\alpha(O) = 1(mod\ Q^{\times 2}) \text{ and } \alpha(0, 0) = b(mod\ Q^{\times 2}) \text{ and}$$

$$\alpha(x, y) = x(mod\ Q^{\times 2})$$

with point at infinity O and non-zero x.

Assign $\bar{E}$ as the curve $y^2 = x(x^2 - 2ax + a^2 - 4b)$ and let $\overline{\Gamma}$ be the set of rational points on curve $\bar{E}$.

Suppose that $\bar{\alpha}$ is a homomorphism such that $\bar{\alpha}: \overline{\Gamma} \to Q^\times/Q^{\times 2}$ that satisfies

$$\bar{\alpha}(O) = 1(mod\ Q^{\times 2}) \text{ and } \bar{\alpha}(0, 0) = a^2 - 4b(mod\ Q^{\times 2}) \text{ and}$$

$$\bar{\alpha}(x, y) = x(mod\ Q^{\times 2})$$

Here, O is point at infinity and x is non-zero.

Denote $N^2 = b_1M^4 + aM^2e^2 + b_2e^4$ as relating equation for Γ where $b_1$and b_2 are divisors of b such that $b = b_1b_2$ with $b_1 \not\equiv 1, b(mod\ Q^{\times 2})$ and $N^2 = b_1M^4 - 2aM^2e^2 + b_2e^4$ as relating equation for $\overline{\Gamma}$ which satisfies that b_1 and b_2 are divisors of $a^2 - 4b$ as $b_1b_2 = a^2 - 4b$ with $b_1 \not\equiv 1, a^2 - 4b(mod\ Q^{\times 2})$.

Suppose that $(M, e\ N)$ is an integral solution of above two equations with $1 = (M, N)$=(M, e)=(N, e) =$(b_1, e) = (b_2, M)$ and $M \neq 0, e \neq 0$.

Lastly, there is gotten $2^r = \frac{\#\alpha(\Gamma)\#\bar{\alpha}(\Gamma)}{4}$ with rank r of E.
We take several notations as follows:

LDV: Legendre symbols of different values([11]).

LSV: Legendre symbols having same value([11]).

$w.i.u.v.1$: with integers u and v and $(u, v) = 1$ ([15]).

$r2.4$: $2^r = \frac{2\cdot 4}{4}$([15]).

$r4.2$: $2^r = \frac{4\cdot 2}{4}$([15]).

$r^{\geq}4.4$: $2^r \geq \frac{4\cdot 4}{4}$([15]).

$r4.4$: $2^r = \frac{4\cdot 4}{4}$([15]).

2 Treatment in Some Curves

In section 2, we will consider results of ranks in curves $y^2 = x^3 \pm p_1 p_2 x$ and others.

Proposition 2.1. Denote E_{2p} as an elliptic curve $y^2 = x^3 + 2px$ where prime p is an odd prime such that $p = 64u^8 + 16u^4v^4 + 9v^8$ with two integers u and v and $(u, v) = 1$ and $p \equiv 1(mod\ 8)$ then, $rank(E_{2(64u^8+16u^4v^4+9v^8)}(Q)) = 2$ is given.

Proof. See [13]. □

Proposition 2.2. Let E_{-pq} be the curve $y^2 = x^3 - pqx$ where two different odd primes p, q are $p = 4u^4 + 28u^2v^2 + 51v^4$ and $q = 4u^4 + 28u^2v^2 + 47v^4$ with integers u and v and $(u,v)=1$ and $p \equiv 3(mod\ 16)$, $q \equiv 15(mod\ 16)$ then, there derived $rank(E_{-(4u^4+28u^2v^2+51v^4)(4u^4+28u^2v^2+47v^4)}(Q)) = 2$.

Proof. See [13]. □

Proposition 2.3. The rank of elliptic curve $E_{-pq}: y^2 = x^3 - pqx$ where different odd primes $p = 25u^4 + 20u^2v^2 + 6v^4$ and $q = 25u^4 + 20u^2v^2 + 2v^4$ with two integers u and v and $(u, v) = 1$ and $p \equiv 3(mod\ 16)$ and $q \equiv 15(mod\ 16)$ is gotten as $rank(E_{-(25u^4+20u^2v^2+6v^4)(25u^4+20u^2v^2+2v^4)}(Q)) = 2$.

Proof. See [13]. □

Proposition 2.4. The rank of elliptic curve $y^2 = x^3 - pqx$ where p, q are different odd primes given as the forms $p = 4u^4 + 20u^2v^2 + 27v^4$ and $q = 4u^4 + 20u^2v^2 + 23v^4$ $w.i.u.v.1$ and $p \equiv 3(mod\ 16)$ and $q \equiv 15(mod\ 16)$ is educed as $rank(E_{-(4u^4+20u^2v^2+27v^4)(4u^4+20u^2v^2+23v^4)}(Q)) = 2.$

Proof. See [15]. □

Proposition 2.5. We take $rank(E_{-(36u^4+36u^2v^2+11v^4)(36u^4+36u^2v^2+7v^4)}(Q)) = 2$ where the curve $E_{-pq}: y^2 = x^3 - pqx$ satisfies that two different odd primes p, q are the forms $p = 36u^4 + 36u^2v^2 + 11v^4$ and $q = 36u^4 + 36u^2v^2 + 7v^4$ $w.i.u.v.1$ and $p \equiv 11(mod\ 16)$ and $q \equiv 7(mod\ 16)$.

Proof. See [15]. □

Proposition 2.6. If an elliptic curve $E_{-pq}: y^2 = x^3 - pqx$ is appointed as two distinct odd primes p and q as $p \equiv 7(mod\ 16)$ and $q \equiv 7(mod\ 16)$ as $-p + q = t^2$ with integer t and $2u^4 + 2pq = v^2$ with integers u and v then, it is deduced that $rank(E_{-(-p+q,2u^4+2pq)}(Q)) = 2.$

Proof. See [15]. □

Proposition 2.7. We gain $rank(E_{(u^4+72u^2v^2+10v^4)(u^4+74u^2v^2+10v^4)}(Q)) = 1$ where E_{pq} is an elliptic curve $y^2 = x^3 + pqx$ that satisfies p and q are different odd primes $p = u^4 + 72u^2v^2 + 10v^4$ and $q = u^4 + 74u^2v^2 + 10v^4$ $w.i.u.v.1$ and $p \equiv 3(mod\ 16)$ and $q \equiv 5(mod\ 16)$.

Proof. See [15]. □

Proposition 2.8. Suppose that E_{pq} is an elliptic curve $y^2 = x^3 + pqx$ with two distinct odd primes p, q as $p = u^4 - 10u^2v^2 + 12v^4$ and $q = u^4 - 10u^2v^2 + 14v^4$ $w.i.u.v.1$ and $p \equiv 3(mod\ 16)$ and $q \equiv 5(mod\ 16)$ then, there comes $rank(E_{(u^4-10u^2v^2+12v^4)(u^4-10u^2v^2+14v^4)}(Q)) = 1.$

Proof. See [15]. □

Proposition 2.9. Assign E_{-2pq} as an elliptic curve $y^2 = x^3 - 2pqx$ where different odd primes p and q are the forms $p \equiv 3(mod\ 16)$ and $q \equiv 7(mod\ 16)$ with $p = 30t^2 + 11$, $q = 60t^2 + 23$ with integer t then, we have $rank(E_{-2(30t^2+11)(60t^2+23)}(Q)) = 1.$

Proof. See [9]. □

Proposition 2.10. Take E_p as an elliptic curve $y^2 = x^3 + px$ where p is a prime$p = u^8 + 12u^4v^4 + 16v^8$ with two integers u and v and (u,v)=1 and $p \equiv 13(mod\ 16)$ then, we are confronted with $rank(E_{(u^8+12u^4v^4+16v^8)}(Q)) = 1$.

Proof. See [9]. □

Proposition 2.11. There is educed that $rank(E_{-p(p-4)}(Q)) = 1$ where elliptic curve $E_{-p(p-4)}$ is given as $y^2 = x^3 - p(p-4)x$ with different odd primes p and $p - 4$ as $p \equiv 15(mod\ 16)$.

Proof. See [8]. □

Proposition 2.12. The consequence $rank(E_{-2(25u^4+154u^2v^2+242v^4)}(Q))$ =1 is induced where E_{-2p} is an elliptic curve $y^2 = x^3 - 2px$ that satisfies an odd prime p is $p = 25u^4 + 154u^2v^2 + 242v^4$ with two integers u and v and (u,v)=1 and $p \equiv 5(mod\ 16)$.

Proof. See [13]. □

3 The Curve $\boldsymbol{E_{-p}}$

In third section, we shall treat the rank of elliptic curve E_{-p}: $y^2 = x^3 - px$. The solvability of equation 1) for Γ and that of 1) for $\bar{\Gamma}$ will not be treated. For this see [13].

Lemma 3.1. (1). If E_{-p} is appointed as an elliptic curve $y^2 = x^3 - px$ where prime p is $p = 6890u^4 + 402u^2v^2 + 9v^4\ w.i.u.v.1$ and $p \equiv 5(mod\ 16)$ then, there deduced that $rank(E_{-(6890u^4+402u^2v^2+9v^4)}(Q)) = 1$.

(2). Denote E_{-p} as an elliptic curve $y^2 = x^3 - px$ where prime p is the form $p = 82u^4 + 90u^2v^2 + 25v^4\ w.i.u.v.1$ and $p \equiv 5(mod\ 16)$ then, there deduced that $rank(E_{-(82u^4+90u^2v^2+25v^4)}(Q)) = 1$.

(3). We appoint that E_{-p} is an elliptic curve $y^2 = x^3 - px$ that satisfies p is a prime as $p = 200u^4 + 240u^2v^2 + 71v^4\ w.i.u.v.1$ and $p \equiv 15(mod\ 16)$ then, we gain $rank(E_{-(200u^4+240u^2v^2+71v^4)}(Q)) = 1$.

(4). Let E_{-p} be an elliptic curve $y^2 = x^3 - px$ where prime p is given as $p = 6562u^4 + 10u^2v^2 + 25v^4\ w.i.u.v.1$ and $p \equiv 5(mod\ 16)$ then, we attain that $rank(E_{-(6562u^4+10u^2v^2+25v^4)}(Q)) = 1$.

(5). Define E_{-p} as an elliptic curve $y^2 = x^3 - px$ which satisfies that p is a prime gotten as the form $p = 72u^4 + 288u^2v^2 + 287v^4\ w.i.u.v.1$ and $p \equiv 7(mod\ 16)$ then, we conclude that $rank(E_{-(72u^4+288u^2v^2+287v^4)}(Q)) = 1$.

(6). Assign E_{-p} as an elliptic curve $y^2 = x^3 - px$ where p is a prime such that $p = 2u^4 + 52u^2v^2 + 337v^4\ w.i.u.v.1$ and $p \equiv 7(mod\ 16)$ then, we gain $rank(E_{-(2u^4+52u^2v^2+337v^4)}(Q)) = 1$.

(7). Suppose that E_{-p} is an elliptic curve $y^2 = x^3 - px$ where prime p is given as $p = 2u^4 + 20u^2v^2 + 49v^4\ w.i.u.v.1$ and $p \equiv 7(mod\ 16)$ then, there educed that $rank(E_{-(2u^4+20u^2v^2+49v^4)}(Q)) = 1$.

(8). If E_{-p} is defined as an elliptic curve $y^2 = x^3 - px$ where prime p is $p = 2410u^4 + 42u^2v^2 + 49v^4\ w.i.u.v.1$ and $p \equiv 5(mod\ 16)$ then, we obtain $rank(E_{-(2410u^4+42u^2v^2+49v^4)}(Q)) = 1$.

$Proof$. (1). It is sufficient that we only check the solvability of following relating equation

$$2)N^2 = -M^4 + (6890u^4 + 402u^2v^2 + 9v^4)e^4 \text{ for } \Gamma$$

due to [12].

Above all, there is a square $9v^4$ in coefficient of e^4.

Since $402u^2v^2$ is factored as $2 \cdot 201u^2v^2 = 2 \cdot 3 \cdot 67u^2v^2$ we should search the term 67^2u^4.

Thus in $-M^4 + 6890u^4e^4$, taking the two values M and e is significant treatment.

Assume that $e = 1$ then, we obtain that $-M^4 + 6890u^4$.

Because our aim is $4489u^4$, if we do a calculation $6890u^4 - 4489u^4$ then, it is $2401u^4$.

Wherefore, we gain the value M and e as $7u$ and 1.

Moreover, from the numeration

$$-(7u)^4 + 6890u^4 + 402u^2v^2 + 9v^4$$

$$= 6890u^4 - 2401u^4 + 402u^2v^2 + 9v^4$$

$$= 4489u^4 + 402u^2v^2 + 9v^4$$

the value N is gotten as $67u^2 + 3v^2$.

Whence, the triple $(7u, 1, 67u^2 + 3v^2)$ satisfies the solution of equation 2).

Henceforth, we take that $\#\alpha(\Gamma) = 4$.

On this account, there is attained that $rank(E_{-(6890u^4+402u^2v^2+9v^4)}(Q)) = 1$

due to $r4.2$.

(2). If we search the solution of next equation

$$2)N^2 = -M^4 + (82u^4 + 90u^2v^2 + 25v^4)e^4 \text{ for } \Gamma$$

then, it is enough to calculate the rank of curve from [12].

The square $25v^4$ is found in coefficient of e^4.

Because we gain the factorization $90u^2v^2 = 2 \cdot 9u^2 \cdot 5v^2$ there ought to be shown $81u^4$.

In numerical value $-M^4 + 82u^4e^4$, setting u and 1 into M and e then, we are confronted with $81u^4$.

Besides, due to enumeration

$$-u^4 + 82u^4 + 90u^2v^2 + 25v^4$$

$$= 81u^4 + 90u^2v^2 + 25v^4$$

the integer N is induced as $9u^2 + 5v^2$.

Therefore, we gain $(u, 1, 9u^2 + 5v^2)$ as the solution of above equation.

Now it follows that $\#\alpha(\Gamma) = 4$.

Hence, we conclude that $rank(E_{-(82u^4+90u^2v^2+25v^4)}(Q)) = 1$ owing to $r4.2$.

(3). Suppose that E_{-p} is an elliptic curve $y^2 = x^3 - px$ where p is a prime such that $p = 200u^4 + 240u^2v^2 + 71v^4\ w.i.u.v.1$ and $p \equiv 15(mod\ 16)$. Put p as $p = 16k + 15$ with integer k then, there are two relating equations for Γ as follows:

$$1)N^2 = M^4 - (16k + 15)e^4 \text{ and}$$

$$2)N^2 = -M^4 + (16k' + 15)e^4.$$

Cutting down on 2) by 16 implies that $0, 1, 4, 9 \equiv N^2 \equiv 15M^4 + 15e^4 \equiv 15, 14(mod\ 16)$. It is unmatched calculation, thus a contradiction is deduced.

And so there educed that $\#\alpha(\Gamma) = 2$.

Now we have the curve $\overline{E_{-p}}$ as $y^2 = x^3 + 4(16k + 15)x$.

Hence, we attain relating equations for $\bar{\Gamma}$:

$$1)N^2 = M^4 + 4(16k + 15)e^4 \text{ and}$$

$$2)N^2 = 2M^4 + 2(16k + 15)e^4 \text{ and}$$

$$3)N^2 = 4M^4 + (16k + 15)e^4.$$

Relating equation 2) is $N^2 = 2M^4 + 2(200u^4 + 240u^2v^2 + 71v^4)e^4$.

First of all, to find the solution of above equation we must treat the coefficient of

e^4.

Because of existence of square term $400u^4$ we can expect the probability that there will be emerged the square of polynomial whose components are variables u and v.

Now there derived that $2 \cdot 20 \cdot 12u^2v^2$, hence there ought to be appeared $144v^4$.

The next thing to do is treating the numerical value $2M^4 + 142v^4e^4$.

Choose M and e as v and 1 then, we are confronted with

$$2v^4 + 142v^4 = 144v^4.$$

We attained our objective.

Furthermore, by the computation

$$2v^4 + 400u^4 + 480u^2v^2 + 142v^4$$

$$= 400u^4 + 480u^2v^2 + 144v^4$$

there educed the value N as $20u^2 + 12v^2$.

Resultantly, the solution of 2) is gotten as $(v, 1, 20u^2 + 12v^2)$.

No solution exists in relating equation 3) since cutting down on it by 4 gives that $1 \equiv N^2 \equiv 15e^4 \equiv 3(mod\ 4)$.

To conclude, it is obtained that $\#\bar{\alpha}(\bar{\Gamma}) = 4$.

Henceforth, we get that $rank(E_{-(200u^4+240u^2v^2+71v^4)}(Q)) = 1$ from $r2.4$.

(4). Owing to [12], the only equation that is indispensable to consider the solvability is

$$2)N^2 = -M^4 + (6562u^4 + 10u^2v^2 + 25v^4)e^4 \text{ for } \Gamma.$$

There exists a square $25v^4$ in coefficient of e^4.

Thus one condition for being appeared the square of polynomial of u and v is gotten.

Next, from the factorization $10u^2v^2 = 2 \cdot 5u^2v^2$ we must find u^4.

Now it is necessary to consider numerical value $-M^4 + 6562u^4e^4$.

Select e as 1 then, we acquire that $-M^4 + 6562u^4$.

From our claim u^4, there must be hold

$$-M^4 + 6562u^4 = u^4.$$

Hence, we get that $M = 9u$.

In addition, on account of computation

$$-(9u)^4 + 6562u^4 + 10u^2v^2 + 25v^4$$

$$= 6562u^4 - 6561u^4 + 10u^2v^2 + 25v^4$$

$$= u^4 + 10u^2v^2 + 25v^4$$

N is educed as $u^2 + 5v^2$.

For this reason, we have the solution of 2) as $(9u, 1, u^2 + 5v^2)$.

Consequentially, there comes $\#\alpha(\Gamma) = 4$.

Therefore, we gain $rank(E_{-(6562u^4+10u^2v^2+25v^4)}(Q)) = 1$ because we face $r4.2$.

(5). Assume that E_{-p} is an elliptic curve $y^2 = x^3 - px$ where$p$ is a prime $p = 72u^4 + 288u^2v^2 + 287v^4\ w.i.u.v.1$ and $p \equiv 7(mod\ 16)$. Let the prime p be $p = 16k + 7$ with integer k then, we obtain relating equations for Γ:

$$1) N^2 = M^4 - (16k + 7)e^4 \text{ and}$$

$$2) N^2 = -M^4 + (16k' + 7)e^4.$$

Taking a reduction modulo 16 in equation 2) shows that $0, 1, 4, 9 \equiv N^2 \equiv 15M^4 + 7e^4 \equiv 7,6,15(mod\ 16)$ and we acquired unmatched congruence and so it cannot possess a solution.

Accordingly, we face that $\#\alpha(\Gamma) = 2$.

Now we attain the curve $\overline{E_{-p}}$ as $y^2 = x^3 + 4(16k + 7)x$.

Thereby, there comes relating equations for $\bar{\Gamma}$ as follows:

$$1) N^2 = M^4 + 4(16k + 7)e^4 \text{ and}$$

$$2) N^2 = 2M^4 + 2(16k + 7)e^4 \text{ and}$$

$$3) N^2 = 4M^4 + (16k + 7)e^4.$$

Equation 2) is rewritten as $N^2 = 2M^4 + 2(72u^4 + 288u^2v^2 + 287v^4)e^4$.

Square term $144u^4$ exists in coefficient of e^4.

Thus, the possibility for being appeared the polynomial's square that is consisted of variables u and v exists.

From the factorization $576u^2v^2 = 2 \cdot 12 \cdot 24u^2v^2$ there has to be emerged $576v^4$.

Suppose that $M = v$ and $e = 1$ then, we are confronted with

$$2v^4 + 574v^4 = 576v^4.$$

It is the result what we searched for.

Besides, owing to enumeration

$$2v^4 + 144u^4 + 576u^2v^2 + 574v^4$$

$$= 144u^4 + 576u^2v^2 + 576v^4$$

the integer N is produced as $12u^2 + 24v^2$.

Eventually, the triple (v, 1, $12u^2 + 24v^2$) is educed as the solution of equation 2).

After cutting down on 3) by 4 shows unmatched congruence $1 \equiv N^2 \equiv 7e^4 \equiv 3(mod\ 4)$.

Thereby, we obtain the conclusion $\#\bar{\alpha}(\bar{\Gamma}) = 4$.

On this account, the result $rank(E_{-(72u^4+288u^2v^2+287v^4)}(Q)) = 1$ is induced because of $r2.4$.

(6). Because of (5) in the above, there is remained only equation

$$2)N^2 = 2M^4 + 2(2u^4 + 52u^2v^2 + 337v^4)e^4 \text{ for } \bar{\Gamma}$$

that is essential to consider the solvability.

First of all, there is a square term $4u^4$ in coefficient of e^4 and so the condition for being shown the square of polynomial for two variables u and v is given.

The term $104u^2v^2$ is factored as $2 \cdot 2 \cdot 26u^2v^2$, hence there should be appeared $676v^4$.

Now it must be hold that $2M^4 + 674v^4 = 676v^4 \cdots\cdots (H)$.

If M and e are determined as v and 1 then, (H) is satisfied.

Furthermore, because of

$$2v^4 + 4u^4 + 104u^2v^2 + 674v^4$$

$$= 4u^4 + 104u^2v^2 + 676v^4$$

we take the value N as $2u^2 + 26v^2$.

Whence, equation 2) possesses a solution (v, 1, $2u^2 + 26v^2$).

And so we are confronted with $\#\bar{\alpha}(\bar{\Gamma}) = 4$.

Thereby, the result $rank(E_{-(2u^4+52u^2v^2+337v^4)}(Q)) = 1$ is gotten on account of $r2.4$

(7). By (5) in the above, the remanent relating equation is only

$$2)N^2 = 2M^4 + 2(2u^4 + 20u^2v^2 + 49v^4)e^4 \text{ for } \bar{\Gamma}.$$

In $4u^4 + 40u^2v^2 + 98v^4$, there is a term $4u^4$ and so we can expect that there will be emerged the square form after putting some integers into M and e.

In the next step, from the factorization $40u^2v^2 = 2 \cdot 2 \cdot 10u^2v^2$, it should be derived $100v^4$.

Now there is remained $2M^4 + 98v^4e^4$.

From our purpose $100v^4$ the relation $2M^4 + 98v^4e^4 = 100v^4$ must be hold.

We appoint that $M = v$ and $e = 1$ then, we are faced with $100v^4$.

In addition, from

$$2v^4 + 4u^4 + 40u^2v^2 + 98v^4$$

$$= 4u^4 + 40u^2v^2 + 100v^4$$

it is given that $N = 2u^2 + 10v^2$.

Consequentially, the triple $(v,\ 1,\ 2u^2 + 10v^2)$ is induced as the solution of equation 2).

Thus, it is gotten that $\#\bar{\alpha}(\bar{\Gamma}) = 4$.

Hence, the result $rank(E_{-(2u^4+20u^2v^2+49v^4)}(Q)) = 1$ is educed owing to $r2.4$.

(8). Owing to [12], it is sufficient that we only examine the solvability of following equation

$$2) N^2 = -M^4 + (2410u^4 + 42u^2v^2 + 49v^4)e^4 \text{ for } \Gamma.$$

There is found the square $49v^4$ in coefficient of e^4.

Next from $42u^2v^2 = 2 \cdot 3 \cdot 7u^2v^2$ there should be deduced $9u^4$.

It is necessary to consider the arithmetical value $-M^4 + 2410u^4e^4$.

Our aim was $9u^4$, thus it must be valid that $-M^4 + 2410u^4e^4 = 9u^4 \cdots\cdots (K)$.

Because of $2410u^4 - 9u^4$(taking $e = 1$) the integer M is given as $7u$.

Namely, the pair (e, M)=$(1, 7u)$ satisfies (K).

Moreover, from the calculation

$$-(7u)^4 + 2410u^4 + 42u^2v^2 + 49v^4$$

$$= 9u^4 + 42u^2v^2 + 49v^4$$

N is gotten as $3u^2 + 7v^2$.

Consequently, the triple $(7u,\ 1,\ 3u^2 + 7v^2)$ is deduced as the solution of 2).

Resultantly, there comes $\#\alpha(\Gamma) = 4$.

Thus, we gain $rank(E_{-(2410u^4+42u^2v^2+49v^4)}(Q)) = 1$ from $r4.2$. □

Remark 3.2. In elliptic curve, integral points and generators are also important thing. In [23], the authors treated that all integral points on elliptic curve $y^2 = x^3 - 30x + 133$ are $(-7\ ,\ 0)$, $(-3\ ,\ \pm 4\ )$, $(2\ ,\ \pm 9\ )$, $(6\ ,\ \pm 13\ )$, $(5143326\ ,\ \pm 116644498677)$. In [7], the authors considered the generators for elliptic curve $E: y^2 = x^3 - nx$. If a positive, non-square, fourth-power-free integer n is gotten as $n = st$ with positive, non-square integers s and t and there are positive integers α, β and m as $t - s = \alpha^2$ and $m^4s - t = \beta^2$ then, the points $G_1 = (-s, s\alpha)$, $G_2 = (m^2s, ms\beta)$ can always be in a system of generators for $E(Q)$ if $m = 2$ or 3([7]).

4 Form $E_{\mp pq}$

In fourth section, we shall research the ranks of $E_{\mp pq}$: $y^2 = x^3 \mp pqx$. In next proof of theorem, we will not manage the solvability of relating equation 1) for Γ and that of 1) and 5) for $\bar{\Gamma}$(in E_{pq}). Refer [13] for this.

Theorem 4.1. (1). Denote E_{-pq} as an elliptic curve $y^2 = x^3 - pqx$ where p and q are distinct odd primes $p = 4u^4 - 20u^2v^2 + 27v^4$ and $q = 4u^4 - 20u^2v^2 + 23v^4$ $w.i.u.v.1$ and $p \equiv 11(mod\ 16)$ and $q \equiv 7(mod\ 16)$ then, there derived that

$$rank(E_{-(4u^4-20u^2v^2+27v^4)(4u^4-20u^2v^2+23v^4)}(Q))$$

$$= rank(E_{2(64u^8+16u^4v^4+9v^8)}(Q))$$

$$= rank(E_{-p(p-4))}(Q)) + rank(E_{-(6890u^4+402u^2v^2+9v^4)}(Q)).$$

(2). Suppose that p and q are distinct odd primes $p = 50u^4 + 20u^2v^2 + v^4$ and $q = 50u^4 + 20u^2v^2 + 3v^4$ $w.i.u.v.1$, $p \equiv 1(mod\ 8)$ and $q \equiv 3(mod\ 16)$ in elliptic curve E_{pq} then, there induced that

$$rank(E_{(50u^4+20u^2v^2+v^4)(50u^4+20u^2v^2+3v^4)}(Q)) \geq$$

$$rank(E_{-(4u^4+28u^2v^2+51v^4)(4u^4+28u^2v^2+47v^4)}(Q))$$

$$= rank(E_{-2(25u^4+154u^2v^2+242v^4)}(Q))$$

$$+rank(E_{-(82u^4+90u^2v^2+25v^4)}(Q)).$$

(3). Assume that p and q are two distinct odd primes $p = 3200u^4 + 160u^2v^2 + v^4$ and $q = 3200u^4 + 160u^2v^2 + 3v^4$ $w.i.u.v.1$ and $p \equiv 1(mod\ 8)$ and $q \equiv 3(mod\ 16)$ in E_{pq} then, we conclude that

$$= rank(E_{(3200u^4+160u^2v^2+v^4)(3200u^4+160u^2v^2+3v^4)}(Q))$$

$$\geq rank(E_{-(25u^4+20u^2v^2+6v^4)(25u^4+20u^2v^2+2v^4)}(Q))$$

$$= 2rank(E_{-(200u^4+240u^2v^2+71v^4)}(Q)).$$

(4). Take two distinct odd primes p and q as $p = u^4 + 6u^2v^2 + 14v^4$ and $q = u^4 + 8u^2v^2 + 14v^4$ $w.i.u.v.1$, $p \equiv 5(mod\ 16)$ and $q \equiv 7(mod\ 16)$ in

E_{pq} then, we gain

$$rank(E_{(u^4+6u^2v^2+14v^4)(u^4+8u^2v^2+14v^4)}(Q))$$

$$= rank(E_{-(4u^4+20u^2v^2+27v^4)(4u^4+20u^2v^2+23v^4)}(Q))$$

$$-rank(E_{-(6562u^4+10u^2v^2+25v^4)}(Q)).$$

(5). If E_{pq} is defined as an elliptic curve $y^2 = x^3 + pqx$ which satisfies that distinct odd primes p and q are $p = 400u^4 - 6u^2v^2 + 11v^4$ and $q = 400u^4 - 6u^2v^2 + 13v^4$ $w.i.u.v.1$ and $p \equiv 5(mod\ 16)$and $q \equiv 7(mod\ 16)$ then, derived consequence is

$$rank(E_{(400u^4-6u^2v^2+11v^4)(400u^4-6u^2v^2+13v^4)}(Q))$$

$$= rank(E_{-(36u^4+36u^2v^2+11v^4)(36u^4+36u^2v^2+7v^4)}(Q))$$

$$-rank(E_{-(72u^4+288u^2v^2+287v^4)}(Q)).$$

(6). We appoint that E_{pq} is an elliptic curve $y^2 = x^3 + pqx$ where p and q are distinct odd primes $p = 48u^4 - 20u^2v^2 + v^4$ and $q = 48u^4 - 20u^2v^2 + 3v^4$ $w.i.u.v.1$ and $p \equiv 13(mod\ 16)$ and $q \equiv 15(mod\ 16)$, then deduced result is

$$rank(E_{(48u^4-20u^2v^2+v^4)(48u^4-20u^2v^2+3v^4)}(Q))$$

$$= rank(E_{-(-p+q,2u^4+2pq)}(Q))$$

$$-rank(E_{-(2u^4+52u^2v^2+337v^4)}(Q)).$$

(7). We appoint that p and q are distinct odd primes as the forms $p = u^4 + 2u^2v^2 + 26v^4$ and $q = u^4 + 4u^2v^2 + 26v^4$ $w.i.u.v.1$ and $p \equiv 13(mod\ 16)$ and $q \equiv 15(mod\ 16)$ in E_{pq} then, we conclude that

$$rank(E_{(u^4+2u^2v^2+26v^4)(u^4+4u^2v^2+26v^4)}(Q))$$

$$= rank(E_{(u^4+72u^2v^2+10v^4)(u^4+74u^2v^2+10v^4)}(Q))$$

$$= rank(E_{-(2u^4+20u^2v^2+49v^4)}(Q)).$$

(8). Suppose that p and q are distinct odd primes $p = 30u^4 - 10u^2v^2 + 9v^4$ and $q = 30u^4 - 10u^2v^2 + 11v^4$ $w.i.u.v.1$ and $p \equiv 13(mod\ 16)$and $q \equiv$

$15(mod\ 16)$ in E_{pq} then, we get the followings:

$$rank(E_{(30u^4-10u^2v^2+9v^4)(30u^4-10u^2v^2+11v^4)}(Q))$$

$$= rank(E_{(u^4-10u^2v^2+12v^4)(u^4-10u^2v^2+14v^4)}(Q))$$

$$= rank(E_{-(2410u^4+42u^2v^2+49v^4)}(Q)).$$

(9). Denote E_{pq} as an elliptic curve $y^2 = x^3 + pqx$ where different odd primes p and q are such that $p = u^4 + 12u^2v^2 - 40v^4$ and $q = u^4 + 12u^2v^2 - 38v^4\ w.i.u.v.1$ and $p \equiv 5(mod\ 16)$ and $q \equiv 7(mod\ 16)$ then, we get

$$rank(E_{(u^4+12u^2v^2-40v^4)(u^4+12u^2v^2-38v^4)}(Q))$$

$$= rank(E_{-2(30t^2+11)(60t^2+23)}(Q))$$

$$= rank(E_{-p(p-4)}(Q)).$$

(10). If E_{pq} is appointed as an elliptic curve $y^2 = x^3 + pqx$ where different odd primes p and q are $p = 72u^4 - 22u^2v^2 - 21v^4$ and $q = 72u^4 - 22u^2v^2 - 19v^4\ w.i.u.v.1$ and $p \equiv 13(mod\ 16)$ and $q \equiv 15(mod\ 16)$ then, it is derived that

$$rank(E_{(72u^4-22u^2v^2-21v^4)(72u^4-22u^2v^2-19v^4)}(Q))$$

$$= rank(E_{(u^8+12u^4v^4+16v^8)}(Q))$$

$$= rank(E_{-2(25u^4+154u^2v^2+242v^4)}(Q)).$$

Proo$f$. (1). From [15], it is enough that we examine the solvability of equations $3)N^2 = (4u^4 - 20u^2v^2 + 27v^4)M^4 - (4u^4 - 20u^2v^2 + 23v^4)e^4$ for Γ and $5)N^2 = 2(4u^4 - 20u^2v^2 + 27v^4)M^4 + 2(4u^4 - 20u^2v^2 + 23v^4)e^4$ for $\bar{\Gamma}$. In the first place, putting 1 into M and e in 3) educes that $4u^4 - 20u^2v^2 + 27v^4 - 4u^4 + 20u^2v^2 - 23v^4 = 4v^4$ and so the triple (1, 1, $2v^2$) is deduced as the solution of relating equation 3) . On this account, we confront to $\#\alpha(\Gamma) = 4$. Next equation in 5), we gain the common numerical value $8u^4 - 40u^2v^2$ in two coefficients of M^4 and e^4, hence suppose that $16u^4 - 80u^2v^2$ is a part of resultant then, there must be appeared $100v^4$. Owing to the existences of $54v^4$ and $46v^4$ in coefficients of M^4 and e^4 we can obtain the term. And all these are possible from the assumption $M = e = 1$. Furthermore, because of the calculation $8u^4 - 40u^2v^2 + 54v^4 + 8u^4 - 40u^2v^2 + 46v^4$ the value N is derived as $4u^2 - 10v^2$. To sum up, the triple (1, 1, $4u^2 - 10v^2$) is produced as the solution of equation 5). Eventually, the conclusion $\#\bar{\alpha}(\bar{\Gamma}) = 4$ is given. As a result, we acquire that $rank(E_{-(4u^4-20u^2v^2+27v^4)(4u^4-20u^2v^2+23v^4)}(Q)) = 2$ on account of r4.4. In addition, from proposition 2.1 and 2.11 and lemma 3.1(1) the proof is done.

(2). On account of [15], we needed to treat the solvability of two relating equations $2) N^2 = (50u^4 + 20u^2v^2 + v^4)M^4 + (50u^4 + 20u^2v^2 + 3v^4)e^4$ for Γ and $10) N^2 = -2(50u^4 + 20u^2v^2 + v^4)M^4 + 2(50u^4 + 20u^2v^2 + 3v^4)e^4$ for $\bar{\Gamma}$. First equation in 2), there exists common arithmetical value $50u^4 + 20u^2v^2$ and so we assume $100u^4$ as a part of resultant then, there ought to be derived the term $4v^4$ after substituting the values into M and e. The bigness of the integers is meaningless in this point. If M and e are selected as 1 in both cases then, we obtain that $v^4 + 3v^4$. It is matched to our aim in the above. In addition, from $50u^4 + 20u^2v^2 + v^4 + 50u^4 + 20u^2v^2 + 3v^4 = 100u^4 + 40u^2v^2 + 4v^4$ the integer N is gotten as $10u^2 + 2v^2$. Resultantly, the solution of equation 2) is given as (1, 1, $10u^2 + 2v^2$). Henceforth, we gain the conclusion $\#\alpha(\Gamma) = 4$. In next equation 10), putting the value 1 into M and e implies that $-100u^4 - 40u^2v^2 - 2v^4 + 100u^4 + 40u^2v^2 + 6v^4 = 4v^4$. Therefore, we take N as $2v^2$. On this account, the solution of relating equation 10) is produced as (1, 1, $2v^2$). For that reason, we are confronted with $\#\bar{\alpha}(\bar{\Gamma}) \geq 4$. Consequentially, the result $rank(E_{(50u^4+20u^2v^2+v^4)(50u^4+20u^2v^2+3v^4)}(Q)) \geq 2$ is gotten owing to $r^{\geq}$4.4. Moreover, because of proposition 2.2 and 2.12 and lemma 3.1(2) we complete the proof of (2).

(3). It were necessary to inquire the solvability of relating equations $2) N^2 = (3200u^4 + 160u^2v^2 + v^4)M^4 + (3200u^4 + 160u^2v^2 + 3v^4)e^4$ and $10) N^2 = -2(3200u^4 + 160u^2v^2 + v^4)M^4 + 2(3200u^4 + 160u^2v^2 + 3v^4)e^4$ for $\Gamma$and $\bar{\Gamma}$ respectively due to [15]. In the first place, we can find the common part $3200u^4 + 160u^2v^2$ in both coefficients of M^4 and e^4. Hence, we take the supposition that $6400u^4 + 320u^2v^2$ comprises the part of resultant then, there should be educed the term $4v^4$ owing to the factorization $320u^2v^2 = 2 \cdot 80 \cdot 2u^2v^2$. Now there exist the terms v^4 and $3v^4$ in coefficients of M^4 and e^4. On this account, if we select the integers M and e as 1 then, we attain our claim. Besides, from the calculation $3200u^4 + 160u^2v^2 + v^4 + 3200u^4 + 160u^2v^2 + 3v^4 = 6400u^4 + 320u^2v^2 + 4v^4$ there derived the value N as $80u^2 + 2v^2$. As a result, the triple (1, 1, $80u^2 + 2v^2$) is gotten as the solution of relating equation 2) . On that account, we have that $\#\alpha(\Gamma) = 4$. In the second place, replacing 1 into M and e yields that $-6400u^4 - 320u^2v^2 - 2v^4 + 6400u^4 + 320u^2v^2 + 6v^4 = 4v^4$. For that reason, the integer N is given as $2v^2$. Wherefore, equation 10) takes the solution (1, 1, $2v^2$). Consequently, we get that $\#\bar{\alpha}(\bar{\Gamma}) \geq 4$. Therefore, $rank(E_{(3200u^4+160u^2v^2+v^4)(3200u^4+160u^2v^2+3v^4)}(Q)) \geq 2$ is deduced because of $r^{\geq}$4.4. Furthermore, by proposition 2.3 and lemma 3.1(3) we accomplish the proof of (3).

(4). We appoint that E_{pq} is an elliptic curve $y^2 = x^3 + pqx$ that satisfies p and q are distinct odd primes as $p \equiv 5(mod\ 16)$ and $q \equiv 7(mod\ 16)$ and $p = u^4 +$

$6u^2v^2+14v^4$ and $q=u^4+8u^2v^2+14v^4\ w.i.u.v.1$. Let p and q be $p=16k+5$ and $q=16k'+7$ with integers k, k' then, relating equations for Γ are induced as follows:

$$1)N^2=M^4+(16k+5)(16k'+7)e^4 \text{ and}$$

$$2)N^2=(16k+5)M^4+(16k'+7)e^4.$$

Equation 2) cannot have a solution since treating a reduction modulo 16 shows unmatched calculation 0, 1, 4, 9 $\equiv N^2\equiv 5M^4+7e^4\equiv$ 12, 5, 7$(mod\ 16)$.

Therefore, we attain that $\#\alpha(\Gamma)=2.$

Next, there is appeared the curve $\overline{E_{pq}}$ as $y^2=x^3-4(16k+5)(16k'+7)x.$

Accordingly, we have relating equations for $\bar{\Gamma}$:

$$1)N^2=M^4-4(16k+5)(16k'+7)e^4 \text{ and}$$

$$2)N^2=-M^4+4(16k+5)(16k'+7)e^4 \text{ and}$$

$$3)N^2=2M^4-2(16k+5)(16k'+7)e^4 \text{ and}$$

$$4)N^2=-2M^4+2(16k+5)(16k'+7)e^4 \text{ and}$$

$$5)N^2=4M^4-(16k+5)(16k'+7)e^4 \text{ and}$$

$$6)N^2=-4M^4+(16k+5)(16k'+7)e^4 \text{ and}$$

$$7)N^2=(16k+5)M^4-4(16k'+7)e^4 \text{ and}$$

$$8)N^2=-(16k+5)M^4+4(16k'+7)e^4 \text{ and}$$

$$9)N^2=2(16k+5)M^4-2(16k'+7)e^4 \text{ and}$$

$$10)N^2=-2(16k+5)M^4+2(16k'+7)e^4 \text{ and}$$

$$11)N^2=4(16k+5)M^4-(16k'+7)e^4 \text{ and}$$

$$12)N^2=-4(16k+5)M^4+(16k'+7)e^4 \text{ and}$$

We cannot find a solution in equations from 2) to 4) since cutting down on these by 4 and p gives that $2)1\equiv N^2\equiv 3(mod\ 4)$ and $3)N^2\equiv 2M^4(mod\ p)$ and $4)N^2\equiv -2M^4(mod\ p)$ respectively and we obtain unmatched congruence in 2) and other two cases there also educed that $3)\left(\frac{2M^4}{p}\right)=-1$ and $4)\left(\frac{-2M^4}{p}\right)=-1$.

In the next step, reducing 6) by q shows that $N^2 \equiv -4M^4 (mod\ q)$ but we also acquire that $\left(\frac{-4M^4}{q}\right) = -1$ and these two results cannot coexist and so it cannot have a solution.

We appoint that equation 8) has a solution then, there appeared the congruences $N^2 \equiv 4(16k' + 7)e^4 (mod\ p)$ and $N^2 \equiv -(16k + 5)M^4 (mod\ q)$ from reducing it by p and q, hence there ought to be emerged that

$$1 = \left(\frac{4(16k\prime+7)e^4}{p}\right) = \left(\frac{q}{p}\right) \text{ and}$$

$$1 = \left(\frac{-(16k+5)e^4}{q}\right) = -\left(\frac{p}{q}\right).$$

LDV is given in the above but there is gotten LSV between p and q. Whence, a contradiction is deduced and so no solution exists in 8).

Suppose that 9) possesses a solution then, the congruences $N^2 \equiv -2(16k' + 7)e^4 (mod\ p)$ and $N^2 \equiv 2(16k + 5)M^4 (mod\ q)$ are derived from cutting down on it by p and q, thus we must obtain that

$$1 = \left(\frac{-2(16k\prime+7)e^4}{p}\right) = -\left(\frac{q}{p}\right) \text{ and}$$

$$1 = \left(\frac{2(16k+5)e^4}{q}\right) = \left(\frac{p}{q}\right).$$

In the above, LDV is produced but we obtain LSV between p and q. Accordingly, we attain a contradiction. Thereby no solution exists in this equation.

Replacing 1 into M and e in equation 10) yields that

$$-2(u^4 + 6u^2v^2 + 14v^4) + 2(u^4 + 8u^2v^2 + 14v^4)$$

$$= -12u^2v^2 + 16u^2v^2 = 4u^2v^2.$$

Therefore, the triple (1, 1, $2uv$) is educed as the solution of 10).

Let equation 12) possess a solution then, there should be induced the congruences $N^2 \equiv (16k' + 7)e^4 (mod\ p)$ and $N^2 \equiv -4(16k + 5)M^4 (mod\ q)$ from reduction of it by p and q. Wherefore, we ought to take that

$$1 = \left(\frac{(16k\prime+7)e^4}{p}\right) = \left(\frac{q}{p}\right) \text{ and}$$

$$1 = \left(\frac{-4(16k+5)e^4}{q}\right) = -\left(\frac{p}{q}\right).$$

We acquire LDV in the above but it is given LSV between p and q. Thus, we get a contradiction. And so 12) cannot take a solution.

If an equation 7) takes a solution then, we confront to $(16k+5)\cdot\big(-2(16k+5)\big) \equiv -2 \in \bar{\alpha}(\bar{\Gamma})(mod\ Q^{\times 2})$ but as we treated in the above equation $4)N^2 = -2M^4 + 2(16k+5)(16k'+7)e^4$ cannot take a solution and so a contradiction is deduced. Therefore, equation 7) cannot possess a solution. And this is also implied to the equation 11).

Accordingly, we attain the conclusion $\#\bar{\alpha}(\bar{\Gamma}) = 4$.

In its turn, $rank(E_{(u^4+6u^2v^2+14v^4)(u^4+8u^2v^2+14v^4)}(Q)) = 1$ is gotten because of $r2.4$.

In addition, owing to proposition 2.4, lemma 3.1(4) we finish the proof of (4) .

(5). Due to (4) in the above, it is enough that we only look into the solvability of following relating equation

$$10)N^2 = -2(400u^4 - 6u^2v^2 + 11v^4)M^4 + 2(400u^4 - 6u^2v^2 + 13v^4)e^4$$

for $\bar{\Gamma}$.

Replace 1 into both M and e gives that $-22v^4 + 26v^4 = 4v^4$.

Accordingly, we acquire the solution of above equation as (1, 1, $2v^2$).

As a result, we obtain that $\#\bar{\alpha}(\bar{\Gamma}) = 4$.

Consequentially, there derived that $r2.4$.

Whence, the result $rank(E_{(400u^4-6u^2v^2+11v^4)(400u^4-6u^2v^2+13v^4)}(Q)) = 1$ is given.

Besides, from proposition 2.5 and lemma 3.1(5) the proof of (5) is done.

(6). Take E_{pq} as an elliptic curve $y^2 = x^3 + pqx$ where p and q are distinct odd primes $p = 48u^4 - 20u^2v^2 + v^4$ and $q = 48u^4 - 20u^2v^2 + 3v^4\ w.i.u.v.1$ and $p \equiv 13(mod\ 16)$ and $q \equiv 15(mod\ 16)$. We appoint that $p = 16k + 13$ and $q = 16k' + 15$ with integers k, k' then, there exist relating equations for Γ:

$$1)N^2 = M^4 + (16k+13)(16k'+15)e^4 \text{ and}$$

$$2)N^2 = (16k+13)M^4 + (16k'+15)e^4.$$

It is impossible that 2) has a solution because doing a reduction modulo 16 in it derived that $0,\ 1,\ 4,\ 9 \equiv N^2 \equiv 13M^4 + 15e^4 \equiv 15,\ 12,\ 13(mod\ 16)$ and the sides are unmatched.

Therefore, we attain that $\#\alpha(\Gamma) = 2$.

Now, the curve $\overline{E_{pq}}$ is educed as $y^2 = x^3 - 4(16k+13)(16k'+15)x$.

Thereby, we get the relating equations for $\bar{\Gamma}$ as follows:

$$1)N^2 = M^4 - 4(16k+13)(16k'+15)e^4 \text{ and}$$

$$2)N^2 = -M^4 + 4(16k+13)(16k'+15)e^4 \text{ and}$$

$$3) N^2=2M^4-2(16k+13)(16k'+15)e^4 \text{ and}$$

$$4) N^2=-2M^4+2(16k+13)(16k'+15)e^4 \text{ and}$$

$$5) N^2=4M^4-(16k+13)(16k'+15)e^4 \text{ and}$$

$$6) N^2=-4M^4+(16k+13)(16k'+15)e^4 \text{ and}$$

$$7) N^2=(16k+13)M^4-4(16k'+15)e^4 \text{ and}$$

$$8) N^2=-(16k+13)M^4+4(16k'+15)e^4 \text{ and}$$

$$9) N^2=2(16k+13)M^4-2(16k'+15)e^4 \text{ and}$$

$$10) N^2=-2(16k+13)M^4+2(16k'+15)e^4 \text{ and}$$

$$11) N^2=4(16k+13)M^4-(16k'+15)e^4 \text{ and}$$

$$12) N^2=-4(16k+13)M^4+(16k'+15)e^4.$$

There is gotten the relation $1\equiv N^2\equiv -M^4+4e^4\equiv 7,3(mod\ 8)$ after taking a reduction modulo 8 in 2) and it is unmatched numeration and so there cannot exist a solution in this equation

None of equations 3) and 4) has a solution since cutting down on it by p educes that 3)$N^2\equiv 2M^4(mod\ p)$ and 4)$N^2\equiv -2M^4(mod\ p)$ but there also gotten 3) $\left(\frac{2M^4}{p}\right)=-1$ and 4) $\left(\frac{-2M^4}{p}\right)=-1$ and doing a pair in each case derives a contradiction.

Taking a reduction modulo 4 in equation 6) leads to $1\equiv N^2\equiv 15e^4\equiv 3(mod\ 4)$ and both LHS and RHS are unmatched, hence no solution exists in this equation.

If equation 8) takes a solution then, we are confronted with $N^2\equiv -(16k+13)e^4(mod\ q)$ and $N^2\equiv 4(16k'+15)M^4(mod\ p)$ because of reducing of it by q and p, hence there must be derived that

$$1=\left(\frac{-(16k+13)e^4}{q}\right)=-\left(\frac{p}{q}\right)$$

and

$$1=\left(\frac{4(16k\prime+15)M^4}{p}\right)=\left(\frac{q}{p}\right).$$

LDV is shown in the above but we are faced with LSV between p and q. Whence, we have a contradiction, thus equation 8) cannot take a solution.

Assume that relating equation 9) has a solution then, there comes $N^2 \equiv 2(16k+13)e^4 (mod\ q)$ and $N^2 \equiv -2(16k'+15)M^4 (mod\ p)$ on account of reduction of it by q and p, therefore we ought to face that

$$1 = \left(\frac{2(16k+13)e^4}{q}\right) = \left(\frac{p}{q}\right) \text{ and}$$

$$1 = \left(\frac{-2(16k\prime+15)M^4}{p}\right) = -\left(\frac{q}{p}\right).$$

In the above, LDV is derived but there is gotten LSV between p and q. Thus, a contradiction is appeared, thus no solution exists in this equation.

Next, setting 1 into M and e in equation 10) gives that

$$-2(48u^4 - 20u^2v^2 + v^4) + 2(48u^4 - 20u^2v^2 + 3v^4)$$

$$= -2v^4 + 6v^4 = 4v^4 .$$

Accordingly, the solution of 10) is given as (1, 1, $2v^2$).

Assign equation 12) possesses a solution then, there comes that $N^2 \equiv -4(16k+13)e^4 (mod\ q)$ and $N^2 \equiv (16k'+15)M^4 (mod\ p)$ due to cutting down on it by q and p. Henceforth, we ought to get that

$$1 = \left(\frac{-4(16k+13)e^4}{q}\right) = -\left(\frac{p}{q}\right) \text{ and}$$

$$1 = \left(\frac{(16k\prime+15)M^4}{p}\right) = \left(\frac{q}{p}\right).$$

We gain LDV in the above but there comes LSV between p and q. And so we take a contradiction. Accordingly, equation 12) has no solution.

Now we appoint that there is a solution in 7) then, we are confronted with $(16k+13) \cdot \big(-2(16k+13)\big) \equiv -2 \in \bar{\alpha}(\bar{\Gamma})(mod\ Q^{\times 2})$ but it is impossible that equation 4)$N^2 = -2M^4 + 2(16k+13)(16k'+15)e^4$ has a solution and so there is emerged a contradiction. Thus, 7) cannot take a solution. And it is similar to the equation 11).

For that reason, we gain $\#\bar{\alpha}(\bar{\Gamma}) = 4$.

Resultantly, the result $rank(E_{(48u^4-20u^2v^2+v^4)(48u^4-20u^2v^2+3v^4)}(Q)) = 1$ is produced due to $r2.4$

Furthermore, from proposition 2.6 and lemma 3.1(6) we accomplish the proof of (6).

(7). On account of (6) in the above, the remaining equation which is needed to take into account the solvability is

$$10)N^2 = -2(u^4 + 2u^2v^2 + 26v^4)M^4 + 2(u^4 + 4u^2v^2 + 26v^4)e^4$$

for $\bar{\Gamma}$.

Suppose that M and e are 1 then, we acquire that $-4u^2v^2 + 8u^2v^2 = 4u^2v^2$.

Whence, the triple (1, 1, $2uv$) is given as the solution of 10).

Resultantly, there comes that $\#\bar{\alpha}(\bar{\Gamma}) = 4$.

Eventually, we say that $rank(E_{(u^4+2u^2v^2+26v^4)(u^4+4u^2v^2+26v^4)}(Q)) = 1$ from $r2.4$.

In addition, due to proposition 2.7 and lemma 3.1(7) we finish the proof of (7).

(8). Owing to (6) in the above, the only remanent equation that is necessary to inquire the solvability is

$$10)N^2 = -2(30u^4 - 10u^2v^2 + 9v^4)M^4 + 2(30u^4 - 10u^2v^2 + 11v^4)e^4$$

for $\bar{\Gamma}$.

Choose M and e as 1 then, we acquire that $-18v^4 + 22v^4 = 4v^4$.

Thus, the triple (1, 1, $2v^2$) is induced as the solution of above equation.

Thereby, we attain that $\#\bar{\alpha}(\bar{\Gamma}) = 4$.

Hence, we have $rank(E_{(30u^4-10u^2v^2+9v^4)(30u^4-10u^2v^2+11v^4)}(Q)) = 1$ because of $r2.4$.

Moreover, by proposition 2.8 and lemma 3.1(8) we accomplish the proof of (8).

(9). By (4) in the above, there is left only following relating equation

$$10)N^2 = -2(u^4 + 12u^2v^2 - 40v^4)M^4 + 2(u^4 + 12u^2v^2 - 38v^4)e^4$$

for $\bar{\Gamma}$ which is indispensable to probe the solvability.

Assume that $M = e = 1$ then, we get that $80v^4 - 76v^4 = 4v^4$.

Thus, we obtain the solution of above equation as (1, 1, $2v^2$)

And hence, it is deduced that $\#\bar{\alpha}(\bar{\Gamma}) = 4$.

And so there comes $rank(E_{(u^4+12u^2v^2-40v^4)(u^4+12u^2v^2-38v^4)}(Q)) = 1$ since we gain $r2.4$.

In addition, from proposition 2.9 and 2.11 the proof is completed.

(10). The only equation which requires to investigate the solvability is

$$10)N^2 = -2(72u^4 - 22u^2v^2 - 21v^4)M^4 + 2(72u^4 - 22u^2v^2 - 19v^4)e^4$$

for $\bar{\Gamma}$ by (6).

Take M and e as 1 then, we obtain the calculation $42v^4 - 38v^4 = 4v^4$.

Therefore, the triple (1, 1, $2v^2$) satisfies the solution of above equation.

For that reason, we are confronted with $\#\bar{\alpha}(\bar{\Gamma}) = 4$.

Now the consequence $rank(E_{(72u^4-22u^2v^2-21v^4)(72u^4-22u^2v^2-19v^4)}(Q)) = 1$ is deduced because we gain $r2.4$.

The proof is completed by proposition 2.10 and 2.12 □

Remark 4.2. In above consequences (2) and (3), the ranks were both at least 2. The maximal rank in E_{pq} is 4. But the form of prime q is $q \equiv 3(\text{mod } 16)$ thus it cannot became 4. The relating equations that have a probability which can take a solution are $4) N^2 = -2M^4 + 2(8k+1)(16k'+3)e^4$ and $7) N^2 = (8k+1)M^4 - 4(16k'+3)e^4$ and $11) N^2 = 4(8k+1)M^4 - (16k'+3)e^4$ for $\bar{\Gamma}$([15]). The values $\bar{\alpha}(P)$ in 7) and 11) in $Q^\times/Q^{\times 2}$ are the same thus if only one equation possesses a solution then, rank became 3. Similarly if there is a solution in equation 4) then, due to algebraic structure the rank also became 3. Hence, in uncertain circumstance, what we can say certain thing is rank is at least 2. It is common result in treating rank of elliptic curve.

Remark 4.3. In elliptic curve E_{-2pq}, the maximal rank is 5. In proposition 2.9, the systematized rank is 1. Searching generalized rank in this form is not simple thing. Even if we pursue the rank of 2 or 3 it is more difficult than the cases of $E_{\pm pq}$. It is because in this curve, the number of relating equations which should be considered the solvability increased.

Remark 4.4. In [10], the author considered the rank of $E_{-2pqs}(p, q, s$ are different odd primes). Under the hypothesis that $p \equiv 5(mod\ 16)$ and $q \equiv 11(mod\ 16)$ and $s \equiv 15(mod\ 16)$ the rank is at most 1. There also found at most rank 1 in this curve E_{-2pqs}. Searching the result of systematized rank 2 or other value is difficult in E_{-2pqs} since there increased the number of relating equations which ought to be treated the solvability.

Remark 4.5. In [14], the author computed the rank of elliptic curve E_{-5p}: $y^2 = x^3 - 5px$ where prime p is $p = 2u^4 + 8u^2v^2 + 3v^4$ with integers u, v and (u, v)=1 and $p \equiv 3(mod\ 16)$ and the case $p = 18u^4 + 72u^2v^2 + 67v^4$ with integers u and v and (u, v)=1 and $p \equiv 3(mod\ 16)$. Both cases, the ranks were 1.

Remark 4.6. In proposition 2.10, the rank was 1. If 2-primary component of III(E_p/Q)(E_p: $y^2 = x^3 + px$) is finite then, ranks of elliptic curves E_p with $p \equiv 3$, 5, 13, 15$(mod\ 16)$ is 1([18]). But without this assumption, it is conjectured but not certainly 1.

Remark 4.7. In [3], the authors showed that if $\epsilon > 0$ then, there exist at most finitely many integers A and B such that $A > |B|^{1+\epsilon} > 0$ for which $E_{A,B}(Q)_{Tors}(E_{A,B}$: $y^2 = x^3 + Ax + B)$ is nontrivial and not isomorphic to $Z/3Z$.

Remark 4.8. Over Q elliptic curve $E: y^2 = x^3 + Ax + B$ has the structure $E(Q) \cong E(Q)_{tors} \oplus Z^r$. But if the background is changed into finite field F_q then, it needs to notice $E(F_q) \simeq Z_n$ or $Z_{n_1} \oplus Z_{n_2}$for some integer $n \geq 1$, or for some integers n_1, $n_2 \geq 1$ with n_1 dividing n_2 ([19, Theorem 4.1]). Of course, we should notice to Hasse principle $| \, q + 1 - \#E(F_q)| \leq 2\sqrt{q}$. In [2], the authors showed that if E/F_q is an elliptic curve where $q = r^2$ such that $r \equiv 1(mod\ 4)$ when q is a square then, E is Legendre isogenous if and only if $|E(F_q)| \in$ 4Z\{$(r+1)^2$}.

5 In $\boldsymbol{E_{-2p}}$

In section 5, we will compare the ranks in $E_{-2p}: y^2 = x^3 - 2px$(p is a prime) with that of curves $E_{\mp pq}$.

Corollary 5.1. (1). Denote prime p as $p = 3411u^4 + 2508u^2v^2 + 484v^4$ $w.i.u.v.1$ and $p \equiv 3(mod\ 16)$ in E_{-2p} then, we conclude that

$$rank(E_{-(4u^4-20u^2v^2+27v^4)(4u^4-20u^2v^2+23v^4)}(Q)) >$$
$$rank(E_{-2(3411u^4+2508u^2v^2+484v^4)}(Q)).$$

(2). Take prime p as $p = 5000u^4 + 100u^2v^2 + v^4$ $w.i.u.v.1$ and $p \equiv 13(mod\ 16)$ in E_{-2p} then, we say that

$$rank(E_{(50u^4+20u^2v^2+v^4)(50u^4+20u^2v^2+3v^4)}(Q)) >$$
$$rank(E_{-2(5000u^4+100u^2v^2+v^4)}(Q)).$$

(3). Let p be the prime gotten as $p = 1843u^4 + 656u^2v^2 + 64v^4$ $w.i.u.v.1$ and $p \equiv 3(mod\ 16)$ in E_{-2p} then, we attain that

$$rank(E_{(3200u^4+160u^2v^2+v^4)(3200u^4+160u^2v^2+3v^4)}(Q)) >$$
$$rank(E_{-2(1843u^4+656u^2v^2+64v^4)}(Q)).$$

(4). If p is given as $p = 2563u^4 + 980u^2v^2 + 100v^4$ $w.i.u.v.1$ and $p \equiv 11(mod\ 16)$ in E_{-2p} then, we get

$$rank(E_{(u^4+6u^2v^2+14v^4)(u^4+8u^2v^2+14v^4)}(Q)) =$$

$$rank(E_{-2(2563u^4+980u^2v^2+100v^4)}(Q)).$$

(5). Define prime p as $p = 1762u^4 + 880u^2v^2 + 121v^4$ $w.i.u.v.1$ and $p \equiv 11(mod\ 16)$ in E_{-2p} then, there comes that

$$rank(E_{(400u^4-6u^2v^2+11v^4)(400u^4-6u^2v^2+13v^4)}(Q)) =$$

$$rank(E_{-2(1762u^4+880u^2v^2+121v^4)}(Q)).$$

(6). Assign p as the form $p = 1251u^4 + 132u^2v^2 + 4v^4$ $w.i.u.v.1$ and $p \equiv 11(mod\ 16)$ in E_{-2p} then, there comes

$$rank(E_{(48u^4-20u^2v^2+v^4)(48u^4-20u^2v^2+3v^4)}(Q)) =$$

$$rank(E_{-2(1251u^4+132u^2v^2+4v^4)}(Q)).$$

(7). Denote p as the prime $p = 1026u^4 + 448u^2v^2 + 49v^4$ $w.i.u.v.1$ and $p \equiv 3(mod\ 16)$ in E_{-2p} then, there induced the conclusion

$$rank(E_{(u^4+2u^2v^2+26v^4)(u^4+4u^2v^2+26v^4)}(Q)) =$$

$$rank(E_{-2(1026u^4+448u^2v^2+49v^4)}(Q)).$$

(8). Set p as the prime $p = 627u^4 + 400u^2v^2 + 64v^4$ $w.i.u.v.1$ and $p \equiv 3(mod\ 16)$ in E_{-2p} then, we attain

$$rank(E_{(30u^4-10u^2v^2+9v^4)(30u^4-10u^2v^2+11v^4)}(Q)) =$$

$$rank(E_{-2(627u^4+400u^2v^2+64v^4)}(Q)).$$

(9). Suppose that p is $p = 576u^4 + 144u^2v^2 + 11v^4$ $w.i.u.v.1$ and $p \equiv 11(mod\ 16)$ in E_{-2p} then, it comes that

$$rank(E_{(u^4+12u^2v^2-40v^4)(u^4+12u^2v^2-38v^4)}(Q)) =$$

$$rank(E_{-2(576u^4+144u^2v^2+11v^4)}(Q)).$$

(10). We appoint that p is $p = 6171u^4 + 592u^2v^2 + 64v^4$ $w.i.u.v.1$ and

$p \equiv 11(mod\ 16)$ in E_{-2p} then, the result

$$rank(E_{(72u^4-22u^2v^2-21v^4)(72u^4-22u^2v^2-19v^4)}(Q)) =$$

$$rank(E_{-2(6171u^4+592u^2v^2+64v^4)}(Q))$$

is induced.

Proof. (1). There is remained only equation

$$4)N^2 = -2M^4 + (3411u^4 + 2508u^2v^2 + 484v^4)e^4 \text{ for } \Gamma$$

that is needed to take into account the solvability from [9].

The square $484v^4$ is in coefficient of e^4 and we get $2508u^2v^2 = 2 \cdot 2 \cdot 11 \cdot 57u^2v^2$.

And so there ought to be emerged $3249u^4$.

Now there is remained $-2M^4 + 3411u^4e^4$.

The claim is $3249u^4$, thus if we do a calculation $3411u^4 - 3249u^4$ then, we obtain that $162u^4$.

This is also factored as $2 \cdot 81u^4$, hence we conclude that $M = 3u$.

To sum up, if we substitute $3u$ and 1 into M and e then, educed numeration is

$$-2(3u)^4 + 3411u^4 + 2508u^2v^2 + 484v^4$$

$$= 3249u^4 + 2508u^2v^2 + 484v^4.$$

Thus, we gain $N = 57u^2 + 22v^2$.

Consequently, we acquire the solution of 4) as $(3u, 1, 57u^2 + 22v^2)$.

Henceforth, it is derived that $\#\alpha(\Gamma) = 4$.

To conclude, the result $rank(E_{-2(3411u^4+2508u^2v^2+484v^4)}(Q)) = 1$ is gotten from $r4.2$.

And by theorem 4.1(1) we complete the proof of (1).

(2). By [13], the left equation which should be inquired the solvability is

$$2)N^2 = -M^4 + 2(5000u^4 + 100u^2v^2 + v^4)e^4 \text{ for } \Gamma.$$

Due to existence of square $10000u^4$ in coefficient of e^4 a potentiality that for being shown the square of variables u, v exists.

From the factorization $200u^2v^2 = 2 \cdot 100u^2v^2$, there has to be shown v^4.

Next, we must consider $-M^4 + 2v^4e^4$.

Select e as 1 then, we attain that $-M^4 + 2v^4$.

Hence, the other value M is educed as v.

Furthermore, from the enumeration

$$-v^4 + 10000u^4 + 200u^2v^2 + 2v^4$$

$$= 10000u^4 + 200u^2v^2 + v^4$$

we get $N = 100u^2 + v^2$.

And so we obtain the solution of 2) as $(v, 1\ , 100u^2 + v^2)$.

At last the conclusion $\#\alpha(\Gamma) = 4$ is given.

Finally, $rank(E_{-2(5000u^4+100u^2v^2+v^4)}(Q)) = 1$ is induced since we take $r4.2$.

In addition, from theorem 4.1(2) we finish the proof of (2).

(3). From [9] the only equation which is essential to probe the solvability is

$$4) N^2 = -2M^4 + (1843u^4 + 656u^2v^2 + 64v^4)e^4 \text{ for } \Gamma.$$

We face the square term $64v^4$ in coefficient of e^4.

Next, from

$$656u^2v^2 = 2 \cdot 41 \cdot 8u^2v^2$$

there should be emerged $1681u^4$.

Now the remanent thing is $-2M^4 + 1843u^4e^4$.

Let $e = 1$ then, we acquire that $-2M^4 + 1843u^4$.

Taking a calculation $1843u^4 - 1681u^4$ then, it is $162u^4$.

Wherefore, there comes $M = 3u$.

Furthermore, from

$$-2(3u)^4 + 1843u^4 + 656u^2v^2 + 64v^4$$

$$= 1681u^4 + 656u^2v^2 + 64v^4$$

the integer N is produced as $41u^2 + 8v^2$.

In sum, the triple $(3u, 1, 41u^2 + 8v^2)$ is educed as the solution of relating equation 4).

Whence, it is gotten $\#\alpha(\Gamma) = 4$.

In its turn, the result $rank(E_{-2(1843u^4+656u^2v^2+64v^4)}(Q)) = 1$ is educed owing to $r4.2$.

Furthermore, by theorem 4.1(3) we finish the proof of (3).

(4). We only needed to treat the solvability of

$$4) N^2 = -2M^4 + (2563u^4 + 980u^2v^2 + 100v^4)e^4 \text{ for } \Gamma$$

from [9].

It is found the term $100v^4$ in coefficient of e^4.

We confront to the factorization $980u^2v^2 = 2 \cdot 49 \cdot 10u^2v^2$, hence there must

be appeared $2401u^4$. Next, we must consider $-2M^4 + 2563u^4e^4$.
Because of our objective in the above the equality

$$2401u^4 = -2M^4 + 2563u^4e^4$$

must be given.
If e is gotten as 1 then, it is educed that $-2M^4 + 2563u^4 = 2401u^4$.
And so there induced that $M = 3u$.
Furthermore, because of

$$-2(3u)^4 + 2563u^4 + 980u^2v^2 + 100v^4$$

$$= 2563u^4 - 162u^4 + 980u^2v^2 + 100v^4$$

the value N is induced as $49u^2 + 10v^2$.
So we get the triple ($3u$, 1 , $49u^2 + 10v^2$) as the solution of 4).
Henceforth, we face $\#\alpha(\Gamma) = 4$.
Resultantly, it is gotten that $rank(E_{-2(2563u^4+980u^2v^2+100v^4)}(Q)) = 1$ since we get $r4.2$.
Now by theorem 4.1(4) the proof of (4) is done.
(5). It is sufficient that we only look into the solvability of

$$4) N^2 = -2M^4 + (1762u^4 + 880u^2v^2 + 121v^4)e^4 \text{ for } \Gamma$$

because of [9].
First of all, the term $121v^4$ is in coefficient of e^4.
And there educed the factorization

$$880u^2v^2 = 2 \cdot 40 \cdot 11u^2v^2.$$

Accordingly, the term $1600u^4$ should be emerged.
In the next step, we should notice the arithmetical value

$$-2M^4 + 1762u^4e^4.$$

Since our claim was $1600u^4$ the relation

$$1600u^4 = -2M^4 + 1762u^4e^4$$

has to be hold.
After putting $e = 1$ there induced $-2M^4 + 1762u^4 = 1600u^4$.
Thereby, we say that $M = 3u$.
Whence, due to calculation

$$-2(3u)^4 + 1762u^4 + 880u^2v^2 + 121v^4$$

$$= 1762u^4 - 162u^4 + 880u^2v^2 + 121v^4$$

it is gotten that $N = 40u^2 + 11v^2$.

On that account, the solution of 4) is produced as $(3u, 1\ , 40u^2 + 11v^2)$.

Whence, there derived $\#\alpha(\Gamma) = 4$.

Therefore, it is given the result $rank(E_{-2(1762u^4+880u^2v^2+121v^4)}(Q)) = 1$ from $r4.2$.

Next, from theorem 4.1(5) the proof of (5) is done.

(6). There is remained only equation

$$4) N^2 = -2M^4 + (1251u^4 + 132u^2v^2 + 4v^4)e^4 \text{ for } \Gamma$$

that is necessary to consider the solvability from [9].

Above all, there is a term $4v^4$ in coefficient of e^4 and it is deduced that

$$132u^2v^2 = 2 \cdot 33 \cdot 2u^2v^2.$$

Thereby, it should be appeared $1089u^4$ after selecting the integers M and e.

Now there is left numerical value $-2M^4 + 1251u^4e^4$.

Because of our purpose $1089u^4$ it must be given

$$1089u^4 = -2M^4 + 1251u^4e^4\ .$$

Suppose that $e = 1$ then, we get

$$-2M^4 + 1251u^4 = 1089u^4.$$

Accordingly, we attain M as $3u$.

Now, from

$$-2(3u)^4 + 1251u^4 + 132u^2v^2 + 4v^4$$

there is induced that $N = 33u^2 + 2v^2$.

For this reason, the triple $(3u, 1, 33u^2 + 2v^2)$ satisfies the solution of equation 4).

Consequentially, the conclusion $\#\alpha(\Gamma) = 4$ is educed.

Eventually, we take the result $rank(E_{-2(1251u^4+132u^2v^2+4v^4)}(Q)) = 1$ because there educed $r4.2$.

Finally, due to theorem 4.1(6) the proof is finished.

(7). On account of [9], it is enough that we only research the solvability of following equation

$$4)N^2 = -2M^4 + (1026u^4 + 448u^2v^2 + 49v^4)e^4 \text{ for } \Gamma.$$

The term $49v^4$ exists and on account of factorization $448u^2v^2 = 2 \cdot 7 \cdot 32u^2v^2$ it must be emerged the term $1024u^4$.

In numerical value $-2M^4 + 1026u^4e^4$ taking $e = 1$ then, we obtain that $-2M^4 + 1026u^4$.

And this ought to be $1024u^4$ and thus the integer M is gotten as u.

Besides, because of

$$-2u^4 + 1026u^4 + 448u^2v^2 + 49v^4$$

we attain that $N = 32u^2 + 7v^2$.

And so there produced the triple $(u, 1 , 32u^2 + 7v^2)$ as the solution of 4).

Whence, there comes that $\#\alpha(\Gamma) = 4$.

Resultantly, we conclude that $rank(E_{-2(1026u^4+448u^2v^2+49v^4)}(Q)) = 1$ since there derived 4.2.

Finally, due to theorem 4.1(7) the proof is finished.

(8). The only equation that has to be researched the solvability is

$$4)N^2 = -2M^4 + (627u^4 + 400u^2v^2 + 64v^4)e^4 \text{ for } \Gamma \text{ from [9].}$$

There exists a square $64v^4$ in coefficient of e^4 and so there is a probability that after setting some integers into M and e the polynomial's square will be emerged.

Now, we gain $400u^2v^2 = 2 \cdot 8 \cdot 25u^2v^2$.

Wherefore, there should be derived the term $625u^4$.

Now it needs to notice $-2M^4 + 627u^4e^4$.

Here, we take $e = 1$ then, we confront to $-2M^4 + 627u^4$.

And so we acquire that $M = u$.

In addition, due to numeration

$$-2u^4 + 627u^4 + 400u^2v^2 + 64v^4$$

$$= 625u^4 + 400u^2v^2 + 64v^4$$

it is gotten that $N = 25u^2 + 8v^2$.

Thereby, the solution of equation 4) is deduced as $(u, 1, 25u^2 + 8v^2)$.

On this account, we say that $\#\alpha(\Gamma) = 4$.

To conclude, it is given that $rank(E_{-2(627u^4+400u^2v^2+64v^4)}(Q)) = 1$ because we have $r4.2$.

At last, we accomplish the proof from theorem 4.1(8).

(9). There is only remained relating equation

$$4)N^2 = -2M^4 + (576u^4 + 144u^2v^2 + 11v^4)e^4 \text{ for } \Gamma$$

that requires to examine the solvability due to [9].

The term $576u^4$ is in coefficient of e^4, thus we can expect an appearance of polynomial's square.

Because of $144u^2v^2 = 2 \cdot 24 \cdot 3u^2v^2$ there must be emerged $9v^4$.

Next, it is remained $-2M^4 + 11v^4e^4$.

Owing to our objective $9v^4$ if we take $e = 1$ then, the relation $-2M^4 + 11v^4 = 9v^4$ is derived.

Thus, the value M is gotten as v.

Moreover, from the computation $-2v^4 + 576u^4 + 144u^2v^2 + 11v^4$ there comes that $N = 24u^2 + 3v^2$.

Accordingly, the triple (v, 1, $24u^2 + 3v^2$) satisfies the solution of 4).

For this reason, we conclude that $\#\alpha(\Gamma) = 4$.

Consequentially, there comes the result $rank(E_{-2(576u^4+144u^2v^2+11v^4)}(Q)) = 1$ on account of $r4.2$.

Whence, we finish the proof by theorem 4.1(9).

(10). There is remained only equation

$$4)N^2 = -2M^4 + (6171u^4 + 592u^2v^2 + 64v^4)e^4 \text{ for } \Gamma$$

which is necessary to probe the solvability owing to [9].

It is found $64v^4$ in coefficient of e^4, hence a potentiality for being shown the square form exists.

On account of expression $592u^2v^2 = 2 \cdot 37 \cdot 8u^2v^2$ there has to be shown the term $1369u^4$.

In the next step, we must consider $-2M^4 + 6171u^4e^4$.

If $e = 1$ then, we get the relation $-2M^4 + 6171u^4 = 1369u^4$ and thus we are confronted with $6171u^4 - 1369u^4 = 4802u^4$.

Accordingly, the integer M is deduced as $7u$.

Furthermore, by the computation $-2(7u)^4 + 6171u^4 + 592u^2v^2 + 64v^4$ we take N as $37u^2 + 8v^2$.

As a result, there is educed ($7u$, 1 , $37u^2 + 8v^2$) as a solution of equation 4) and it follows that $\#\alpha(\Gamma) = 4$.

Thus, we face the result $rank(E_{-2(6171u^4+592u^2v^2+64v^4)}(Q)) = 1$ from $r4.2$.

Henceforth, the proof is done from theorem 4.1(10). □

Remark 5.2. If the congruent elliptic curve $E_n: y^2 = x^3 - n^2x$ has positive Mordell-Weil rank then, positive integer n is called a congruent number([5]). In [5], the authors treated that $p_1, p_2, \ldots, p_t$ are distinct primes congruent to 3 modulo 8 as $\left(\frac{p_j}{p_i}\right) = -1$ for $i > j$ and if t is an even positive integer then, $n = 2p_1p_2 \cdots p_t$ is a non-congruent number([5]).

6 Examples

In section 6, the examples of previous results will be submitted. Treating the primality can be done in [4].

There exist examples from lemma 3.1(1) to (8):

(p, u, v)
(48197, 1, 7)
(2738117, 1, 23)
(16562597, 7, 1)

(p, u, v)
(197, 1, 1)
(7477, 3, 1)
(17957, 1, 5)

(p, u, v)
(8111, 1, 3)
(182431, 1, 7)
(1068751, 1, 11)

(p, u, v)
(8677, 1, 3)
(531637, 3, 1)
(43054117, 9, 1)

(p, u, v)
(647, 1, 1)
(496007, 9, 1)
(6097031, 17, 1)

(p, u, v)
(2887, 5, 1)
(7687, 7, 1)
(279751, 19, 1)

(p, u, v)
(71, 1, 1)
(31751, 11, 1)
(172871 17, 1)

(p, u, v)
(122117, 1, 7)
(724901, 1, 11)

Examples of theorem 4.1(1) are induced as follows:

(p, q, u, v)
(11, 7, 2, 1)
(2971, 2647, 7, 3)
(56171, 56167, 11, 1)

Select the pair (u, v) as (11, 3) in the above then, we gain the primality as $(O$, $X)(X$ denotes the primality isn't hold and O means it is satisfied.)

There presented examples from theorem 4.1(2) to (10):

(p, q, u, v)
(881, 883, 2, 1)
(15761, 15923, 4, 3)

(p, q, u, v)
(57041, 57203, 2, 3)
(842321, 842483, 4, 3)

(p, q, u, v)
(149, 167, 3, 1)
(10181, 10631, 3, 5)

(p, q, u, v)
(1237, 1399, 1, 3)
(249541, 249703, 5, 3)

(p, q, u, v)
(29, 31, 1, 1)
(173, 1423, 1, 5)
(25229, 82351, 1, 13)

(p, q, u, v)
(701, 751, 5, 1)
(3181, 3631, 5, 3)
(26861, 30911, 9, 5)

(p, q, u, v)
(29, 31, 1, 1)
(71549, 71551, 7, 1)

(p, q, u, v)
(149, 151, 3, 1)
(24469, 24631, 11, 3)

(p, q, u, v)
(29, 31, 1, 1)
(1051469, 1051471, 11, 1)

We are faced with examples from corollary 5.1(1) to (10):

(p, u, v)
(21187, 1, 2)
(167443, 1, 4)
(2146387, 1, 8)
(19088323, 1, 14)
(97845907, 13, 1)
(222875203, 1, 26)
(285615427, 17, 1)
(299463187, 1, 28)
(649689283, 1, 34)
(1012830787, 1, 38)
(1333989859, 25, 1)
(1818937363, 1, 44)

(p, u, v)
(5101, 1, 1)
(5981, 1, 3)
(12301, 1, 7)
(19661, 1, 9)
(31741, 1, 11)
(50461, 1, 13)

(p, u, v)
(58243, 1, 5)
(155251, 3, 1)
(187651, 1, 7)

(p, u, v)
(3643, 1, 1)
(89563, 1, 5)
(738043, 1, 9)

(p, u, v)
(19483, 1, 3)
(6325387, 1, 15)

(p, u, v)
(102523, 3, 1)
(953179, 5, 7)
(3010123, 7, 1)

(p, u, v)
(1523, 1, 1)
(140627, 1, 7)
(652499, 5, 1)
(4223027, 1, 17)

(p, u, v)
(1091, 1, 1)
(173891, 1, 7)

(p, u, v)
(47963, 3, 1)
(363611, 5, 1)
(1390043, 7, 1)

(p, u, v)
(6827, 1, 1)
(188843, 1, 7)
(40535947, 9, 1)

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