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On t-Derivations of Lattices

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Abstract

In this paper, we introduce the notion of t-derivation for a lattice and investigate some related properties. Moreover, we characterize modular lattices and distributive lattices by isotone t-derivations.

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1 Introduction

The notion of lattice theory introduced by Birkhoff [3]. Balbes and Dwinger [1] gave the concept of distributive lattices and Hoffmann introduced the notion of partially ordered set (Poset). The application of lattice theory plays an important role in different areas such as information science [6], information retrieval [4], information access controls [13] and cryptanalysis [5].

Derivations is a very interesting research topic in the theory of different algebraic structures. After the derivation on a ring was defined by Posner in [12], many authors studied the derivation theory in different algebraic structures. In 2004, Jun and Xin [8] applied the notion of derivation in ring theory to BCI-algebras. Thereafter, M. A. Javed and M. Aslam [10] studied f-derivations in BCI-algebras as its generalization.

Recently the notion of derivation introduced in rings and near rings has been studied by various researchers in the context of lattices (see [1, 2, 14]. In 2008, Xin et al. [14] introduced the notion of derivation in lattices and discussed its properties. After that, many authors generalized this concept in lattices. For example Yilmaz and Özturk [15] introduced the notion of f-derivation on lattices.

The notion of t-derivations in BCI-algebras and complicated subtraction algebras are introduced in [9, 11]. In this paper, the notion of t-derivation on lattices is introduced, which is a generalization of derivation in lattices. Further we studied its properties in the context of t-derivations and characterized modular lattices and distributive lattices by isotone t-derivations.

2 Preliminaries

Definition 2.1. ([3]) Let L be a nonempty set endowed with operations $\wedge$ and $\vee$. If $(L, \wedge, \vee)$ satisfies the following conditions for all $x, y, z \in L:$

(1) $x \wedge x = x, x \vee x = x$.

(2) $x \wedge y = y \wedge x, x \vee y = y \vee x$.

(3) $(x \wedge y) \wedge z = x \wedge (y \wedge z), (x \vee y) \vee z = x \vee (y \vee z)$.

(4) $(x \wedge y) \vee x = x, (x \vee y) \wedge x = x$.

Then L is called lattice.

Definition 2.2. ([3]) A lattice L is called a distributive lattice if one of the following two identities hold for all $x, y, z \in L:$

(5) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

(6) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

In any lattice, the conditions (5) and (6) are equivalent.

Definition 2.3. ([3]) Let L be a lattice. A binary relation $\leq$ on L is defined by $x \leq y$ if and only if $x \wedge y = x$ and $x \vee y = y$.

Definition 2.4. ([1]) A lattice L is called a modular lattice if it satisfies the following condition for all $x, y, z \in L$. If $x \leq y$ then $x \vee (y \wedge z) = (x \vee y) \wedge z$.

Lemma 2.5. *([14]) Let L be a lattice. Let the binary relation $\leq$ be as in Definition 2.3. Then $(L, \leq)$ is a partially ordered set (poset) and for any $x, y \in L$, $x \wedge y$ is the g.l.b. of $\{x, y\}$ and $x \vee y$ is the l.u.b. of $\{x, y\}$.*

Definition 2.6. ([14]) Let L be a lattice. A function $d : L \to L$ on a lattice L is called a derivation if

$$d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y))$$

for all $x, y \in L$.

Lemma 2.7. *[14] Let L be a lattice and d be a derivation on L. Then the following hold:*

(1)$dx \le x$;

(2)$dx \wedge dy \le d(x \wedge y) \le dx \vee dy$;

(3) *If L has a least element 0 and a greatest element 1, then* $d0 = 0$, $d1 \le 1$.

Definition 2.8. Let L_1 and L_2 be lattices. A function $f : L_1 \to L_2$ is called increasing if $x \le y$ implies $fx \le fy$ for all $x, y \in L_1$.

3 t-Derivations on Lattices

In this section, we introduce the notion of t-derivations for lattices.

Definition 3.1. Let L be a lattice. Then for any $t \in L$, we define a self map $D_t : L \to L$ by $D_t(x) = x \wedge t$ for all $x \in L$.

Definition 3.2. Let L be a lattice and D_t be a mapping on L. A function $D_t : L \to L$ is a t-derivation if

$$D_t(x \wedge y) = (D_t(x) \wedge y) \vee (x \wedge D_t(y))$$

for all $x, y \in L$.

Definition 3.3. Let L be a lattice and D_t be a t-derivation on L. Then D_t is called isotone t-derivation if it is increasing.

Example 3.4.

Let $L = \{0, a, b, 1\}$ be a lattice shown by the Hasse diagram of Fig 1. Define mapping D_t as follows:
For $t = 0$, $D_t(x) = 0 \;\; forall \;\; x \in L$

$$For\ t = a,\ D_t(x) = \begin{cases} 0 & \text{for } x = 0 \\ a & \text{for } x = a \ or\ b\ or\ 1 \end{cases}$$

$$For\ t = b,\ D_t(x) = \begin{cases} 0 & \text{for } x = 0 \\ a & \text{for } x = a \\ b & \text{for } x = b\ or\ 1 \end{cases}$$

$$For\ t = 1,\ D_t(x) = \begin{cases} 0 & \text{for } x = 0 \\ a & \text{for } x = a \\ b & \text{for } x = b \\ 1 & \text{for } x = 1 \end{cases}$$

Then it is easy to verify that D_t is a t-derivation.

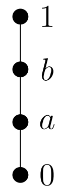

Fig 1

Proposition 3.5. *Let L be a lattice and D_t is a t-derivation on L. Then the following identities hold for all $x, y \in L$:*

(a) $D_t(x) \leq x$.

(b) $D_t(x) \wedge D_t(y) \leq D_t(x \wedge y) \leq D_t(x) \vee D_t(y)$.

(c) *If L has a least element 0, then* $D_t(0) = 0$.

(d) *If L has a greatest element 1 and D_t is an increasing function, then* $D_t(x) = (D_t(1) \wedge x) \vee D_t(x)$.

Proof. (a): For all $x \in L$,

$$D_t(x) = D_t(x \wedge x) = (D_t(x) \wedge x) \vee (x \wedge D_t(x)) = D_t(x) \wedge x.$$

which implies

$$D_t(x) \leq x$$

(b) For all $x, y \in L$, we have $D_t(x \wedge y) = (D_t(x) \wedge y) \vee (x \wedge D_t(y))$. Since $D_t(y) \leq y$ for all $y \in L$, therefore $D_t(x) \wedge D_t(y) \leq D_t(x) \wedge y$ similarly $D_t(x) \wedge D_t(y) \leq x \wedge D_t(y)$. Thus $D_t(x) \wedge D_t(y) \leq D_t(x \wedge y)$. Also $D_t(x) \wedge y \leq D_t(x)$ and $x \wedge D_t(y) \leq D_t(y)$, therefore

$$D_t(x \wedge y) \leq D_t(x) \vee D_t(y)$$

(c) Since 0 is the least element, then (a) gives $0 \leq D_t(x) \leq x = 0$, which implies $D_t(0) = 0$.

(d) Note that $D_t(x) \leq x \leq 1$ for all $x \in L$, so
$D_t(x) = D_t(1 \wedge x) = (D_t(1) \wedge x) \vee (1 \wedge D_t(x)) = (D_t(1) \wedge x) \vee D_t(x)$. □

Proposition 3.6. *Let L be a lattice and D_t is a t-derivation on L. Then the following hold for all $x, y \in L$:*

(a) $D_t(x) = (D_t(x \vee y) \wedge x) \vee D_t(x)$.

(b) *If $y \leq x$ and $D_t(x) = x$ then $D_t(y) = y$.*

Proof. (a) Let $x, y \in L$, then by Definition 2.1 (4), we have
$D_t(x) = D_t((x \vee y) \wedge x) = (D_t(x \vee y) \wedge x) \vee ((x \vee y) \wedge D_t(x))$ so the last relation along with Proposition 3.5 (a) implies
$D_t(x) = (D_t(x \vee y) \wedge x) \vee D_t(x)$.

(b) Let $y \leq x$ and $D_t(x) = x$, then $D_t(y) = D_t(x \wedge y) = (D_t(x) \wedge y) \vee (x \wedge D_t(y))$. Since $D_t(y) \leq y \leq x$, therefore $D_t(y) = y \vee D_t(y) = y$.

□

Theorem 3.7. *Let L be a lattice with greatest element 1. Let D_t be a t-derivation, on L, then*
(*a*) *If $D_t(1) \geq x$, then $D_t(x) = x$.*
(*b*) *If $D_t(1) \leq x$, then $D_t(1) \leq D_t(x)$.*
(*c*) *$D_t(1) = 1$ if and only if $D_t(x) = x$.*

Proof. (*a*) Let $D_t(1) \geq x$ for $x \in L$. Using Proposition 3.6 (*a*) and hypothesis, we have $D_t(x) = (D_t(1) \wedge x) \vee D_t(x) = x \vee D_t(x) = x$.

(*b*) Let $D_t(1) \leq x$ for $x \in L$. Then Proposition 3.6 (*a*) along with Definition 2.1 (4) implies $D_t(x) \wedge D_t(1) = ((D_t(1) \wedge x) \vee D_t(x)) \wedge D_t(1) = (D_t(1) \vee D_t(x)) \wedge D_t(1) = D_t(1)$, which implies $D_t(1) \leq D_t(x)$.

(*c*) Let $D_t(x) = x$, then obviously $D_t(1) = 1$. Conversely let $D_t(1) = 1$. Since $x \leq 1$ and $D_t(1) = 1$, then (*b*) implies $D_t(x) = x$. □

Theorem 3.8. *Let L be a lattice and D_t is a t-derivation on L. Then the following hold for all $x, y \in L$:*
(*a*) *$D_t^2(x) = D_t(x)$.*
(*b*) *$D_t(x) = x$ if only if $D_t(x \vee y) = (x \vee D_t(y)) \wedge (D_t(x) \vee y)$.*

Proof. (*a*) Take, $D_t^2(x) = D_t(D_t(x)) = D_t(x \wedge D_t(x)) = (D_t(x) \wedge D_t(x)) \vee (x \wedge D_t(D_t(x))) = D_t(x) \vee (x \wedge D_t^2(x))$. Since $D_t(D_t(x)) \leq D_t(x) \leq x$, so $D_t^2(x) = D_t(x)$.

(*b*) Let $D_t(x) = x$. Then $D_t(x \vee y) = x \vee y = (x \vee y) \wedge (x \vee y) = (x \vee D_t(y)) \wedge (D_t(x) \vee y)$. Conversely, let $D_t(x \vee y) = (x \vee D_t(y)) \wedge (D_t(x) \vee y)$. Replacing y by x in the last equation, we get $D_t(x) = x$. □

Theorem 3.9. *Let L be a lattice, D_t be a t-derivation on L. Then the following statements are equivalent:*
(*a*) *D_t is isotone t-derivation on L.*
(*b*) *$D_t(x) \vee D_t(y) \leq D_t(x \vee y)$.*

Proof. (*a*) ⇒ (*b*) Since D_t is isotone t-derivation, therefore $D_t(x) \leq D_t(x \vee y)$ and $D_t(y) \leq D_t(x \vee y)$. Hence $D_t(x) \vee D_t(y) \leq D_t(x \vee y)$.

(*b*) ⇒ (*a*) Suppose that $D_t(x) \vee D_t(y) \leq D_t(x \vee y)$. For $x \leq y$, $D_t(x) \leq D_t(x) \vee D_t(y) \leq D_t(x \vee y) = D_t(y)$, which implies $D_t(x) \leq D_t(y)$. Hence D_t is isotone. □

Definition 3.10. Let L be a lattice and D_t be t-derivation. Define a set $F_{D_t}(L) = \{x \in L : D_t(x) = x\}$.

Proposition 3.11. *Let L be a lattice, D_t be an isotone t-derivation. If $x, y \in F_{D_t}(L)$, then $x \vee y \in F_{D_t}(L)$.*

Proof. Since D_t is isotone t-derivation therefore

$$(x \vee y) = D_t(x) \vee D_t(y) \leq D_t(x \vee y).$$

Proposition 3.5 (a) yield $D_t(x \vee y) = (x \vee y)$ and hence $x \vee y \in F_{D_t}(L)$. □

Proposition 3.12. *Let L be a lattice and D_{t_1} and D_{t_2} be two isotone t-derivations on L. Then $D_{t_1} = D_{t_2}$ if and only if $F_{D_{t_1}}(L) = F_{D_{t_2}}(L)$.*

Proof. It is obvious that $D_{t_1} = D_{t_2}$ implies $F_{D_{t_1}}(L) = F_{D_{t_2}}(L)$. Conversely let $F_{D_{t_1}}(L) = F_{D_{t_2}}(L)$ and $x \in L$. By Theorem 3.8 (a), $D_{t_1}(x) \in F_{D_{t_1}}(L) = F_{D_{t_2}}(L)$ and so $D_{t_2}D_{t_1}(x) = D_{t_1}(x)$. Similarly we can get $D_{t_1}D_{t_2}(x) = D_{t_2}(x)$. Since D_{t_1} and D_{t_2} are isotone t-derivations, we have $D_{t_2}D_{t_1}(x) \leq D_{t_2}(x) = D_{t_1}D_{t_2}(x)$ and so $D_{t_2}D_{t_1}(x) \leq D_{t_1}D_{t_2}(x)$. Similarly we can get $D_{t_1}D_{t_2}(x) \leq D_{t_2}D_{t_1}(x)$, this shows that $D_{t_1}D_{t_2}(x) = D_{t_2}D_{t_1}(x)$. It follows that $D_{t_1}(x) = D_{t_2}D_{t_1}(x) = D_{t_1}D_{t_2}(x) = D_{t_2}(x)$, that is $D_{t_1} = D_{t_2}$. □

Theorem 3.13. *Let L be a lattice with greatest element 1. Let D_t be a t-derivation on L for all $x \in L$, then the following statements are equivalent:*

(a) *D_t is isotone t-derivation on L.*

(b) *$D_t(x) = x \wedge D_t(1)$.*

(c) *$D_t(x \wedge y) = D_t(x) \wedge D_t(y)$.*

Proof. $(a) \Rightarrow (b)$ Suppose that D_t is isotone, then $D_t(x) \leq D_t(1)$. Since $D_t(x) \leq x$, so $D_t(x) \leq x \wedge D_t(1)$. Also Proposition 3.5 (d) gives $D_t(x) = (D_t(1) \wedge x) \vee D_t(x)$, which implies $D_t(1) \wedge x \leq D_t(x)$. Thus $D_t(x) = x \wedge D_t(1)$.

$(b) \Rightarrow (c)$ Suppose that $D_t(x) = x \wedge D_t(1)$. Then $D_t(x) \wedge D_t(y) = (x \wedge D_t(1)) \wedge (y \wedge D_t(1)) = (x \wedge y) \wedge D_t(1) = D_t(x \wedge y)$.

$(c) \Rightarrow (a)$ Suppose that $D_t(x \wedge y) = D_t(x) \wedge D_t(y)$ and $x \leq y$. Then $D_t(x) = D_t(x \wedge y) = D_t(x) \wedge D_t(y)$, implies $D_t(x) \leq D_t(y)$. Thus D_t is isotone t-derivation on L.

□

Theorem 3.14. *Let L be a modular lattice and D_t be a t-derivation on L. Then the following statements are equivalent:*

(1) *D_t is isotone t-derivation,*

(2) *$D_t(x \wedge y) = D_tx \wedge D_ty$,*

(3)*$D_tx = x$ implies $D_tx \vee D_ty = D_t(x \vee y)$.*

Proof. (1) $\Rightarrow$ (2) Let D_t is isotone t-derivation and $x \wedge y \leq x$, $x \wedge y \leq y$ implies $D_t(x \wedge y) \leq D_tx$, $D_t(x \wedge y) \leq D_ty$ respectively. Thus $D_t(x \wedge y) \leq D_tx \wedge D_ty$. Since L is modular and $D_tx \wedge y \leq D_tx \leq x$ we have $D_t(x \wedge y) = (D_tx \wedge y) \vee (x \wedge D_ty) =$

$((D_tx \wedge y) \vee D_ty) \wedge x \geq (D_tx \wedge y) \wedge x = D_tx \wedge y \geq D_tx \wedge D_ty$.

(2) $\Rightarrow$ (1) Assume that $x \leq y$. Then $D_tx = D_t(x \wedge y) = D_tx \wedge D_ty$, therefore $D_tx \leq D_ty$.

(1) $\Rightarrow$ (3) Assume that $D_tx = x$ and D_t is isotone. Since L is modular and by Proposition 3.6(a) we get $D_ty = (D_t(x \vee y) \wedge y) \vee D_ty = (D_ty \vee y) \wedge D_t(x \vee y) = y \wedge D_t(x \vee y)$ hence $D_tx \vee D_ty = D_tx \vee (y \wedge D_t(x \vee y)) = (D_tx \vee y) \wedge D_t(x \vee y) = (x \vee y) \wedge D_t(x \vee y) = D_t(x \vee y)$.

(3) $\Rightarrow$ (1) Assume $x \leq y$, then by Theorem 3.8 (i) $D_t(D_tx) = D_tx$ by hypothesis, $D_t(D_tx \vee y) = D_t(D_tx) \vee D_ty = D_tx \vee D_ty$. Otherwise $x \leq y$ implies $D_tx \leq x \leq y$ thus $D_t(D_tx \vee y) = D_ty$. And so $D_ty = D_tx \vee D_ty$, hence $D_tx \leq D_ty$. □

Theorem 3.15. *Let L be a distributive lattice and D_t be a t-derivation on L. Then the following statements are equivalent:*
(a) D_t is isotone t-derivation on L.
(b) $D_t(x) \vee D_t(y) = D_t(x \vee y)$.

Proof. (a) $\Rightarrow$ (b) Suppose that D_t is isotone t-derivation. Since $D_tx \leq D_t(x \vee y)$ and $D_ty \leq D_t(x \vee y)$. Proposition 3.6 (a) implies $D_tx = (D_t(x \vee y) \wedge x) \vee D_tx = (D_t(x \vee y) \vee D_tx) \wedge (x \vee D_tx) = D_t(x \vee y) \wedge x$. Thus $D_tx \vee D_ty = (D_t(x \vee y) \wedge x) \vee (D_t(x \vee y) \wedge y) = D_t(x \vee y) \wedge (x \vee y) = D_t(x \vee y)$.

(b) $\Rightarrow$ (a) Assume $x \leq y$, then $D_t(y) = D_t(x \vee y) = D_t(x) \vee D_t(y)$, hence $D_t(x) \leq D_t(y)$, so D_t is isotone t-derivation. □

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