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A Note on the Optimal CIs for Scale Parameter of Truncated Exponential Model Based on Records

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Abstract

Confidence intervals (CIs) play fundamental task in statistical inferences and statsitical decision making. For this sake, many researches have been conducted to propose optimal CIs. One method to construct an optimal CI is use the optimal estimators. In this paper, we consider a new method to construct uniformly better confidence intervals for the scale parameter of the exponential distribution based on record data. In this regard, we use the maximum posterior coverage probability to obtain a new Bayes and Empirical Bayes confidence intervals. The results presented by the use of simulated data and a real data analysis.

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1 Introduction

Exponential model has a crucial position in lifetime modeling. Due to its importance in reliability and life time data, many authors have done research about this distribution. Among them are Balakrishnan and Basu (1995), Chen and Bhattacharyya (1998), Gupta et al. (2002), Childs et al. (2003), Jaheen (2004), Chandrasekar et al. (2004), Balakrishnan et al. (2007) and Doostparast (2009).

Suppose that $X_1, X_2, ..., X_n$ are independently and identically distributed (iid) random variables from the population X. If X_j value exceed than all of previous observations, it is named an upper record value .Record values are very important in many real life applications such as life-tests,hydrology, industry, climate events,seismology, sports events, and economics. Theory and application of record values was firstly initiated by Chandler's (1952) . And further developments appeared in Glick (1978), Nagaraja (1988), Balakrishnan et al. (1995), Arnold et al. (1998), Arabi Belaghi et al. (2012), (2014a, 2014b).

Now let $Y_1 = y_1, ..., Y_n = y_n$ be the first n observed upper record values from p.d.f Y . Then the likelihood function of the observations is given by

$$L(\theta; \boldsymbol{y}) = \prod_{i=1}^{n-1} h(y_i; \theta) f_\theta(y_n), \tag{1}$$

where $\boldsymbol{y} = (y_1, \cdots, y_n)$, and $h(y_i; \theta)$ is the hazard function at y_i ($h(y_i; \theta) = f_\theta(y_i) / [1 - F_\theta(y_i)]$).

CI is an important tool in statistical inferences. So far, many works has been emerged to construct optimal CIs. Stein (1962) first suggested improved estimators to construct an improved confidence set and then his work was continued by Brown (1966) and Joshi (1967)for the mean of multivariate normal model. Hwang and Casella (1982) developed underlying works to obtain minimax confidence sets for the mean of multivariate normal using the positive part of the Stein estimator and showed that when $p \geq 4$, their proposed CIs have uniformly greater coverage probability (CP) and are minimax. Hwang and Chen (1986) further proceed the problem of improving confidence sets for the coefficients of a linear model with spherically symmetric errors. Recently Petropoulos (2011) has considered new classes of CIs for the scale parameter of two parameter exponential distribution based on *iid* samples and his work was surpassed for the location parameter of that distribution by Jiang and Wong (2012). Further Kourouklis and Petropoulos (2012) have constructed two new classes of improved CIs for the variance of a normal distribution assuming the mean to be unknown. They used a generalized Bayes estimator and showed that their CIs satisfy Kubokawa's (1994) conditions when the observations are independent to have greater CP in the origin uniformly. More recently, Arabi

Belaghi et al. (2014) extended the results of Kubokawa for the case of dependant random variable from any bivariate distribution the get uniformly better CIs for the scale parameter of Burr XII model.
The format of this paper is as follows: In Section 2 we give the ML, UMVU, Bayes and Empirical Bayes estimators of the scale parameter. Section 3 contains new Bayes and empirical Bayes CIs. In Section 4 we conduct a simulation study and a real example for comparison and illustration purpose. This paper will be concluded with a discussion in Section 5.

2 Estimators of the Parameters

Let $Y_1, \cdots, Y_n$ be the first n ordered upper records from the two parameters exponential distribution with the following pdf:

$$f_\sigma(y) = \sigma e^{-\sigma(y-\eta)}, \quad \sigma > 0, \quad y > \eta$$

Then by (1) we have

$$L(\sigma; \boldsymbol{y}) = \sigma^{-n} e^{-\sigma(y_n-\eta)}, \quad y_1 > \eta. \tag{1}$$

It follows that the maximum likelihood estimator (MLE) of σ and η are given by

$$\hat{\eta}_{ML} = Y_1, \tag{2}$$

$$\hat{\sigma}_{ML} = \frac{n}{Y_n - Y_1}. \tag{3}$$

It is easy to prove that the distribution of $T = Y_n - Y_1$ is Gamma with shape parameter $n-1$ and scale σ denoted by $T \sim \Gamma(n-1, \sigma)$ and independent of Y_1 . It turns out that the uniformly minimum variance unbiased estimator (UMVUE) of σ can be emerge as

$$\hat{\sigma}_U = \frac{n-2}{T}. \tag{4}$$

Now it is assume the $\Gamma(\nu, \gamma)$ as the natural conjugate prior distribution with the following pdf

$$\pi(\sigma) = \frac{\gamma^\nu}{\Gamma(\nu)} \sigma^{\nu-1} e^{-\gamma\sigma}, \quad \alpha, \gamma > 0, \tag{5}$$

Further it is easy to follow that the posterior distribution of σ given $\boldsymbol{y}$ is

$$\pi(\sigma | \boldsymbol{y}) = \frac{(\gamma + y_n)^{n+\nu}}{\Gamma(n+\nu)} \sigma^{n+\nu-1} e^{-(\gamma+T)\sigma}. \tag{6}$$

Thus by taking into account the square error loss function (SEL), the Bayes estimator (BE) of σ denoted by $\hat{\sigma}_B$, is given by

$$\hat{\sigma}_B = \frac{n+\nu-1}{\gamma+T}. \tag{7}$$

Further the marginal density of T given σ can be deduced as

$$f_T(t;\nu,\gamma) = \frac{\Gamma(n+\nu-1)}{\Gamma(n-1)\Gamma(\nu)} \frac{t^{n-2}}{(t+\gamma)^{n+\nu-1}}, t>0$$

Denoting the MLEs of ν and γ respectively by $\hat{\nu}_{ML}$ and $\hat{\gamma}_{ML}$, there exists a relation between them, namely, $\hat{\gamma}_{ML} = \frac{1}{n-1}\hat{\nu}_{ML}T$ and therefore the empirical Bayes estimator (EBE) of σ can be derived as

$$\hat{\sigma}_{EB} = \frac{n-1}{T}. \tag{8}$$

3 Some New CI's

In this section we try to propose a new CIs that has the same ratio of endpoints as the shortest CI while having greater CP.

As it is known (Akhlaghi and Parsian (1986)), the shortest CI of σ is $\left(\frac{a}{T}, \frac{b}{T}\right)$, where a and b can be calculated by solving the following system simultaneously

$$\begin{cases} \int_a^b g(t)\,dt = 1-\gamma \\ g(a) = g(b) \end{cases}, \tag{1}$$

where $g(.)$ is the density of $\Gamma(n-1,2)$. Our main purpose is to obtain a function $\psi(T)$ in order to get an interval of the form

$$(a\psi(T), b\psi(T)) \tag{2}$$

with greater CP than the shortest CI while having the same ratio of endpoints by making use of (3.1).

The posterior CP of (3.2) is given by

$$\int_{a\psi(T)}^{b\psi(T)} \pi(\sigma|T=t)d\sigma = \int_{a\psi(t)}^{b\psi(t)} \frac{(\gamma+t)^{n+\nu}}{\Gamma(n+\nu)} \sigma^{n+\nu-2} e^{-(\gamma+t)\sigma} d\sigma. \tag{3}$$

Since the integrand in (3.3) is a unimodal function, the value of $\psi(T)$ that maximizes the posterior CP, will be the unique solution of the equation

$$\frac{\partial}{\partial\psi(T)} \int_{a\psi(t)}^{b\psi(T)} \frac{(\gamma+t)^{n+\nu}}{\Gamma(n+\nu)} \sigma^{n+\nu-2} e^{-(\gamma+t)\sigma} d\sigma = 0, \tag{4}$$

that implies

$$\psi(T) = \frac{(n+\nu-1)\ln\frac{b}{a}}{(b-a)(\gamma+T)}. \tag{5}$$

Therefore, the interval (3.2) will be of the form

$$I_B = \left(\frac{a(n+\nu-1)\ln\frac{b}{a}}{(b-a)(\gamma+T)}, \frac{b(n+\nu-1)\ln\frac{b}{a}}{(b-a)(\gamma+T)}\right). \tag{6}$$

Note that from $g(a) = g(b)$ it can be immediately observed that

$$I_B = \left(\frac{a(n+\nu-1)}{(n-1)(\gamma+T)}, \frac{b(n+\nu-1)}{(n-1)(\gamma+T)}\right). \tag{7}$$

The authors used the values of Akhlaghi and Parsian (1986) to the CP of Bayes CI given by I_B to verify whether it is uniformly dominant over shortest CI. Unfortunately this does not happen for all values of the parameter space except for some values of ν, γ and n. On the other hand, using the empirical Bayes estimators of ν and γ will give a new CI namely, an empirical Bayes CI, (EBCI) of the form

$$I_{EB} = \left(\frac{a}{\left(1-\frac{1}{n-1}\right)T}, \frac{b}{\left(1-\frac{1}{n-1}\right)T}\right). \tag{8}$$

It is obvious that although, EBCI has the same ratio of endpoints as the usual shortest CI with bigger CP, it just depends on the number of records. So providing a broader class of CIs motivated us to use another random ancillary variable to obtain a uniformly improved class of CIs that includes the EBCI as well. In this regard, we use the covariate Y_{n-1} to get its hidden information in order to formulate a uniformly improved class of confidence intervals.

In this regards, we use Theorem 5.2 and Remark 5.1 in Arabi Belaghi et al. (2014)to construct an improved CI over the usual shortest CI of the form $I_0 = (\frac{a}{Y_n-Y_1}, \frac{b}{Y_n-Y_1})$, where $g(a) = g(b)$.

Define the new CI for σ as

$$I(\phi(W)) = \left(\frac{a}{\phi(W)(Y_n-Y_1)}, \frac{b}{\phi(W)(Y_n-Y_1)}\right), \tag{9}$$

where $W = \frac{Y_{n-1}-Y_1}{Y_n-Y_1}$.

We show that this CI has the same ratio of endpoints as the shortest CI but greater CP.

Theorem 3.1 *Suppose that:*

(i) $\phi(w)$ *is non-decreasing on* $(0,1)$ *and* $lim_{w\to 1}\phi(w) = 1$,

(ii) $\phi(w) > 1 - \frac{1}{n-1}$ *on a set of* w *values where* $\phi(w)$ *is non constant.*

Then, $I(\phi(W))$ *has the same ratio of endpoints as the shortest CI but greater CP.*

Proof
See Arabi Belaghi et al. (2014).

Remark 3.1 *Note that such a* $\phi(W)$ *as is* $\phi(W) = \max\left\{W, 1 - \frac{1}{n-1}\right\}$.

4 Numerical Examples

In this section, we conduct a simulation study to show the superiority of our proposed confidence intervals. Firstly, we write the following R code to obtain the shorted confidence intervals arguments (a, b).

```
################## chisq.hdi <- function(df, alpha = 0.05, tol =
1e-7){

# df is chi squared degrees of freedom # nu error probabilities
#(i.e., 1-alpha = confidence level)  tolerance level for
 # |alpha - alpha(i)|.  outputs (L, U, nu) where L, U are the lower and
#upper endpoints and nu is the final #iteration value for given
#nu.

if(nu <= 0 | nu >= 1){stop("The confidence level must be between 0
and 1")} if(df <= 0){stop("The degrees of freedom must be
positive")}

if(df <= 2){ chisqhdi <- c(0, qchisq(1-nu, df)) alphai <- alpha }
else { c <- dchisq(df-2, df) vmin <- 0 vmax <- c alphai <- 1
while(abs(alphai - alpha) > tol){ vmid <- (vmin + vmax) / 2 a <-
uniroot(function(x) dchisq(x, df) - vmid, lower = 0, upper =
df-2)$root b <- 1/uniroot(function(x) dchisq(1/x, df) - vmid,
lower = 0, upper = 1/(df-2))$root alphai <- pchisq(a, df) +
pchisq(b, df, lower.tail = FALSE) if(alphai > alpha){ vmax <- vmid
} else { vmin <- vmid } } chisqhdi <- c(a, b) }

c(chisqhdi, alpha)

}
 ############
```

Then, we use this function in the simulation study. The results are given in table 2 based on 10, 000 simulations for various parameters.

Table 1: Simulated C.P of Shortest (S.CP) and Improved (I.CI) CI

$n = 4, \alpha = 0.05$	$\sigma = 1, \mu = 0$	$\sigma = 1, \mu = 1$	$\sigma = 5, \mu = 2$	$\sigma = 50, \mu = 20$
$S.CP$	0.956	0.949	0.951	0.957
$I.CP$	0.980	0.977	0.972	0.978
$n = 4, \nu = 0.1$	$\sigma = 1, \mu = 0$	$\sigma = 1, \mu = 1$	$\sigma = 5, \mu = 2$	$\sigma = 50, \mu = 20$
$S.CP$	0.887	0.897	0.893	0.899
$I.CP$	0.942	0.943	0.938	0.946

It can be clearly seen that the coverage probability of the improved CI always higher than the usual shortest CI. Now we consider the following real data example.

Proschan (1963) registered data on intervals between failures (in hours) of the air conditioning system of a fleet of 13 Boeing 720 jet airplanes. He showed that the exponential distribution can be adequately fitted for failure time of the air-conditioning system for each of the planes. Here, for the sake of illustration, the planes 8045 is selected and the corresponding failure time data are presented as follows:

102, 209, 14, 57, 54, 32, 67, 59, 134, 152,27, 14, 230, 66, 61,34

For the illustration purpose, we add 5 to all of the data to get the two parameter exponential with $\eta = 5$ and $\sigma = 1/\sigma = 1/87$. The extracted upper records are $107, 214, 235$ so the ML and UMVU estimator of $\sigma = \frac{1}{\sigma}$ are $3/128$ and $1/128$, respectively. Also, $W = 0.83$ and $\phi(W) = \max\left(0.83, 1 - \frac{1}{3-1}\right) = 0.83$. By using the R defined function we have

```
chisq.hdi(4,0.1)
 [1] 0.167637 7.864299 0.100000
> chisq.hdi(4,0.05)
[1] 0.08464386 9.53025455 0.04999998
 > chisq.hdi(4,0.01)
[1]  0.017449734 13.285443896  0.009999934
```

Therefore, 0.90% and 0.95% and 0.99%shortest and improved CI are

5 Discussion

In this paper improvement of the usual well-known CIs was considered for the scale parameter of the two parameters exponential distribution based on record

Table 2: Shortest (S.CP) and Improved (I.CI) CI for σ

	0.90%	0.95%	0.99%
$S.CP$	$(0.000654832, 0.03071992)$	$(0.0003306401, 0.03722756)$	$(6.816302e-05, 0.05189627)$
$I.CP$	$(0.0007889543, 0.03701195)$	$(0.0003983615, 0.04485248)$	$(8.212412e-05, 0.06252562)$

observations. We proposed a new method "maximum posterior coverage probability" to get uniformly better confidence interval resulted in the empirical Bayes confidence intervals. We then used the results of Arabi Belaghi et al. (2014) method to prove the superiourity of our proposed confidence interval. Finally,a simulation study and a real example proposed in order to illustrate the inferential method developed here.

Note that in this paper the authors generalized the results of Kubokawa (1994) for the bivariate distribution. Further development for the multivariate case is in progress by the authors and hope get the expected results soon.

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