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Probabilistic Programming Approach with Dependent Random Parameters

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Abstract

In this paper, the chance-constrained programming (CCP) technique is considered, when exist some dependent exponential distributed random parameters. Firstly, a proposed bivariate exponential distribution is presented. Secondly, a transforming method to convert chance constraints to its equivalent deterministic ones. That through two cases: (i) dividual (joint) constraints, (ii) individual constraints.
Finally a numerical example is introduced to illustrate the converting method of the chance constraints to its equivalent deterministic ones.

Keywords: Bivariate exponential distributions, CCP, Downton bivariate distribution, Farle and other bivariate distribution, transformation techniques, individual constraints, dividual (joint) constraints

1. Introduction

(CCP) technique is considered one of the most applicable technique of the stochastic programming [7,13,14,22]. It was first introduced by Charnes and Cooper (1959). In the literature of (CCP), various models and approaches have been suggested by several researches [4,5,7,8,9,19,20,21,22,23].

The most of researches have dealt with independent exponential distributed random parameters $\tilde{a}_{ij}$, or $\tilde{b}_i$, $i = 1,2,\dots,m$, $j = 1,2,\dots,n$. [3,7,15]. Recently, few researches have dealt with dependent some random parameters as Hafez and other [14,15,16], Ismail and other [17], El-Dash [8,9,10].

In this paper, the following linear chance constraints are investigated.

$$P_r\left\{\sum_{j=1}^{n} a_{ij}\, x_j \le \tilde{b}_i \ , \qquad i = 1,2\right\} \ge \gamma \tag{1.1}$$

$$P_r\left\{\sum_{j=1}^{2} \tilde{a}_{ij}\, x_j + \sum_{j=3}^{n} a_{ij}\, x_j \le b_i\right\} \ge \gamma_i \ , \qquad i = 3,4,\dots,m \tag{1.2}$$

Where $\tilde{a}_{ij}$ or $\tilde{b}_i$ are random parameters and a_{ij}, b_i, c_j are constants and $0 < \gamma$, $\gamma_i < 1$ and $x_j \ge 0$ donate to tolerance measures and decision variables respectively.

2. A proposed Bivariate Exponential Distribution

In this section a bivariate exponential distribution is presented. This distribution is an extension of Farlie, Gumbel and Morgenstern bivariate exponential distribution when $\lambda_i > 0$, $i = 1,2$ rather than $\lambda_i = 1$ [1,2,24].This distribution is very important for insurance, economic, demographic, reliability sector, etc [17,22].
Let $\tilde{a}_{i1}$ and $\tilde{a}_{i2}$ are exponential random variables with parameters $\lambda_{i1}, \lambda_{i2} > 0$ respectively and correlation coefficient ρ , $-\frac{1}{4} < \rho < \frac{1}{4}$.

Definition (2.1): Let $f(\tilde{a}_{i1}, \tilde{a}_{i2})$, $F(a_{i1}, a_{i2})$ are joint density and cumulative function of $(\tilde{a}_{i1}, \tilde{a}_{i2})$ respectively, then:

$$f(\tilde{a}_{i1}, \tilde{a}_{i2}) = \lambda_1\lambda_2\, e^{-(\lambda_1\tilde{a}_1+\lambda_2\tilde{a}_2)}\left\{1 + \alpha\left(2e^{-\lambda_1\tilde{a}_1} - 1\right)\left(2e^{-\lambda_2\tilde{a}_2} - 1\right)\right\} |\alpha| < 1 \ , \quad \tilde{a}_1, \tilde{a}_1, \lambda_1, \lambda_2 > 0 \tag{2.1}$$

$$F(a_1, a_2) = P_r(\tilde{a}_1 < a_1, \tilde{a}_2 < a_2) = \left[1 - e^{-\lambda_1 a_1}\right]\left[1 - e^{-\lambda_2 a_2}\right]\left[1 + \alpha\, e^{-(\lambda_1 a_1+\lambda_2 a_2)}\right] \tag{2.2}$$

$$-\frac{1}{4} < \rho < \frac{1}{4} \ \longrightarrow \ \rho = \frac{1}{4}\alpha \tag{2.3}$$

it is easy to prove that:

$$\int_0^\infty\int_0^\infty f(\tilde{a}_1, \tilde{a}_2)\, d\tilde{a}_1\, d\tilde{a}_2 = 1 \ , \quad F(\infty, \infty) = 1$$

It is noted that when $\tilde{a}_{i1}, \tilde{a}_{i2}$ are independent then $\rho = \alpha = 0$ in turn:

$$f(\tilde{a}_{i1}, \tilde{a}_{i2}) = \lambda_1 \lambda_2 \, e^{-(\lambda_1 \tilde{a}_1 + \lambda_2 \tilde{a}_2)} \quad , \tag{2.4}$$

$$F(a_1, a_2) = \left[1 - e^{-\lambda_1 a_1}\right]\left[1 - e^{-\lambda_2 a_2}\right] \tag{2.5}$$

3. Probability Distribution of $\tilde{Z} = \sum_{j=1}^{2} \tilde{a}_{ij} x_j$

Let $(\tilde{a}_{i1}, \tilde{a}_{i2})$ are two dependent exponential distributed random variables and follow the suggested bivariate distribution in (2.1) - (2.3). The following theorem gives the probability distribution of
$\tilde{Z}_i = \sum_{j=1}^{2} \tilde{a}_{ij} x_j \;,\; x_j \geq 0 \;,\; j = 1,2, \ldots, n$

Theorem (3.1): Let $f(\tilde{z}_i)$ and $F(z_i)$ donate to a density and cumulative functions of $\tilde{z}_i$ respectively, then [11]:

$$f(\tilde{z}_i) = c_1 \lambda_1 \lambda_2 (1 + \alpha) \left[e^{\frac{-\lambda_1}{x_1}\tilde{z}} - e^{\frac{-\lambda_2}{x_2}\tilde{z}}\right] - 2\lambda_1 \lambda_2 \alpha \left\{ c_2 \left[e^{\frac{-2\lambda_1}{x_1}\tilde{z}} - e^{\frac{-\lambda_2}{x_2}\tilde{z}}\right] + c_3 \left[e^{\frac{-\lambda_1}{x_1}\tilde{z}} - e^{\frac{-2\lambda_2}{x_2}\tilde{z}}\right] - c_1 \left[e^{\frac{-2\lambda_1}{x_1}\tilde{z}} - e^{\frac{-2\lambda_2}{x_2}\tilde{z}}\right] \right\} \tag{3.1}$$

$$F(z_i) = c_1 (1 + \alpha) \left[\lambda_2 x_1 \left(1 - e^{\frac{-\lambda_1}{x_1} z_i}\right) - 2\lambda_1 \lambda_2 \left(1 - e^{\frac{-\lambda_2}{x_2} z_i}\right)\right] - \alpha \left\{ c_2 \left[\lambda_2 x_1 \left(1 - e^{\frac{-2\lambda_1}{x_1} z_i}\right) - 2\lambda_1 x_2 \left(1 - e^{\frac{-\lambda_2}{x_2} z_i}\right)\right] + c_3 \left[2\lambda_2 x_1 \left(1 - e^{\frac{-\lambda_1}{x_1} z_i}\right) - \lambda_1 x_2 \left(1 - e^{\frac{-2\lambda_2}{x_2} z_i}\right)\right] - c_1 \left[\lambda_2 x_1 \left(1 - e^{\frac{-2\lambda_1}{x_1} z_i}\right) - \lambda_1 x_2 \left(1 - e^{\frac{-2\lambda_2}{x_2} z_i}\right)\right] \right\} \tag{3.2}$$

where:

$$C_1 = (\lambda_2 x_1 - \lambda_1 x_2)^{-1}, \quad C_2 = (\lambda_2 x_1 - 2\lambda_1 x_2)^{-1}, \text{and} \quad C_3 = (2\lambda_2 x_1 - \lambda_1 x_2)^{-1} \tag{3.3}$$

Proof: Since $\tilde{Z}_i = \tilde{a}_{i1} x_1 + \tilde{a}_{i2} x_2$, $x_1, x_2 \geq 0$, and $\tilde{k} = \tilde{a}_{i2} x_j$ in turn $|J| = \begin{vmatrix} \frac{1}{x_1} & \frac{-1}{x_1} \\ 0 & \frac{1}{x_2} \end{vmatrix} = \frac{1}{x_1 x_2}$

$$f(\tilde{z}_i, \tilde{k}) = \frac{\lambda_1 \lambda_2}{x_1 x_2} \, e^{-(\lambda_1 \tilde{a}_{i1} + \lambda_2 \tilde{a}_{i2})} \left\{1 + \alpha \left(2e^{-\lambda_1 \tilde{a}_1} - 1\right)\left(2e^{-\lambda_2 \tilde{a}_2} - 1\right)\right\}$$

$$= \frac{\lambda_1 \lambda_2}{x_1 x_2} \, e^{\frac{-\lambda_1}{x_1}\tilde{z} - \left(\frac{\lambda_2 x_1 - \lambda_1 x_2}{x_1 x_2}\right)\tilde{k}} \left\{1 + \alpha \left[4e^{\frac{-\lambda_1}{x_1}\tilde{z} - \left(\frac{\lambda_2 x_1 - \lambda_1 x_2}{x_1 x_2}\right)\tilde{k}} - 2e^{\frac{-\lambda_1}{x_1}\tilde{z} + \frac{\lambda_1}{x_1}\tilde{k}} - 2e^{\frac{-\lambda_2}{x_2}\tilde{k}} + 1\right]\right\} \longrightarrow$$

$$f(\tilde{z}_i) = \frac{\lambda_1\lambda_2}{x_1x_2} e^{\frac{-\lambda_1}{x_1}\tilde{z}} \int_0^{\tilde{z}} e^{-\left(\frac{\lambda_2x_1-\lambda_1x_2}{x_1x_2}\right)\tilde{k}}\, d\tilde{k}$$

$$+ \frac{\lambda_1\lambda_2\alpha}{x_1x_2}\left\{4e^{\frac{-2\lambda_1}{x_1}\tilde{z}} \int_0^{\tilde{z}} e^{\frac{-2(\lambda_2x_1-\lambda_1x_2)}{x_1x_2}\tilde{k}} d\tilde{k} - 2\, e^{\frac{-2\lambda_1}{x_1}\tilde{z}} \int_0^{\tilde{z}} e^{\frac{-(\lambda_2x_1-\lambda_1x_2)}{x_1x_2}\tilde{k}} d\tilde{k}\right.$$

$$\left. -2e^{\frac{-\lambda_1}{x_1}\tilde{z}} \int_0^{\tilde{z}} e^{\frac{-(2\lambda_2x_1-\lambda_1x_2)}{x_1x_2}\tilde{k}} d\tilde{k} + e^{\frac{-\lambda_1}{x_1}\tilde{z}} \int_0^{\tilde{z}} e^{\frac{-(\lambda_2x_1-\lambda_1x_2)}{x_1x_2}\tilde{k}} d\tilde{k}\right\}$$

$$= c_1\lambda_1\lambda_2(1+\alpha)\left[e^{\frac{-\lambda_1}{x_1}\tilde{z}} - e^{\frac{-\lambda_2}{x_2}\tilde{z}}\right] - 2\lambda_1\lambda_2\alpha\left\{c_2\left[e^{\frac{-2\lambda_1}{x_1}\tilde{z}} - e^{\frac{-\lambda_2}{x_2}\tilde{z}}\right] + c_3\left[e^{\frac{-\lambda_1}{x_1}\tilde{z}} - e^{\frac{-2\lambda_2}{x_2}\tilde{z}}\right]\right.$$

$$\left. -c_1\left[e^{\frac{-2\lambda_1}{x_1}\tilde{z}} - e^{\frac{-2\lambda_2}{x_2}\tilde{z}}\right]\right\}$$

Also

$$F(z_i) = P_r(\tilde{z}_i \le z_i) = \int_0^{z_i} f(\tilde{z}_i)\, d\tilde{z}_i$$

$$= c_1\lambda_1\lambda_2(1+\alpha)\int_0^{z_i}\left[e^{\frac{-\lambda_1}{x_1}\tilde{z}_i} - e^{\frac{-\lambda_2}{x_2}\tilde{z}_i}\right] d\tilde{z}_i - 2\lambda_1\lambda_2\alpha\left\{c_2\int_0^{z_i}\left[e^{\frac{-2\lambda_1}{x_1}\tilde{z}_i} - e^{\frac{-\lambda_2}{x_2}\tilde{z}_i}\right] d\tilde{z}_i\right.$$

$$\left. +c_3\int_0^{z_i}\left[e^{\frac{-\lambda_1}{x_1}\tilde{z}_i} - e^{\frac{-2\lambda_2}{x_2}\tilde{z}_i}\right] d\tilde{z}_i - c_1\int_0^{z_i}\left[e^{\frac{-2\lambda_1}{x_1}\tilde{z}_i} - e^{\frac{-2\lambda_2}{x_2}\tilde{z}_i}\right] d\tilde{z}_i\right\}$$

$$= c_1(1+\alpha)\left[\lambda_2x_1\left(1 - e^{\frac{-\lambda_1}{x_1}z_i}\right) - \lambda_1x_2\left(1 - e^{\frac{-\lambda_2}{x_2}z_i}\right)\right]$$

$$-\alpha\left\{c_2\left[\lambda_2x_1\left(1 - e^{\frac{-2\lambda_1}{x_1}z_i}\right) - 2\lambda_1x_2\left(1 - e^{\frac{-\lambda_2}{x_2}z_i}\right)\right]\right.$$

$$+c_3\left[2\lambda_2x_1\left(1 - e^{\frac{-\lambda_1}{x_1}z_i}\right) - \lambda_1x_2\left(1 - e^{\frac{-2\lambda_2}{x_2}z_i}\right)\right]$$

$$\left. - c_1\left[\lambda_2x_1\left(1 - e^{\frac{-2\lambda_1}{x_1}z_i}\right) - 2\lambda_1x_2\left(1 - e^{\frac{-2\lambda_2}{x_2}z_i}\right)\right]\right\},$$

$$i = 1,2,\dots,m$$

4. The equivalent deterministic constraints

Case (i): Let $\tilde{b}_1, \tilde{b}_2$ follow bivariate exponential distribution in (2.1), (2.2), then [10, 14]:

$$P_r\left(\sum_{j=1}^{n} a_{1j}\,x_j \ge \tilde{b}_1\,, \sum_{j=1}^{n} a_{2j}\,x_j \ge \tilde{b}_2\right) = F\left(\sum_{j=1}^{n} a_{1j}\,x_j, \sum_{j=1}^{n} a_{2j}\,x_j\right) \ge \gamma \longrightarrow$$

$$\left[1-e^{-\lambda_1\sum_{j=1}^{n}a_{1j}x_j}\right]\left[1-e^{-\lambda_2\sum_{j=1}^{n}a_{2j}x_j}\right]\left[1+\alpha e^{-\left(\lambda_1\sum_{j=1}^{n}a_{1j}x_j+\lambda_2\sum_{j=1}^{n}a_{2j}x_j\right)}\right]\geq\gamma \qquad (4.1)$$

Case (ii): Let $(\tilde{a}_{i1},\tilde{a}_{i2})$ follow the proposed distribution in (2.1), (2.2) and consider the chance constraint in (1.3):

$$P_r\left(\tilde{a}_{i1}x_1+\tilde{a}_{i2}x_2+\sum_{j=3}^{n}a_{ij}\,x_j\leq b_i\right)\geq\gamma_i \quad , \qquad i=1,2,\ldots,m$$

from the relationship in (3.2), the above constraints are equivalent to the following deterministic ones:

$$\begin{aligned}c_1(1+\alpha)&\left[\lambda_2x_1\left(1-e^{\frac{-\lambda_1}{x_1}\left(b_i-\sum_{j=3}^{n}a_{ij}x_j\right)}\right)-\lambda_1x_2\left(1-e^{\frac{-\lambda_2}{x_2}\left(b_i-\sum_{j=3}^{n}a_{ij}x_j\right)}\right)\right]\\&-\alpha\left\{c_2\left[\lambda_2x_1\left(1-e^{\frac{-2\lambda_1}{x_1}\left(b_i-\sum_{j=3}^{n}a_{ij}x_j\right)}\right)\right.\right.\\&\left.-2\lambda_1x_2\left(1-e^{\frac{-\lambda_2}{x_2}\left(b_i-\sum_{j=3}^{n}a_{ij}x_j\right)}\right)\right]\\&+c_3\left[2\lambda_2x_1\left(1-e^{\frac{-\lambda_1}{x_1}\left(b_i-\sum_{j=3}^{n}a_{ij}x_j\right)}\right)-\lambda_1x_2\left(1-e^{\frac{-2\lambda_2}{x_2}\left(b_i-\sum_{j=3}^{n}a_{ij}x_j\right)}\right)\right]\\&\left.-c_1\left[\lambda_2x_1\left(1-e^{\frac{-2\lambda_1}{x_1}\left(b_i-\sum_{j=3}^{n}a_{ij}x_j\right)}\right)-2\lambda_1x_2\left(1-e^{\frac{-2\lambda_2}{x_2}\left(b_i-\sum_{j=3}^{n}a_{ij}x_j\right)}\right)\right]\right\}\geq\gamma_i \quad , \quad i=1,2,\ldots,m \qquad (4.2)\end{aligned}$$

5. Numerical example

Convert the following chance constraints to its equivalent deterministic ones:

(i) $P_r\left(x_1+2x_2\geq\tilde{b}_1\,,3x_1+x_2\leq\tilde{b}_2\right)\geq0.9$
(ii) $P_r(\tilde{a}_1x_1+\tilde{a}_2x_2+5x_3\leq100)\geq0.9$

$(\tilde{b}_1,\tilde{b}_2)$ follow proposed distribution with $\rho=0.2$, $\lambda_1=0.125$, $\lambda_2=0.200$ respectively, also $(\tilde{a}_{i1},\tilde{a}_{i2})$ follow proposed distribution with $\rho^{\backslash}=0.1$, $\lambda_1^{\backslash}=0.4$, $\lambda_2^{\backslash}=0.20$.

Solution:

(i) From (2.3), then $\alpha=4\rho=4(0.2)=0.8$

$$P_r\left(x_1+2x_2\geq\tilde{b}_1\,,3x_1+x_2\leq\tilde{b}_2\right)$$

$$=\int_{0}^{x_1+2x_2}\int_{3x_1+x_2}^{\infty}\lambda_1\lambda_2e^{-(\lambda_1\tilde{b}_1+\lambda_2\tilde{b}_2)}\left\{1+\alpha\left(2e^{-\lambda_1\tilde{b}_1}-1\right)\left(2e^{-\lambda_2\tilde{b}_2}-1\right)\right\}d\tilde{b}_1\,d\tilde{b}_2$$

$$\longrightarrow$$

$$\lambda_2(1+\alpha)(3x_1+x_2)\left(e^{-\lambda_1(x_1+2x_2)}-1\right)+\lambda_2(3x_1+x_2)\left(e^{-2\lambda_1(x_1+2x_2)}-1\right)\geq0.9$$

$$\longrightarrow$$

$$(1.08x_1 + 0.36x_2)\left(e^{-(0.125x_1+0.25x_2)} - 1\right) + (0.6x_1 + 0.2x_2)\left(e^{-(0.25x_1+0.5x_2)} - 1\right) \geq 0.9 \quad (1)$$

(ii) From (2.3), we have $\alpha^{\backslash} = 4\rho^{\backslash} = 4(0.1) = 0.4$, in turn

$$P_r(\tilde{a}_1x_1 + \tilde{a}_2x_2 + 5x_3 \leq 100) = P_r(\tilde{z} \leq 100 - 5x_3) \quad (2)$$

from (3.2) and substituting in (2), we have:

$$\left(\frac{0.7}{0.1x_1-0.2x_2}\right)\left[0.2x_1\left(1-e^{\frac{-(40-2x_3)}{x_1}}\right) - 0.4x_2\left(1-e^{\frac{-(20-x_3)}{x_2}}\right)\right] - 0.4\left\{\left(\frac{1}{0.2x_1-0.8x_2}\right)\left[0.2x_1\left(1-e^{\frac{-(80-4x_3)}{x_1}}\right)\right.\right.$$
$$\left.-0.8x_2\left(1-e^{\frac{-(20-x_3)}{x_2}}\right)\right] + \left(\frac{1}{x_1-x_2}\right)\left[x_1\left(1-e^{\frac{-(40-2x_3)}{x_1}}\right) - x_2\left(1-e^{\frac{-(40-2x_3)}{x_2}}\right)\right] - \left(\frac{1}{x_1-2x_2}\right)\left[x_1\left(1-e^{\frac{-(40-2x_3)}{x_1}}\right)\right.$$
$$\left.\left.-2x_2\left(1-e^{\frac{-(40-2x_3)}{x_2}}\right)\right]\right\} \geq 0.9 \quad (3)$$

It is noted that the equivalent deterministic constraints in (1), (3) are nonlinear and may be approximated to linear constraints.

References

[1] Albert, W. and Ingram, O., *A Multivariate Exponential Distribution*, Mathematical Note no. 450, Mathematics research laboratory, Stanford university, California, 1966. https://doi.org/10.21236/ad0634335

[2] Alvin C. Rencher, W. F. Christensen, *Methods of Multivariate Analysis*, Wiley-Interscience, New York, 2012. https://doi.org/10.1002/9781118391686

[3] Biswal, M., Biswal, N. and Li, D., Probabilistic Linear Programming Problems with Exponential Random Variables: A Technical Note, *European Journal of Operational Research*, **111** (1998), 589-597. https://doi.org/10.1016/s0377-2217(97)90319-2

[4] Charnes, A. and Cooper, W., Chance-Constrained Programming, *Management Science*, **6** (1) (1959), 73-79. https://doi.org/10.1287/mnsc.6.1.73

[5] Charnes; Cooper; Karwan and Wallace, A Chance-Constrained Goal Programming Model to Evaluate Response Resources for Marine Pollution Disasters, *Journal of Environmental Economics and Management*, 6 (1979), 244-274. https://doi.org/10.1016/0095-0696(79)90007-x

[6] Charnes; Cooper; Kirby and Raike, Selected Recent Developments in Chance–Constrained Programming, The 41st National Meeting of ORSA in New Orleans, 1972.

[7] El-Dash, A., Chance-Constrained and Nonlinear Goal Programming, Ph.D. Thesis, Applied Mathematics Дepartment, North Wales Уniversity, UK, 1984.

[8] El-Dash, A., Two Probabilistic Models for Solving an Inventory Oxygen Problem, *Journal of the Operational Research Society*, **40** (11) (1989), 961-969. https://doi.org/10.1057/jors.1989.169

[9] El-Dash, A., Chance Constrained Programming with Generalized Exponential Distributed Random Parameters, *IJRRAS*, **37** (1) (2018).

[10] El-Dash, A. and Hafez, N., Joint Chance-Constrained Programming with Generalized Exponential Random Parameters, *IJRRAS*, **38** (1) (2019).

[11] El-Dash, A., Probabilistic Linear Problems with Bivariate Exponential Distributed Random Parameters, *IJRRAS*, **41** (1) (2019).

[12] El-Dash, A., Chance-constrained Programming with Dependent Exponential Distributed Random Parameters, to appear.

[13] Geletu, A. Et al., Advances and Application of Chance-Constrained Approaches to Systems Optimization under Uncertainly, *International Journal of Systems Science* **44** (2012) 1209- 1232. https://doi.org/10.1080/00207721.2012.670310

[14] Hafez, N., A Joint Chance Constrained Programming Approach and Its Applications, Ph.D. thesis, Math. and Statistics Department, Helwan University, Egypt, 2018.

[15] Hafez, N., El-Dash, A. and Al-Behery, Chance Constrained Programming with Independent or Dependent Exponential Input Coefficient, *IJRRAS*, **34** (2) (2018).

[16] Hafez, N., El-Dash, A. and Al-Behery, Joint Chance Constrained Programming with Dependent Parameters, *Advances and Application in Statistics Journal*, **53** (1) (2018), 1-12. https://doi.org/10.17654/as053010001

[17] Heinz, K., *Statistics for Business and Economics*, Third edition, Harper-Collins college publishers, 1994.

[18] Ismail, M., El Hefnawy, A. and Saad, A., New Deterministic Solution to a Chance Constrained Linear Programming Model with Weibull Random Coefficients, *Future Business Journal*, **4** (1) (2018), 109-120. https://doi.org/10.1016/j.fbj.2018.02.001

[19] Jagannathan, R., Chance-Constrained Programming with Joint Constraints, *Journal of Operation Research*, **22** (2) (1974), 358-372. https://doi.org/10.1287/opre.22.2.358

[20] Linderoth, J., Chance-Constrained Programming, IE 495-Lecture 22, April 2003.

[21] Sengupta, J., Chance-Constrained Linear Programming with Chi-Square Type Deviates, *Management Science*, **14** (3) (1972), 337-349. https://doi.org/10.1287/mnsc.19.3.337

[22] Sengupta, J., *Stochastic Programming Methods and Applications*, North Holland Publishing Company, Amsterdam.

[23] Sengupta, J. and Tintner, G., A Review of Stochastic Linear Programming, *Review of the International Statistical Institute*, **39** (1971), 197. https://doi.org/10.2307/1402174

[24] Sibuya, M., Bivariate extreme statistics, I, *Annals of the Institute of Statistical Mathematics*, **11** (1960), 195 - 210. https://doi.org/10.1007/bf01682329

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