International Journal of Contemporary Mathematical Sciences
Vol. 15, 2020, no. 4, 199 – 205
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ijcms.2020.91449

An Approach to Probabilistic Linear Goal Programming (PLGP)

Afaf El-Dash

Dept. of Mathematics and Applied Statistics
Faculty of Commerce, Helwan University, Cairo, Egypt

Abstract

In this paper, a suggested approach to solve (PLGP) problems is presented, when some (or all) random aspiration levels $\tilde{b}_i$ follow generalized exponential distributions $\mathbf{GE}_{(\lambda_i,\mu_i,\alpha_i)}$. This approach allows to obtain: i) best compromise solution of (PLGP) problem, ii) upper limits of the random deviational variables $\tilde{d}_i^-, \tilde{d}_i^+$ and its probabilities, and iii) the actual tolerance measures γ_i^* (or the actual risk measures $R^* = (1-\gamma_i^*)$). Finally a numerical example is presented to illustrate the procedure steps.

Keywords: Chance Constrained Programming (CCP), $\mathbf{GE}_{(\lambda_i,\mu_i,\alpha_i)}$, Linear Goal Programming (LGP), Probabilistic Programming (PP), Stochastic Programming (SP).

1. Introduction

Up to now, there are many areas of (GP) which have not been completely researched, such as (PLGP).

The (LGP) model becomes a (PLGP) model when some or all of the parameters are random variables with certain distributions [5, 12]. The (PLGP) technique is one of the most important techniques for optimal decision making under uncertainty. Where there are many problems in practical applications of (GP) having random variable parameters [5, 6]. Charnes, Cooper, Neihaus and Sholtz (1968) have jointly developed a manpower planning model which considers the effects of Markov process from period to period

[3]. Contini (1968) used generalized inverse method to study (CCGP) problems, when the vector of targets $\tilde{b}_i$ represents random variables having a normal distributions [4]. Lee (1972) presented some applications of (PLGP) technique in management and economic sectors. Also Keown, Keown and Taylor III (1978) presented some applications in banking sector [11]. El-Dash (1984) presented the mathematical definition of random deviational variables and approach to transform (PLGP) model to another deterministic one, when some random aspiration levels follow exponential distributions with one or two random parameters [5]. As yet, the studies were introduced to solve (PGP) problems still very few.
In this paper the El-Dash approach of (PLGP) is developed when some aspiration levels are random variables and follow $\mathbf{GE}_{(\lambda_i,\mu_i,\alpha_i)}$ [7, 8].

2. PLGP Model

Let the general LGP model as following:

Find X, d^-, d^+, such that [10, 9]:

$$\text{Lexico. Min. } A = \{g_1(d^-,d^+), g_2(d^-,d^+), \dots, g_t(d^-,d^+), \dots, g_k(d^-,d^+)\} \tag{2.1}$$

$$\text{S.T.} \quad \sum_{j=1}^{n} a_{ij}x_j + d_i^- - d_i^+ = b_i \quad , \quad i = 1,2,\dots,M \tag{2.2}$$

$$x_j\ , d_i^-\ , d_i^+ \geq 0 \quad , \quad d_i^-.\, d_i^+ = 0 \quad , \quad i = 1,2,\dots,M \tag{2.3}$$

Where $g_t(d^-,d^+)$ is the objective function with priority t, $t = 1,2,\dots k$ and b_i, x_j, d_i^-, d_i^+ are aspiration levels, decision variables, and under and over deviational variables respectively, $i = 1,2,\dots,M$,
$j = 1,2,\dots,n$.
When some (or all) b_i are random variables with a certain probability distributions and denoted by $\tilde{b}_i$, in turn, the goals in (2.2) may be classified to 3 sets of goals:

$$G_1: \quad \sum_{j=1}^{n} a_{ij}x_j - \tilde{d}_i^+ = \tilde{b}_i \quad , \quad i = 1,2,\dots,m_1 \tag{2.4}$$

$$G_2: \quad \sum_{j=1}^{n} a_{ij}x_j + \tilde{d}_i^- = \tilde{b}_i \quad , \quad i = m_1{+}1, m_1{+}2,\dots,m_2 \tag{2.5}$$

$$G_3: \quad \sum_{j=1}^{n} a_{ij}x_j + d_i^- - d_i^+ = b_i \quad , \quad i = m_2+1, m_2+2,\dots,M \tag{2.6}$$

G_1, G_2 are probabilistic goal's sets, where $\tilde{d}_i^+, \tilde{d}_i^-$ are random over and under deviational variables [5] and:

$$\tilde{d}_i^+ = \max\{0, \textstyle\sum_{j=1}^{n} a_{ij}x_j - \tilde{b}_i\} \quad , \quad i = 1,2,\dots,m_1 \tag{2.7}$$

$$\tilde{d}_i^- = \max\{0, \tilde{b}_i - \sum_{j=1}^{n} a_{ij}x_j\} \quad , \quad i = m_1+1, m_1+2, \dots, m_2 \tag{2.8}$$

In this paper, $\tilde{b}_i \sim GE_{(\lambda_i,\mu_i,\alpha_i)}$ with cumulative function $F(b_i)$ and its inverse function $F^{-1}(b_i)$ as following [7, 8]:

$$F(\tilde{b}_i) = \left[1 - e^{-(b_i-\mu_i)/\lambda_i}\right]^{\alpha_i} , \quad \tilde{b}_i > \mu_i \quad , \quad i = 1,2, \dots, m_2 \tag{2.9}$$

$$F^{-1}(\tilde{b}_i) = \left[\mu_i - \lambda_i \ \ln\left(1 - \tilde{b}_i^{\ 1/\alpha_i}\right)\right] \quad , \quad i = 1,2, \dots, m_2 \tag{2.10}$$

In next section, the G_1, G_2 are transformed to deterministic ones [5, 7].

3. Transformation G_1 and G_2 to Deterministic Goals

Let a given certain tolerance measures γ_i , $0 \le \gamma_i < 1$, $i = 1,2, \dots, m_2$. The goals (2.4),(2.5) can be written as following chance constraints [13]:

$$P_r\left(\sum_{j=1}^{n} a_{ij}x_j \le \tilde{b}_i\right) = \gamma_i \quad \longrightarrow \tag{2.11}$$

$$\sum_{j=1}^{n} a_{ij}x_j = F^{-1}(1-\gamma_i) \quad , \ i = 1,2, \dots, m \tag{2.12}$$

also

$$P_r\left(\sum_{j=1}^{n} a_{ij}x_j \ge \tilde{b}_i\right) = \gamma_i \quad \longrightarrow \tag{2.13}$$

$$\sum_{j=1}^{n} a_{ij}x_j = F^{-1}(\gamma_i) \quad , \ i = m_1+1, m_1+2, \dots, m_2 \tag{2.14}$$

<u>**Note:**</u> the equation in (2.11),(2.13) may be inequalities $\ge \gamma_i$ or $\le \gamma_i$.

The constraints in (2.12),(2.14) are transformed to deterministic goals as following [5, 9]:

$$\sum_{j=1}^{n} a_{ij}x_j - d_i^+ = F^{-1}(1-\gamma_i) \ , \ i = 1,2, \dots, m_1 \tag{2.15}$$

$$\sum_{j=1}^{n} a_{ij}x_j + d_i^- = F^{-1}(\gamma_i) \quad , \quad i = m_1+1, m_1+2, \dots, m_2 \tag{2.16}$$

it is noted that the above constraints are linear.
Here

$$d_i^+ = \max\left\{0, \sum_{j=1}^{m} a_{ij}x_j - F^{-1}(1-\gamma_i)\right\} \quad , \quad i = 1,2, \dots, m_1 \tag{2.17}$$

$$d_i^- = \max\left\{0, F^{-1}(\gamma_i) - \sum_{j=1}^{n} a_{ij}x_j\right\} \quad , \quad i = m_1+1, m_1+2, \dots, m_2 \tag{2.18}$$

to satisfy the constraints (2.11),(2.13), that is required :

$$\min d_i^+ \quad , \ i = 1,2, \dots, m_1 \tag{2.19}$$

$$\min d_i^- \quad , \; i = m_1{+}1, m_1{+}2, \dots, m_2 \tag{2.20}$$

In turn, the deterministic (LGP) model (2.1),(2.6),(2.15),(2.16), can be solved by sequential method or modified simplex method [9, 10] and determine best compromise solution $\left(x^*, d^{+^*}, d^{-^*}\right)$. The following theorem states the relationship between $\tilde{d}_i^+, \tilde{d}_i^-$ and $d_i^{+^*}, d_i^{-^*}$ respectively.

Theorem.

Assume that $d_i^+ > 0 \quad , \; i = 1,2,\dots,m_1$ or $d_i^- > 0 \quad , \; i = m_1{+}1, m_1{+}2, \dots, m_2$ are the upper limits of $\tilde{d}_i^+$ or $\tilde{d}_i^-$ respectively, then:

1) $$\min \; P_r\left(0 \le \tilde{d}_i^+ < d_i^+\right) = \left[1 - (1-\gamma_i)^{1/\alpha_i}\right]\left[1 - e^{-d^{+^*}/\lambda_i}\right] \quad , \; i = 1,2,\dots,m_1 \tag{3.1}$$

and the actual tolerance measures

$$\gamma_i^* = \gamma_i - \left[1 - (1-\gamma_i)^{1/\alpha_i}\right]\left[1 - e^{-d^{+^*}/\lambda_i}\right] \quad , \; i = 1,2,\dots,m_1 \tag{3.2}$$

2) $$\min \; P_r\left(0 \le \tilde{d}_i^- < d_i^-\right) = \left(1 - \gamma_i^{1/\alpha_i}\right)\left(e^{d^{-^*}/\lambda_i} - 1\right) \quad , \; i = m_1{+}1, m_1{+}2, \dots, m_2 \tag{3.3}$$

and the actual tolerance measures

$$\gamma_i^* = \gamma_i - \left\{\left(1 - \gamma_i^{1/\alpha_i}\right)\left(e^{d^{-^*}/\lambda_i} - 1\right)\right\} \quad , \; i = m_1{+}1, m_1{+}2, \dots, m_2 \tag{3.4}$$

in turn

$$R^* = 1 - \gamma_i^* \quad , \; i = 1,2,\dots,m_2$$

Proof.

The function $P_r\left(0 \le \tilde{d}_i^+ < d_i^+\right) \; , \; i = 1,2,\dots,m_1$ or $P_r\left(0 \le \tilde{d}_i^- < d_i^+\right) \; , \; i = m_1{+}1, m_1{+}2, \dots, m_2$ are monotonic increasing functions of d_i^+ or $d_i^- > 0$ respectively [5], in turn:

1) $$\min \; P_r\left(0 \le \tilde{d}_i^+ < d_i^+\right) = P_r\left(0 \le \tilde{d}_i^+ < d_i^{+^*}\right)$$

$$= \int_{F^{-1}(1-\gamma_i)}^{F^{-1}(1-\gamma_i)+d_i^{+^*}} f\left(\tilde{b}_i\right) d\tilde{b}_i$$

$$= \left\{1 - e^{\ln\left[1-(1-\gamma_i)^{\frac{1}{\alpha_i}} - d_i^{+^*}\right]}\right\} - \left\{1 - e^{\ln\left[1-(1-\gamma_i)^{\frac{1}{\alpha_i}}\right]}\right\}$$

$$= \left(1 - e^{-d^{+^*}/\lambda_i}\right) - (1-\gamma_i)^{\frac{1}{\alpha_i}}\left(1 - e^{-\frac{d^{+^*}}{\lambda_i}}\right) \quad , \quad i = 1,2,\dots,m_1$$

$$= \left[1 - (1-\gamma_i)^{1/\alpha_i}\right]\left[1 - e^{-d^{+^*}/\lambda_i}\right] \quad , \; i = 1,2,\dots,m_1$$

2) by same way:

$$\min \quad P_r\left(0 \le \tilde{d}_i^- < d_i^-\right) = P_r\left(0 \le \tilde{d}_i^- < d_i^{-*}\right)$$

$$= \int_{F^{-1}(\gamma_i)-d_i^{-*}}^{F^{-1}(\gamma_i)} f\left(\tilde{b}_i\right) d\tilde{b}_i = \left\{1 - e^{\ln\left(1-\gamma_i^{1/\alpha_i}\right)}\right\} - \left\{1 - e^{\ln\left(1-\gamma_i^{1/\alpha_i}\right)+d^{-*}/\lambda_i}\right\}$$

$$= \left(1 - \gamma_i^{1/\alpha_i}\right)\left(e^{d^{-*}/\lambda_i} - 1\right) \qquad , \quad i = m_1+1, m_1+2, \dots, m_2$$

and actual tolerance measures

$$\gamma_i^* = \gamma_i - \left[\left(1 - \gamma_i^{\frac{1}{\alpha_i}}\right)\left(e^{\frac{d^{-*}}{\lambda_i}} - 1\right)\right] \qquad , \quad i = m_1+1, m_1+2, \dots, m_2$$

4. A Numerical Example

Consider the following constraints:

$$x_1 + x_2 \le 5 \qquad (1)$$
$$3x_1 + 4x_2 \ge 24 \qquad (2)$$
$$x_1 + 2x_2 \le \tilde{b} \qquad (3)$$
$$x_1, x_2 \ge 0 \quad , \quad \tilde{b} \sim GE_{(\lambda=3,\mu=1,\alpha=2)} \qquad (4)$$

The decision maker wants to construct deterministic LGP model to satisfy the following objectives according to its priorities.
First priority: satisfy constraints (1),(2),
Second priority: satisfy constraint (3) with tolerance measure $\gamma = 0.9$

Find $x_1, x_2, d_1^+, d_2^-, d_3^+$ such that:

$$\text{Lexico.} \quad \min A = \{a_1, a_2\} = \{(d_1^+ + d_2^-)(d_3^+)\}$$
$$\text{S.T.} \quad x_1 + x_2 - d_1^+ = 5$$
$$3x_1 + 4x_2 + d_2^- = 24$$
$$x_1 + 2x_2 - d_3^+ = F^{-1}(0.10) = 2.1404$$
$$x_1, x_2, d_1^+, d_2^-, d_3^+ \ge 0$$

The best compromise solution of the above model by using sequential method [9, 10] is:

$$\{a_1^* = 1, a_2^* = 9.86\} , \quad x_1^* = 0, x_2^* = 6, d_1^+ = 1, d_2^- = 0, d_3^+ = 9.86$$

in turn, the actual tolerance measure:

$$\gamma_3^* = \gamma_i - \left[1 - (1-\gamma_i)^{1/\alpha_i}\right]\left[1 - e^{-d_3^+/\lambda}\right]$$

$$= 0.9 - [1 - 0.68377][0.96262]$$

$$= 0.59559 \simeq 0.6$$

5. Conclusions

This paper presents a new approach to transform (PLGP) model to deterministic (LGP) model, when some aspiration levels follow $\mathbf{GE}_{(\lambda_i,\mu_i,\alpha_i)}$, in turn, it is solved by sequential method or modified simplex method. This approach allows to determine the best compromise solution and the actual tolerance and risk measures respectively.

References

[1] Al-Shammari, Minwir, Masri, Hatem (Eds.), *Multiple Criteria Decision Making in Finance, Insurance and Investment*, Springer, 2015. https://doi.org/10.1007/978-3-319-21158-9

[2] Belaid, Abdelaziz and Davide, Stochastic Goal Programming Model: Theory and Applications, *Journal of Multi-Criteria Decision Analysis*, **19** (2012), 185-200. https://doi.org/10.1002/mcda.1466

[3] A. Charnes, W. W. Cooper, R. J. Niehaus, A Goal Programming Model for Manpower Planning, Mang. Sci. Research report No. 115, Carnegie Mellon University, 1967. https://doi.org/10.21236/ad0664501

[4] Contini, G.: Stochastic Approach to Goal Programming, *Operations Research*, **16** (1968), 576-586. https://doi.org/10.1287/opre.16.3.576

[5] El-Dash, A., Chance-Constrained and Nonlinear Goal Programming, Ph.D. Thesis, Applied Mathematics Department, North Wales university, UK (1984).

[6] El-Dash, A. et al., Stochastic Goal Programming for Repair and Maintenance of Flying System, Proceeding of the 16th international conference for statistics, computer science, social and demographic research, Ain Shams University, **1** (1991), 345-366.

[7] El-Dash, A., Chance Constrained Programming (CCP) with Generalized Exponential Distributed Random Parameters, *IJRRAS*, **37** (1) (2018).

[8] Gupta, R. and Kundu, D., Theory & Methods: Generalized exponential distributions, *Australian and New Zealand Journal of Stat.*, **41** (2) (1999), 137-188.

[9] Ignazio, J., *Linear Programming in Single and Multiple-Objective Systems*, Prentice-Hall, Inc. London, 1982.

[10] Lee, S., *Goal Programming for Decision Analysis*, Auerbach Publishers Inc., New York, 1972.

[11] Keown, Keown and Taylor, Chance-Constrained Goal Programming Model for Bank Liquidity Management,, *Decision Sciences*, January (1978), 93-106. https://doi.org/10.1111/j.1540-5915.1978.tb01369.x

[12] Sengupta, J., *Stochastic Programming Methods and Applications*, North Holland Publishing Company, Amsterdam (1972).

[13] Tetsuzo, T., Tamaki, T., Mosahiz, I., *Multi-Objective Programming and Goal-Programming*, Springer, Japan, 2003. https://doi.org/10.1007/978-3-540-36510-5

Received: July 1, 2020; Published: July 20, 2020