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The Gap g_n Between Two Consecutive Primes Satisfies $g_n = \mathcal{O}({p_n}^{2/3})$

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Abstract

The following is proven using arguments that do not revolve around the Riemann Hypothesis or Sieve Theory. If p_n is the n^{th} prime and $g_n = p_{n+1} - p_n$, then $g_n = \mathcal{O}({p_n}^{2/3})$.

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1 Introduction

The study of maximal gaps between consecutive primes is an important subject that is actively pursued and the Bertrand's postulate [10] is one of its first consequences. In 1850, Chebyshev proved the Bertrands postulate [14], and P. Erdös presented a simplified proof in 1932 [13]. Strong results were also obtained in the generalizations of Bertrand's Postulate. In 2006, M. El Bachraoui proved the existence of a prime in the interval $[2n, 3n]$ [3]. In 2011, Andy Loo exhibited a proof that shows not only the existence of a prime between $3n$ and $4n$, but also the infinitude of the number of primes in this interval when n goes to infinity [11]. Pierre Dusart gave the best known result in this category when he improved in 2016 his previous work by showing that there is a prime between x and $(x + x/(25\log^2 x))$ for $x \geq 468991632$ [2].

"On 25*th* October 1920 G. H. Hardy read Cramér's paper *"On the distribution of primes"* to the Cambridge Philosophical Society. Here Cramér develops a statistical approach to this question showing that for any $\epsilon > 0$

$$p_{n+1} - p_n = O(p^\epsilon)$$

for 'most' p_n: in fact for all but at most $x^{1-3\epsilon/2}$ of the primes $p_n \leq x$."[4]. As a result of the Prime Number Theorem alone, we have $p_{n+1} - p_n < \epsilon p_n$ for all $\epsilon > 0$.
By the Prime Number Theorem with error term, we obtain

$$p_{n+1} - p_n < \tfrac{p_n}{(\log p_n)^c} \quad \text{for some positive constant } c.$$

Significant works have been done on the upper bound of the gap between consecutive primes by various authors without assuming an unproved hypothesis. Hoheisel was the first to show in 1930 the existence of a constant $\delta > 0$ (mainly $\delta = 1/33000$) such that $p_{n+1} - p_n = O({p_n}^{1-\delta})$ [7]. Heilbronn [6], and Tchudakoff [16], both improved on the value of δ. Ingham [9] made a significant progress that contributed to the first solutions surrounding the problem of existence of a prime between two consecutive cubes.

Based on observations uniquely centered on the midpoint m of two consecutive primes p_n, p_{n+1} and the largest odd multiple of p_n not exceeding m^2, we exercise basic proof techniques to show that the gap g_n between two consecutive primes satisfies $g_n = O({p_n}^{2/3})$. It is indeed shown precisely that

$${g_n}^3 < 16{p_n}^2.$$

Better results are obtained under the assumption of the Riemann Hypothesis. Harald Cramér proved that if the Riemann hypothesis holds, then the gap g_n satisfies $g_n = O(\sqrt{p_n}\log p_n)$. Results of the form $g_n = \mathcal{O}({p_n}^\theta)$ with different $\theta < 1$ were given in the past. Among these where θ is close to $1/2$ are namely from [8] $\theta = 7/12$, Dr Brown gave an alternative proof, [5] $\theta = 7/12$, [12] $\theta = 1051/1920$, [15] $\theta = 6/11$, The best unconditional bound is known to Baker, Harman and Pintz, who proved the existence of x_0 such that there is a prime in the interval $[x, x + O(x^{21/40})]$ for $x > x_0$, [1].

The key ideas that allow us to give an explicit and unconditional result using methods that can be easy exercises in first course in elementary number theory can be summarized as following. The use of the midpoint m of the two consecutive primes p_n and p_{n+1}, combined with few other elementary manipulations of the largest odd multiple of p_n not exceeding m^2.

Lemma 4 establishes the relation

$$g_n < 2\sqrt{2p_n(c_n+1)}. \tag{1}$$

Proceedings from Lemma 1 to Lemma 3 lead to a modest upper bound of c_n.

$$c_n < p_n/g_n. \tag{2}$$

Theorem 1 is a combination of (1) and (2), that gives the desired result and implies the existence of a prime between two consecutive cubes for all positive integer $n > 16$.

2 Main results

Consider two consecutive primes p_n and p_{n+1}, set m to be their point fixed, once for all.
There exists a positive integer b such that $m - b = p_n$ and $m + b = p_{n+1}$.

$$(m-b)(m+b) = p_n p_{n+1}. \tag{3}$$
$$m^2 - b^2 = p_n p_{n+1}. \tag{4}$$
$$m^2 - p_n p_{n+1} = b^2. \tag{5}$$
$$m^2 - p_n p_{n+1} > 0. \tag{6}$$
$$p_n p_{n+1} < m^2. \tag{7}$$

- Let αp_n denote the largest odd multiple of p_n not exceeding m^2.
- Let βp_{n+1} denote the largest odd multiple of p_{n+1} not exceeding m^2.
- Set c_n to be the number of odd multiples of p_n between $p_n p_{n+1}$ and m^2.
- Set c_{n+1} to be the number of odd multiples of p_{n+1} between $p_n p_{n+1}$ and m^2.

Combining (7) with the previous four sentences shows that αp_n and βp_{n+1} are between $p_n p_{n+1}$ and m^2, and

$$\alpha p_n = p_n(p_{n+1} + 2c_n). \tag{8}$$
$$\beta p_{n+1} = p_{n+1}(p_n + 2c_{n+1}). \tag{9}$$
$$\beta p_{n+1} - \alpha p_n = 2(p_{n+1}c_{n+1} - p_n c_n). \tag{10}$$

Lemma 2.1. $c_{n+1} \leq c_n$.

Proof. In the contrary, suppose that $c_{n+1} > c_n$.

$$\alpha p_n = p_n(p_{n+1} + 2c_n) \qquad \text{(by (8))}. \tag{11}$$

$$\beta p_{n+1} = p_{n+1}(p_n + 2c_{n+1}) \qquad \text{(by (9))}. \tag{12}$$

$$\geq p_{n+1}(p_n + 2(c_n + 1)). \tag{13}$$

$$\geq p_{n+1}(p_n + 2c_n) + 2p_{n+1}. \tag{14}$$

$$\geq \alpha p_n + 2p_{n+1}. \tag{15}$$

$$\geq \alpha p_n + 2p_n. \tag{16}$$

$$\beta p_{n+1} > m^2 \text{ (by definition, } \alpha p_n + 2p_n > m^2.) \tag{17}$$

Observe that (17) is in contradiction with the definition of βp_{n+1}, that is the largest odd multiple of p_{n+1} not exceeding m^2; Therefore, $c_{n+1} \leq c_n$.

□

- Set $X_n = (m^2 - p_n) mod(2p_n)$
- and $X_{n+1} = (m^2 - p_{n+1}) mod(2p_{n+1})$.

Lemma 2.2. $\beta p_{n+1} - \alpha p_n = X_n - X_{n+1}$.

Proof. The value X_n gives the distance between m^2 and the largest odd multiple of p_n not exceeding m^2. Hence, $m^2 - X_n$ is the largest odd multiple of p_n not exceeding m^2; That is αp_n by definition.

$$\alpha p_n = m^2 - X_n. \tag{18}$$

Similarly, X_{n+1} represents the distance between m^2 and the largest odd multiple of p_{n+1} not exceeding m^2. Thus, $m^2 - X_{n+1}$ is the largest odd multiple of p_{n+1} not exceeding m^2; That is βp_{n+1} by definition.

$$\beta p_{n+1} = m^2 - X_{n+1}. \tag{19}$$

(18) and (19) give

$$\beta p_{n+1} - \alpha p_n = X_n - X_{n+1}. \tag{20}$$

□

Corollary 2.2.1. $\beta p_{n+1} - \alpha p_n < 2p_n$.

Proof. By Lemma 2.2

$$\beta p_{n+1} - \alpha p_n = X_n - X_{n+1}. \tag{21}$$

$$\text{Since} \quad X_n = (m^2 - p_n) mod(2p_n) < 2p_n \tag{22}$$

$$\text{and} \quad X_{n+1} = (m^2 - p_{n+1}) mod(2p_{n+1}) > 0, \tag{23}$$

$$\text{then} \quad \beta p_{n+1} - \alpha p_n < 2p_n. \tag{24}$$

□

Corollary 2.2.2. $\beta p_{n+1} - \alpha p_n = 2c_n g_n$.

Proof. It is proven unconditionally in Lemma 2.2 that

$$\beta p_{n+1} - \alpha p_n = X_n - X_{n+1}.$$

The result above is obtained without reference to $c_{n+1} = c_n$ or $c_{n+1} < c_n$; Therefore,

$$\text{if } c_{n+1} = c_n, \text{ then } \quad \beta p_{n+1} - \alpha p_n = X_n - X_{n+1}; \quad (25)$$

$$\text{Otherwise (that is if } c_{n+1} < c_n), \quad \beta p_{n+1} - \alpha p_n = X_n - X_{n+1}. \quad (26)$$

Moreover,

$$X_n - X_{n+1} = (m^2 - p_n)mod(2p_n) - (m^2 - p_{n+1})mod(2p_{n+1}). \quad (27)$$

$$X_n - X_{n+1} = ((\frac{p_n + p_{n+1}}{2})^2 - p_n)mod(2p_n) - ((\frac{p_n + p_{n+1}}{2})^2 - p_{n+1})mod(2p_{n+1}). \quad (28)$$

It is clear from (28) that $X_n - X_{n+1}$ is independent of c_n and c_{n+1} and it is solely in terms of the given consecutive primes p_n and p_{n+1}.
Therefore, the value $X_n - X_{n+1}$ does not change whether $c_{n+1} = c_n$ or $c_{n+1} < c_n$.
The previous sentence with (25) and (26) imply that
a)The quantity $\beta p_{n+1} - \alpha p_n$ if $c_{n+1} = c_n$, is the same as it would be if $c_{n+1} < c_n$.
Statement a) above combined with Lemma 2.1, imply the following.
b)The quantity $\beta p_{n+1} - \alpha p_n$ can be obtaining by letting $c_{n+1} = c_n$.

$$\text{Thus, from} \quad \beta p_{n+1} - \alpha p_n = 2(p_{n+1}c_{n+1} - p_n c_n) \quad \text{(by (10))}. \quad (29)$$

$$\text{We obtain} \quad \beta p_{n+1} - \alpha p_n = 2(p_{n+1}c_n - p_n c_n) \quad \text{(Stat b) } c_{n+1} = c_n). \quad (30)$$

$$\text{So that} \quad \beta p_{n+1} - \alpha p_n = 2c_n g_n. \quad (31)$$

□

Lemma 2.3. $c_n < p_n/g_n$.

Proof.

$$\beta p_{n+1} - \alpha p_n < 2p_n \quad \text{(Corollary 2.2.1)}. \quad (32)$$

$$\beta p_{n+1} - \alpha p_n = 2c_n g_n \quad \text{(Corollary 2.2.2)}. \quad (33)$$

$$\text{(32) and (33) imply that} \quad 2c_n g_n < 2p_n. \quad (34)$$

$$c_n < p_n/g_n. \quad (35)$$

□

Lemma 2.4. $g_n < 2\sqrt{2p_n(c_n + 1)}$.

Proof. We have $\alpha p_n = p_n(p_{n+1} + 2c_n)$ is the largest odd multiple of p_n not exceeding m^2.

$$p_n(p_{n+1} + 2c_n) + 2p_n > m^2. \tag{36}$$
$$m^2 < p_n(p_{n+1} + 2c_n + 2). \tag{37}$$
$$m^2 - p_n p_{n+1} < 2p_n(c_n + 1). \tag{38}$$
$$m^2 - p_n(2m - p_n) < 2p_n(c_n + 1). \tag{39}$$
$$m^2 - 2mp_n + {p_n}^2 < 2p_n(c_n + 1). \tag{40}$$
$$(m - p_n)^2 < 2p_n(c_n + 1). \tag{41}$$
$$(g_n/2)^2 < 2p_n(c_n + 1). \tag{42}$$
$$(g_n/2)^2 < 2p_n(c_n + 1). \tag{43}$$
$$g_n < 2\sqrt{2p_n(c_n + 1)}. \tag{44}$$

□

Theorem 2.1. $g_n = \mathcal{O}({p_n}^{2/3})$.

Proof.

By Lemma 2.3, $\quad c_n < p_n/g_n;$ That is $\quad c_n + 1 < 2p_n/g_n. \qquad (45)$

By Lemma 2.4, $\quad g_n < 2\sqrt{2p_n(c_n + 1)}. \qquad (46)$

With (45) and (46), $\quad g_n < 2\sqrt{2p_n \frac{2p_n}{g_n}}. \qquad (47)$

$$ {g_n}^3 < 16{p_n}^2. \tag{48}$$
$$ g_n \ll {p_n}^{2/3}. \tag{49}$$

□

3 Primes between two consecutive cubes for all positive integers

Let $K = 16^{1/3}$ and let $N > K^3$ (that is $N > 16$) be a positive integer.
Let p_n be the largest prime less than N^3. Then

$$p_n < N^3 < p_{n+1}. \tag{50}$$
$$p_{n+1} < Kp_n^{2/3} + p_n \quad \text{(by (48))}. \tag{51}$$
$$\text{Since} \quad p_n < N^3 \qquad \text{(by (50))}, \tag{52}$$
$$\text{then} \quad K{p_n}^{2/3} < KN^2. \tag{53}$$

(51), (52) and (53) yield $\quad p_{n+1} < N^3 + KN^2. \qquad (54)$

$$p_{n+1} < N^3 + 16^{1/3}N^2. \tag{55}$$
$$p_{n+1} < N^3 + 3N^2 + 3N. \tag{56}$$

$$N^3 < p_{n+1} < (N+1)^3 - 1 \qquad \text{for all } N > 16. \tag{57}$$

The presence of (57) and a manual verification for $N \leq 16$ complete the argument.

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