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Generalized Somewhat Λ-Sets

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Abstract

The notion of a generalized somewhat Λ-set is introduced. The basic properties of these sets are developed. It is shown that the class of generalized Λ-sets is properly contained in the class of generalized somewhat Λ-sets. Conditions sufficient for a function to preserve generalized somewhat Λ-sets are investigated.

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1 Introduction

In 1986 Maki [5] introduced the notions of a Λ-set and a generalized Λ-set. These concepts have been generalized and extended by a number of authors. For example, Ganster, Jafari, and Noiri [4] investigated pre-Λ-sets and Caldas, Jafari and Noiri [3] studied Λ-generalized sets. In this note we continue this line of investigation by introducing the concepts of a somewhat Λ-set and a generalized somewhat Λ-set The basic properties of these sets are developed. For example, it is shown that a set A is a generalized somewhat Λ-set if and only if $\Lambda_{sw}(A) \subseteq \mathrm{Cl}(A)$, where $\Lambda_{sw}(A)$ denotes the intersection of all somewhat open sets containing A. Also it is established that the class of generalized Λ-sets is properly contained in the class of generalized somewhat Λ-sets and that the generalized somewhat Λ-sets are closed under union but not under intersection. Conditions sufficient for a function to preserve generalized somewhat Λ-sets are investigated. For example, it is shown that, if $f : X \to Y$ is contra somewhat Λ-continuous and open, then $f^{-1}(B)$ is a generalized somewhat Λ-set whenever B is a generalized somewhat Λ-set.

2 Preliminaries

The symbols X and Y represent topological spaces with no separation properties assumed unless explicitly stated. All sets are considered to be subsets of topological spaces. The closure and interior of a set A are signified by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively.

Definition 2.1 *A subset A of a space X is said to be somewhat open* [2] *if $A = \emptyset$ or there exists $x \in A$ and an open set U such that $x \in U \subseteq A$. A set is called somewhat closed if its complement is somewhat open.*

The collection of all somewhat open sets in a space X will be denoted by $SW(X)$ and the intersection of all somewhat open sets in a space X containing a set A will be denoted by $\Lambda_{sw}(A)$. The collection of all open sets containing a set A will be denoted by $\Lambda(A)$. Maki [5] used the notation A^{Λ} and also the notation $\ker(A)$ (the kernel of A) is used.

Definition 2.2 *A subset A is a space X is said to be a Λ-set* [5] *if $A = \Lambda(A)$.*

Definition 2.3 *A set A is said to be a generalized Λ-set (briefly a ${}_g\Lambda$-set)* [5] *if $\Lambda(A) \subseteq F$ whenever $A \subseteq F$ and $F$is closed.*

3 Somewhat Λ-sets and the operator Λ_{sw}

Definition 3.1 *A set A is said to be a somewhat Λ-set if $A = \Lambda_{sw}(A)$.*

The following lemma gives the basic properties of the operator Λ_{sw}.

Lemma 3.2 *The following statements hold for subsets A and B of a space X and a collection $\{A_\alpha : \alpha \in \mathcal{A}\}$ of subsets of X:*

(a) $A \subseteq \Lambda_{sw}(A)$.

(b) *If $A \subseteq B$, then $\Lambda_{sw}(A) \subseteq \Lambda_{sw}(B)$.*

(c) $\Lambda_{sw}(\Lambda_{sw}(A)) = \Lambda_{sw}(A)$.

(d) *If $A \in SW(X)$, then $A = \Lambda_{sw}(A)$.*

(e) $\Lambda_{sw}(\cup\{A_\alpha : \alpha \in \mathcal{A}\}) = \cup\{\Lambda_{sw}(A_\alpha) : \alpha \in \mathcal{A}\})$.

(f) $\Lambda_{sw}(\cap\{A_\alpha : \alpha \in \mathcal{A}\}) \subseteq \cap\{\Lambda_{sw}(A_\alpha) : \alpha \in \mathcal{A}\}$.

Proof. (a), (b), (d), and (f) are immediate consequences of the definition of Λ_{sw}.

(c). Since $A \subseteq \Lambda_{sw}(A)$, it follows from (b) that $\Lambda_{sw}(A) \subseteq \Lambda_{sw}(\Lambda_{sw}(A))$. For the reverse inclusion, assume that $x \notin \Lambda_{sw}(A)$. Then there exists $U \in SW(X)$ such that $A \subseteq U$ and $x \notin U$. Then $\Lambda_{sw}(A) \subseteq U$ and hence $\Lambda_{sw}(\Lambda_{sw}(A)) \subseteq U$. Thus $x \notin \Lambda_{sw}(\Lambda_{sw}(A))$, which proves that $\Lambda_{sw}(\Lambda_{sw}(A)) \subseteq \Lambda_{sw}(A)$.

(e) Let $\{A_\alpha : \alpha \in \mathcal{A}\}$ be a family of subsets of X. Since for every $\alpha \in \mathcal{A}$, $\Lambda_{sw}(A_\alpha) \subseteq \Lambda_{sw}(\cup_{\alpha\in\mathcal{A}} A_\alpha)$, $\cup_{\alpha\in\mathcal{A}}\Lambda_{sw}(A_\alpha) \subseteq \Lambda_{sw}(\cup_{\alpha\in\mathcal{A}} A_\alpha)$. For the reverse inclusion, assume that $x \notin \cup_{\alpha\in\mathcal{A}}\Lambda_{sw}(A_\alpha)$. Then for every $\alpha \in \mathcal{A}$, $x \notin \Lambda_{sw}(A_\alpha)$. Thus for every $\alpha \in \mathcal{A}$ there exists $U_\alpha \in SW(X)$ for which $A_\alpha \subseteq U_\alpha$ and $x \notin U_\alpha$. Then $\cup_{\alpha\in\mathcal{A}} U_\alpha \in SW(X)$, $\cup_{\alpha\in\mathcal{A}} A_\alpha \subseteq \cup_{\alpha\in\mathcal{A}} U_\alpha$ and $x \notin \cup_{\alpha\in\mathcal{A}} U_\alpha$. Therefore $x \notin \Lambda_{sw}(\cup_{\alpha\in\mathcal{A}} A_\alpha)$, which proves that $\Lambda_{sw}(\cup_{\alpha\in\mathcal{A}} A_\alpha) \subseteq \cup_{\alpha\in\mathcal{A}}\Lambda_{sw}(A_\alpha)$.

4 Generalized somewhat Λ-sets

Definition 4.1 *A subset A of a space X is said to be a generalized somewhat Λ-set (briefly a ${}_g\Lambda_{sw}$-set) if $\Lambda_{sw}(A) \subseteq F$ whenever $A \subseteq F$ and F is a closed subset of X.*

Since for every set A, $\Lambda_{sw}(A) \subseteq \Lambda(A)$, every ${}_g\Lambda$-set is a ${}_g\Lambda_{sw}$-set. Also every Λ-set is a Λ_{sw}-set. The following example shows that a ${}_g\Lambda_{sw}$-set is not necessarily a ${}_g\Lambda$-set and that a Λ_{sw}-set is not necessarily a Λ-set.

Example 4.2 *Let $X = \{a, b, c\}$ have the topology $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. Since $\Lambda_{sw}(\{c\}) = \{c\}$, but $\Lambda(\{c\}) = X$, $\{c\}$ is a Λ_{sw}-set but not a Λ-set. Also $\{c\}$ is a ${}_g\Lambda_{sw}$ set but not a ${}_g\Lambda$-set.*

Theorem 4.3 *A subset A of a space X is a ${}_g\Lambda_{sw}$-set if and only if $\Lambda_{sw}(A) \subseteq$ Cl(A).*

Proof. Assume A is a ${}_g\Lambda_{sw}$-set. Since $A \subseteq \mathrm{Cl}(A)$, $\Lambda_{sw}(A) \subseteq \mathrm{Cl}(A)$. Assume $\Lambda_{sw}(A) \subseteq \mathrm{Cl}(A)$. If $A \subseteq F$, where F is closed, then $\Lambda_{sw}(A) \subseteq \mathrm{Cl}(A) \subseteq F$ and hence A is a ${}_g\Lambda_{sw}$-set.

Theorem 4.4 *If A is a ${}_g\Lambda_{sw}$-set, then $\Lambda_{sw}(A) - A$ does not contain a nonempty somewhat open set (or equivalently does not contain a nonempty open set).*

Proof. Assume A is a ${}_g\Lambda_{sw}$-set and U is a nonempty somewhat open set such that $U \subseteq \Lambda_{sw}(A) - A$. Then there exists a nonempty open set V such that $V \subseteq U$ Thus $V \subseteq X - A$ and $V \subseteq \Lambda_{sw}(A)$. Also, since $A \subseteq X - V$, which is closed, $\Lambda_{sw}(A) \subseteq X - V$ and hence $V \subseteq (X - \Lambda_{sw}(A)) \cap \Lambda_{sw}(A)$ and thus $V = \emptyset$. This contradiction implies that $\Lambda_{sw}(A) - A$ does not contain a nonempty somewhat open set.

Theorem 4.5 *Assume $\Lambda_{sw}(A)$ is open. If $\Lambda_{sw}(A) - A$ does not contain a nonempty somewhat open set, then A is a ${}_g\Lambda_{sw}$-set.*

Proof Suppose A is not a ${}_g\Lambda_{sw}$-set. Then for some closed set F, $A \subseteq F$, but $\Lambda_{sw}(A) \cap (X - F)$ is a nonempty open (and hence somewhat open) subset of $\Lambda_{sw}(A) - A$.

Corollary 4.6 *If $\Lambda_{sw}(A)$ is open, then A is a ${}_g\Lambda_{sw}$-set if and only if $\Lambda_{sw}(A) - A$ does not contain a nonempty somewhat open set (or equivalently does not contain a nonempty open set).*

The ${}_g\Lambda_{sw}$-sets are obviously closed under union. However, as the next example shows, they are not closed under intersection.

Example 4.7 *Let $X = \{a, b, c\}$ have the topology $\tau = \{X, \emptyset, \{a, b\}\}$. Let $A = \{a, c\}$ and $B = \{b, c\}$. Since the only closed set containing either A or B is X, both A and B are ${}_g\Lambda_{sw}$-sets. However, $A \cap B = \{c\}$ is not a ${}_g\Lambda_{sw}$-set.*

5 Preserving ${}_g\Lambda_{sw}$-sets

Theorem 5.1 *If $f : X \to Y$ is continuous and has the property that $\Lambda_{sw}(f(A)) \subseteq f(\Lambda_{sw}(A))$ for every ${}_g\Lambda_{sw}$-set $A \subseteq X$, then $f(A)$ is a ${}_g\Lambda_{sw}$-set whenever A is a ${}_g\Lambda_{sw}$-set.*

Proof. Let $A \subseteq X$ be a ${}_g\Lambda_{sw}$-set and assume that $f(A) \subseteq F$, where F is closed. Then $A \subseteq f^{-1}(F)$ and, since $f^{-1}(F)$ is closed, $\Lambda_{sw}(A) \subseteq f^{-1}(F)$ and hence $f(\Lambda_{sw}(A)) \subseteq F$. Since $\Lambda_{sw}(f(A)) \subseteq f(\Lambda_{sw}(A))$ by assumption, $\Lambda_{sw}(f(A)) \subseteq F$, which proves that $f(A)$ is a ${}_g\Lambda_{sw}$-set.

Example 5.2 *Let $X = \{a, b, c\}$ have the topologies $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{X, \emptyset, \{a, b\}\}$. The identity mapping $f : (X, \tau) \to (X, \sigma)$ is continuous but does not have the property that $\Lambda_{sw}(f(A) \subseteq f(\Lambda_{sw}(A))$ for every ${}_g\Lambda_{sw}$-set $A \subseteq X$.*

Definition 5.3 *A function $f : X \to Y$ is said to be contra-1-somewhat continuous* [2] *provided that for every closed set $F \subseteq Y$ such that $f^{-1}(F) \neq \emptyset$, there exists a nonempty open set $U \subseteq X$ such that $U \subseteq f^{-1}(F)$*

Theorem 5.4 [2] *A function $f : X \to Y$ is contra-1-somewhat continuous if and only if $f^{-1}(V)$ is somewhat closed for every open set $V \subseteq Y$.*

The proof of the following result is analogous to that of Theorem 5.1.

Theorem 5.5 *If $f : X \to Y$ is contra-1-somewhat continuous and has the property that $\Lambda_{sw}(f(A)) \subseteq f(\Lambda_{sw}(A))$ for every ${}_g\Lambda_{sw}$-set $A \subseteq X$, then $f(A)$ is a ${}_g\Lambda_{sw}$-set whenever A is a ${}_g\Lambda_{sw}$-set.*

Corollary 5.6 *If $f : X \to Y$ is either contra-1-continuous or continuous and has the property that $\Lambda_{sw}(f(A)) \subseteq f(\Lambda_{sw}(A))$ for every ${}_g\Lambda_{sw}$-set $A \subseteq X$, then $f(A)$ is a ${}_g\Lambda_{sw}$-set whenever A is a ${}_g\Lambda_{sw}$-set.*

Corollary 5.7 *If $f : X \to Y$ is either contra-1-continuous or continuous and $f(\Lambda_{sw}(A))$ is a Λ_{sw}-set for every ${}_g\Lambda_{sw}$-set $A \subseteq X$, then $f(A)$ is a ${}_g\Lambda_{sw}$-set whenever A is a ${}_g\Lambda_{sw}$-set.*

Corollary 5.8 *If $f : X \to Y$ is contra-continuous and has the property that $\Lambda_{sw}(f(A)) \subseteq f(\Lambda_{sw}(A))$ for every ${}_g\Lambda_{sw}$-set $A \subseteq X$, then $f(A)$ is a ${}_g\Lambda_{sw}$-set whenever A is a ${}_g\Lambda_{sw}$-set.*

Definition 5.9 *A function $f : X \to Y$ is said to be somewhat open (respectively, m-somewhat open) if $f(A)$ is somewhat open for every open (respectively, somewhat open) set A.*

Lemma 5.10 *If $f : X \to Y$ is injective and m-somewhat open, then $\Lambda_{sw}(f(A)) \subseteq f(\Lambda_{sw}(A))$ for every $A \subseteq X$.*

Proof. Let $A \subseteq X$. Then $\Lambda_{sw}(f(A)) = \cap\{W \subseteq Y : f(A) \subseteq W, W \in SW(Y)\} \subseteq \cap\{f(U) : A \subseteq U, U \in SW(X)\} = f(\cap\{U \subseteq X : A \subseteq U, U \in SW(X)\}) = f(\Lambda_{sw}(A))$.

Theorem 5.11 *If $f : X \to Y$ is contra-1-continuous, injective and m-somewhat open, then $f(A)$ is a ${}_g\Lambda_{sw}$-set whenever A is a ${}_g\Lambda_{sw}$-set.*

Lemma 5.12 *If $f : X \to Y$ is open, then $f^{-1}(Cl(B)) \subseteq Cl(f^{-1}(B))$ for every $B \subseteq Y$.*

Definition 5.13 *A function $f : X \to Y$ is said to be contra somewhat Λ-continuous if $f^{-1}(F)$ is a Λ_{sw}-set for every closed set $F \subseteq Y$.*

Theorem 5.14 *If $f : X \to Y$ is contra somewhat Λ-continuous and open, then $f^{-1}(B)$ is a ${}_g\Lambda_{sw}$-set whenever B is a ${}_g\Lambda_{sw}$-set.*

Proof. Assume B is a ${}_g\Lambda_{sw}$-set in Y. Using Theorem 4.3 and Lemma 5.12 we obtain $\Lambda_{sw}(f^{-1}(B)) \subseteq \Lambda_{sw}(f^{-1}(\mathrm{Cl}(B))) = f^{-1}(\mathrm{Cl}(B)) \subseteq \mathrm{Cl}(f^{-1}(B))$. Then by Theorem 4.3 $f^{-1}(B)$ is a ${}_g\Lambda_{sw}$-set.

Theorem 5.15 *If $f : X \to Y$ is somewhat open and has the property that $\Lambda_{sw}(f^{-1}(B)) \subseteq f^{-1}(\Lambda_{sw}(B))$ for every ${}_g\Lambda_{sw}$-set $B \subseteq Y$ and $\Lambda_{sw}(f^{-1}(B))$ is open for every ${}_g\Lambda_{sw}$-set $B \subseteq Y$,then $f^{-1}(B)$ is a ${}_g\Lambda_{sw}$-set whenever B is a ${}_g\Lambda_{sw}$-set.*

Proof Let B be ${}_g\Lambda_{sw}$-set in Y. Assume $f^{-1}(B) \subseteq F$, where F is closed in X. Since B is a ${}_g\Lambda_{sw}$-set , it follows from Theorem 4.4 that $\Lambda_{sw}(B) - B$ does not contain a nonempty somewhat open set. Suppose $\Lambda_{sw}(f^{-1}(B)) \nsubseteq F$. Then $f(\Lambda_{sw}(f^{-1}(B)) \cap (X-F)) \subseteq f(\Lambda_{sw}(f^{-1}(B)) \cap (X-f^{-1}(B)) = f(\Lambda_{sw}(f^{-1}(B)) - f^{-1}(B)) \subseteq f(f^{-1}(\Lambda_{sw}(B)) - f^{-1}(B)) = f(f^{-1}(\Lambda_{sw}(B) - B)) \subseteq \Lambda_{sw}(B) - B$. Therefore $f(\Lambda_{sw}(f^{-1}(B)) \cap (X - F))$ is a nonempty somewhat open subset of $\Lambda_{sw}(B) - B$, which is a contradiction. Thus $\Lambda_{sw}(f^{-1}(B)) \subseteq F$ which proves that $f^{-1}(B)$ is a ${}_g\Lambda_{sw}$-set.

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