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Inverse Optimization for Probabilistic Programming Problem to Minimize the Value of Poverty Measure

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Abstract

The problem of poverty is one of the most important problems which has serious and negative impacts on many countries especially developing and poor countries. Decision- makers are trying to develop plans and programs to minimize poverty levels in the areas that suffer from. (N. M. Albehery, 2019) introduced the model to minimize the poverty level in a population using the inverse optimization programming problem by determining the optimal distribution of income aids that can be given to the poor people in a population when the budget of income aids is a constant value and the introduced model had been applied using Egypt real data in 2014/2015. In this paper, the inverse optimization programming problem will be suggested to minimize the poverty level when the budget of income aids be random variable with a known distribution. The suggested model will be applied using Egypt real data in 2017/2018.

Keywords: Poverty measures and lines, mathematical programming problems, dual programming problem, stochastic and probabilistic programming problems, chance constraint programming problems and inverse optimization programming problem

1. Introduction

Many programs and plans are developed by decision makers to minimize poverty levels such as increasing the income of poor people in a population. (B. Bidani and M. Ravallion, 1992) introduced nonlinear programming model and (N. M. Albehery, 2003) suggested the deterministic and probabilistic nonlinear goal programming model to determine the optimal distribution of income aids that give to the poor people to minimize the poverty levels.

Inverse optimization problem is one of the relatively new area of research and has wide applications in many fields. The optimization problem is to find $x^{*} \epsilon$ x such that the objective function f (x, c) is optimal at x^{*}. If the coefficients of decision variables of the objective function or in the right hand side of the problem can be adjusted as little as possible, the given feasible solution becomes optimal solution. (D. Burton and P. L. Toint, 1992), (J. Zhang and Z. Liu, 1996), (R. Ahuja and J.B. Orlin, 1998), (C. Yang and J. Zhang, 1999), (C. Heuberger, 2004), (N. Mohamed, 2006), (J. Zhang and C. Xu, 2010), (Y. Jiang, X. Xian, L. Zhang and J. Zhang, 2011), (S. Jain and N. Arya, 2013), (H. A. Mohammed and N. M. Albehery, 2016) and (N. M. Albehery, 2019) discussed the inverse optimization problems through definition, finding algorithms to solve problems and applying the model using data from different fields.

In this paper, a brief review of the model that had been introduced by (N. M. Albehery, 2019) is presented in section 2. The stochastic and probabilistic programming problems is introduced in section 3. In section 4, the inverse optimization problem will be used to make the feasible possible solution of income aids optimal solution when the budget of income aids is random variable with known distribution and make the minimum value of poverty measure is optimum. In section 5, the real data sets collected in Egypt in 2017/2018 will be used to apply the model. In section 6, the conclusion is provided.

2. A brief review of the introduced model by (N. M. Albehery, 2019)

(N. M. Albehery, 2019) introduced the following mathematical programming model:

$$\left.\begin{array}{ll} \mathbf{Min}\ \mathbf{P}_{\alpha}(\mathbf{x},\mathbf{S},\mathbf{z}) = \sum_{j=1}^{m}\left(\frac{n_j}{n}\right)\sum_{i=1}^{q_j}\left(1-\frac{x_{ji}+s_{ji}}{z_j}\right)^{\alpha} & ,\alpha \geq 0,\ \ j=1,2,\dots,m \\ \mathbf{S.T} & \\ \sum_{j=1}^{m} a_{ij}\, s_j = S & ,i=1,2,\dots,n \\ s_j \geq 0 & ,j=1,2,\dots,m \end{array}\right\} \quad (1)$$

Here $\alpha = 1$, S is the budget of income aids which is constant value and fixed by the decision maker, s_j is the amount of income aids that the decision maker needs to determine to give to the poor people in area j, j=1, 2, …, m, a_{ij} is the parameter of the decision variable s_j , j=1,2,…,m in constraint i, i=1, 2, …, n, these parameters are the percentage of poor people, the percentage of the illiteracy and the percentage of dropout from education in area j, and n is the number of constraints in the problem. P(x, z) is called Foster's index where x ∈ [0, z), z is a poverty line that is the border line between poor and non-poor people, z ∈ (0, ∞) (J. Foster, J. Greer and E. Thorbock, 1984). Here:

$$\mathbf{P_{\alpha}(x,z) = \frac{1}{n}\sum_{i=1}^{q}\left(\frac{z-x_i}{z}\right)^{\alpha}}, \quad \boldsymbol{\alpha \geq 0} \qquad \textbf{(2)}$$

n is the population size, q is the number of poor people in a population. When $\alpha = 0$, the Foster's index is called the headcount ratio, when $\alpha = 1$, it is called the poverty gap ratio, When $\alpha = 2$, it is called the severity of poverty. Also, the Foster index can be expressed as a weighted sum of foster indices in m categories or groups of a population as the following:

$$\mathbf{P_{\alpha}(x,z) = \sum_{j=1}^{m}\left(\frac{n_j}{n}\right)P_{\alpha}(x^j,z)}, \quad \boldsymbol{\alpha \geq 0}, \qquad (3)$$

m is the number of categories (areas or groups of people) in a population, n_j is a number of people in category j. The introduced model in (1) had been solved and found $\left(P_{\alpha}^{*}, S_j^{*}\right), j = 1,2,\dots,m$ is the optimal solution. Also, the introduced model assumed feasible possible solution for the decision variables $\left(S_j^{0}\right), j = 1,2,\dots,m$. And the inverse optimization programming problem had been used to make $\left(P_{\alpha}^{*}, S_j^{0}\right), j = 1,2,\dots,m$ is optimal solution for the programming problem by adjusting the coefficients in the objective function $\left(c_j\right), j = 1,2,\dots,m$ by the new coefficient parameter $\left(d_j^{*}\right), j = 1,2,\dots,m$ as follows:

$$\left.\begin{array}{ll}
\mathbf{Min\ P_{\alpha_{inv}} = \|c-d\|_p} & \\
\mathbf{S.T} & \\
\quad \mathbf{\sum_{j\in\bar{J}} a_{ji}\, Y_j \geq d_i} & \mathbf{, i = 1,2,\dots,n} \\
\quad \mathbf{\sum_{j\in J} a_{ji}\, Y_j = d_i} & \mathbf{, i = 1,2,\dots,n} \\
\quad \mathbf{Y_j \geq 0} & \mathbf{, j = 1,2,\dots,m}
\end{array}\right\} \qquad \textbf{(4)}$$

Here: Y_j is the optimal dual decision variable j , $j = \mathbf{1}, \mathbf{2}, \dots, m$, $J = \{j \,|s_j^0 = 0\}$, $\bar{J} = \{j \,|s_j^0 > 0\}$, $\|c - d\|_p$ is the vector norm of degree p, where $p = 1$. The new value of $d_i^*, i = 1,2,\dots,m$ had been calculated using the following equation:

$$d_i^* = \begin{cases} \mathbf{c_j} - |\mathbf{c_j^*}| & \textbf{if} \quad \mathbf{c_j^*} > 0, \ \mathbf{s_j^0} > 0, i = 1,2,\dots,m \\ \mathbf{c_j} + |\mathbf{c_j^*}| & \textbf{if} \quad \mathbf{c_j^*} < 0, \ \mathbf{s_j^0} > 0, j = 1,2,\dots,n \\ \mathbf{c_j} & \text{O.W}, \end{cases} \qquad \textbf{(5)}$$

$$\mathbf{c_j^*} = \mathbf{c_j} - \sum_{j \in J} \mathbf{a_{ji}} \mathbf{Y_j^*} \quad , \mathbf{i} = \mathbf{1}, \mathbf{2}, \dots, \mathbf{m}, \ \boldsymbol{j} = \mathbf{1}, \mathbf{2}, \dots, \boldsymbol{n} \qquad \textbf{(6)}$$

Real data set collected in Egypt in 2014/2015 had been used to apply the above model.

In model (1), the decision maker assumed that the budget of income aids "S" is constant value, but this case is not realistic. In this paper, we will assume that the budget of income aids "S" is random variable with known distribution.

3. Stochastic and probabilistic programming problems

Mathematical Programming approaches underwent a rapid development in the last years. Stochastic Programming approach is one of these approaches where some or all of the model parameters are random variables. Randomness may exists in some or all in the right hand side input, left hand side input or in the objective function coefficients. In this paper, we are concerned with the case when the right hand side input coefficients are random.

Consider the following linear programming model:

$$\left.\begin{array}{lll} \text{Min. } f(x,c) = \sum_{j=1}^{m} c_j x_j, & j = 1,2,3,\dots,m & \\ \text{S.T. } \sum_{i=1}^{n} a_{ij}\ x_j \le b_i, & i = 1,2,\dots,I & ,j = 1,2,3,\dots, \\ \quad \sum_{i=1}^{n} a_{ij}\ x_j < \widehat{b_\iota}, & i = I+1,2,\dots,n & ,j = 1,2,3,\dots,m \\ \quad x_j \ge 0, j = 1,2,3,\dots,m & & \end{array}\right\} \qquad \textbf{(7)}$$

We refer to the stochastic parameters as $\widehat{b_\iota}$ and the mathematical linear programming model in (7) is called stochastic linear programming model. If the probability distribution of $\widehat{b_\iota}$ is known or can be approximated, the model in (7) will be probabilistic linear programming model. (Charnes and Cooper, 1959) is the first researchers who introduced Chance constrained programming technique (CCP). This technique is using to transform the stochastic model into deterministic one when the distribution of random parameter is known, then the deterministic model can be solved using appropriate mathematical programming technique. Also, the chance constrained programming technique assumed that the stochastic constraints will hold with probability at least ∂. Where ∂ is called tolerance measure (Liu, 2002) and it refers to the confidence level provided as an appropriate safety margin by the decision-maker. CCP can be used to transform the random constraint in model (7) after determining the value of ∂ as the following (El- Dash, A, 1984):

$$P\left(\sum_{i=1}^{n} a_{ij}\ x_j > \widehat{b_\iota}\right) = \partial_i \ \rightarrow \ F_i\left(\sum_{i=1}^{n} a_{ij}\ x_j\right) = 1 - \partial_i, \qquad (8)$$
$$\sum_{i=1}^{n} a_{ij}x_j = F_i^{-1}(1 - \partial_i)\,, i = I + 1, 2, 3, .., n\,, j = 1, 2, 3, .., m \qquad (9)$$

where: $\widehat{b_\iota}$ is random variable with known probability distribution, F_i is the cumulative probability distribution and F_i^{-1} is the inverse of cumulative probability distribution of $\widehat{b_\iota}$.

4. The inverse optimization for probabilistic programming problem to find the optimal distribution of income aids

(N. M. Albehery, 2019) used the inverse optimization programming problem to make the alternative distribution of income aids is the optimal distribution and make the poverty level is optimum value when the budget of income aids is given as constant value. In this paper, the inverse optimization programming problem will be used to make the alternative distribution of income aids is the optimal distribution when the budget of income aids is random variable with known distribution as the following:

First: The main objective function of our model is minimizing the poverty measure (Foster poverty index) "$P_\alpha(x, S, z)$" by increasing the income of poor people in a population as the following:

$$\mathbf{Min\ P_\alpha(x, S, z)} = \sum_{j=1}^{m}\left(\frac{n_j}{n}\right)\sum_{i=1}^{q_j}\left(1 - \frac{x_{ji}+s_{ji}}{z_j}\right)^{\alpha}\ ,\ \alpha \geq 0 \qquad (10)$$

Where**,** m is the number of areas or groups in a population, j= 1, 2, …, m, n is the population size, n_j is the size of population in area j, x_j is the mean income of poor people in area j, s_j is the amount of income aid that the decision maker needs to give to area j, z_j is the poverty line which is the border line between poor and non-poor people in area j, q_j is the number of poor people in area j.

Second: Some constraints will be defined as the following:

$$\sum_{j=1}^{m} a_{ij}\, s_j \leq \hat{S} \qquad , i = 1, 2, \dots, n \qquad (11)$$
$$s_j \geq 0 \qquad , j = 1, 2, \dots, m$$

where: $\hat{\mathbf{S}}$ is the budget of income that is random variable with known distribution. And s_j is the amount of income aids that will be determined to give to the poor people in area j, j=1, 2, …, m, $\mathbf{s_j = \sum_{i=1}^{q_j} s_{ji}}$, a_{ij} is the parameter of the decision variable s_j , j=1,2,…,m in constraint i, i=1, 2, …, n. This parameter can be defined as the percentage of poor people, the percentage of the illiteracy, the percentage of unemployment and the percentage of dropout from education in area j, and n is the number of constraints in the problem.

Third: The programming model will be probabilistic programming model and takes the following form:

$$\left.\begin{array}{ll} \mathbf{Min\ P_\alpha(x,S,z)} = \sum_{j=1}^{m}\left(\frac{n_j}{n}\right)\sum_{i=1}^{q_j}\left(1-\frac{x_{ji}+s_{ji}}{z_j}\right)^{\alpha} & ,\alpha \geq 0,\ \ j=1,2,\dots,m \\ \mathbf{S.T} & \\ \sum_{j=1}^{m} a_{ij}\, s_j \leq \hat{S} & ,i=1,2,\dots,n \\ s_j \geq 0 & ,j=1,2,\dots,m \end{array}\right\} \quad (12)$$

If $\alpha \leq 1$ the objective function is linear function and when $\alpha = 2$ the objective function is nonlinear function. To solve the probabilistic programming model in (12), the chance constrained programming technique will be used to transform the random constraint in this model after determining the value of tolerance measure $\boldsymbol{\partial}$ as defined in equations (8) and (9) is section 3. The model will be deterministic model as follows:

$$\left.\begin{array}{ll} \mathbf{Min\ P_\alpha(x,S,z)} = \sum_{j=1}^{m}\left(\frac{n_j}{n}\right)\sum_{i=1}^{q_j}\left(1-\frac{x_{ji}+s_{ji}}{z_j}\right)^{\alpha} & ,\alpha \geq 0,\ \ j=1,2,\dots,m \\ \mathbf{S.T} & \\ \sum_{j=1}^{m} a_{ij}\, s_j = \boldsymbol{F^{-1}}(1-\partial) & ,i=1,2,\dots,n \\ s_j \geq 0 & ,j=1,2,\dots,m \end{array}\right\} \quad (13)$$

where: $\boldsymbol{F^{-1}(1-\partial)}$ is the inverse of cumulative distribution of variable S.

Fourth: By solving the model in (13), we consider $(P_\alpha^*, S_j^*), j = 1,2,\dots,m$ is the optimal solution. If the decision maker wants to make an alternative feasible possible values of decision variables $(S_j^0), j = 1,2,\dots,m$ is optimal values and keeps the minimum value of the optimal objective function P_α^* as the same value. Then he wants to make $(P_\alpha^*, S_j^0), j = 1,2,\dots,m$ is the optimal solution for the problem defined in (13). The inverse optimization programming problem defined in equations (5) and (6) in section 2 will be used to adjust the coefficients in the objective function $(c_j), j = 1,2,\dots,m$ by finding the new coefficient parameter $(d_j^*), j = 1,2,\dots,m$ to make $(P_\alpha^*, S_j^0), j = 1,2,\dots,m$ optimal solution.

5. Applied example

In this paper, we will divide Egypt into urban area and rural areas, and the optimal values of the income aids will be distributed to the poor people in these areas to decrease the foster poverty measure "poverty gap ratio" $P_1(x, S, z)$. The relative poverty line will be used as the border line between poor and non-poor people in a population and it is defined as upper limit of relative poverty line; $Z_+ =$ 2/3 of the median of annual household expenditure (N. M. Albehery and T. Wang, 2011). Data collected from the Household income, expenditure and consumption survey (HIECS) for the round 2017/2018 that was presented by "Central Agency for Public Mobilization and Statistics (CAPMAS)" the government agency to collect data in Egypt will be used to apply our model. Also, we assume the budget of income aids $\hat{\mathbf{S}}$ is random variable with continues uniform distribution that takes the following form:

$f(\hat{S}) = \frac{1}{b-a}$, $a \leq \hat{S} \leq b$, let a = 50 million and b = 100 million of Egyptian pound.

The following table contains the data used to construct the probabilistic programming model in (12):

Indicators	Urban Egypt (1)	Rural Egypt (2)
Sample size = n	11452	13519
Percentage of poor people	24.58%	28.29%
Percentage of illiteracy	17.7%	32.2%
Percentage of unemployment	12.1%	9.5%
Upper limit of relative poverty line = z_+	20976.898 EP	19388.28 EP

The probabilistic model defined in (12) will be the linear programming model as the following:

$$\text{Min } P_1(x, S, z) = 0.00219\ S_1 + 0.00279\ S_2$$
$$\text{S.T.} \quad 24.58\%\ S_1 + 28.29\%\ S_2 \leq \hat{S}$$
$$17.7\% S_1 + 32.2\%\ S_2 \leq \hat{S}$$
$$12.1\%\ S_1 + 9.5\%\ S_2 \leq \hat{S}$$
$$s_j \geq 0 \quad , j = 1, 2 \quad \text{And } \hat{S} \sim U[50, 100]$$

Using the chance constrained programming technique (CCP) to transform the random constraint and let $\boldsymbol{\partial} = 0.9$ in model (13) into deterministic constraint, the model will be deterministic programming model as the following:

$$\text{Min } P_1(x, S, z) = 0.00219\ S_1 + 0.00279\ S_2$$
$$\text{S.T.} \quad 24.58\%\ S_1 + 28.29\%\ S_2 = 55$$
$$17.7\%\ S_1 + 32.2\%\ S_2 = 55$$
$$12.1\%\ S_1 + 9.5\%\ S_2 = 55 \qquad , s_j \geq 0 \quad , j = 1, 2$$

By solving the above model using Excel solver, the optimal solution will be
$\boldsymbol{S_1^* = 7.4\ million\ EGP, S_2^* = 13.01\ million\ EGP\ and\ Min\ P_1^* = 0.0053.}$
Using the dual programming problem for the above linear programming problem using the binding constraints and by solving the dual problem using Excel solver, the optimal solution of dual variables will be:
$\boldsymbol{Y_1^* = 0.00727,\ Y_2^* = 0.00228\ and\ P_1^* = 0.0053.}$
Assume the decision maker is willing to make alternative possible distribution of the income aids where the feasible possible amount of income aids that are given to urban and rural Egypt respectively is:
$\boldsymbol{S_1^0 = 8.3 million\ EGP, S_2^0 = 12\ million\ EG}P$. And he wants to keep the minimum of gap poverty ratio is optimum at $\boldsymbol{P_1^* = 0.0053.}$
Then the inverse optimization for the above linear programming problem will be:

$$\mathbf{Min\ P_{1_{inv}} = \|c - d\|}$$
$$\mathbf{S.T.} \quad \mathbf{24.58\%\ Y_1 + 28.29\%\ Y_2 = d_1}$$
$$\mathbf{17.7\%\ Y_1 + 32.2\%\ Y_2 = d_2}$$
$$\mathbf{Y_j \geq 0 \quad , j = 1, 2}$$

$d_i^*, i = 1,2$ are the optimal value of the new coefficient of decision variables in the objective function where $Y_j^*, j = 1,2$ are the optimal dual decision variables can be calculated using equations (5) and (6). The new coefficient of the decision variables of the objective function will be (0.00243, 0.00202) instead of (0.00219, 0.00279) that make the feasible possible values of the decision variables (8.3, 12) are the optimal solution for the suggested model and satisfy the minimum value of the poverty gap ratio at 0.0053.

6. Conclusion

In this paper, inverse optimization programming problem is using to make the feasible possible distribution of income aids is the optimal distribution when the budget of income aids is random variable with continuous uniform distribution to make the poverty gap ratio in urban and rural Egypt is minimum value. In our applied example, the chance constraint programming technique is used to transform the probabilistic constrained into deterministic one and the coefficients of the decision variables in the objective function are adjusted to be new coefficients.

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