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Pseudo NP-Injective Rings

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Abstract

Let R be a ring. A right R-module M is called *pseudo nonessential principally injective* (briefly, pseudo NP-*injective*) if, for each nonessential principal right ideal kR of R, any R-monomorphism from kR to M can be extended to an R-homomorphism from R to M. If R_R is pseudo NP-injective, then we call R a *right pseudo* NP-*injective ring*. In this paper, we give some characterizations and properties of right NP-injective rings.

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Keywords: pseudo NP-injective rings, NP-injective rings and P-injective rings

1. Introduction

Let R be a ring. A right R-module M is called *principally injective* (or P-*injective*) [8], if every R-homomorphism from a principal right ideal of R to M can be extended to an R-homomorphism from R to M. Equivalently, $l_M r_R(a) = Ma$ for all $a \in R$ where l and r are left and right annihilators, respectively.

In [9], Nicholson, Park, and Yousif extended this notion of principally injective rings to the one for modules. In [5], W. Junchao introduced the definition of Jcp-injective rings, a ring R is called right Jcp-injective if for each $a \in R \setminus Z_r$, any R-homomorphism from aR to R can be extended to an R-homomorphism from R to R. In [13], A right R-module M is called NP-*injective* if, for each nonessential principal right ideal kR of R, any R-homomorphism from kR to M can be extended to an R-homomorphism from

R to M. In this note we introduce the definition of right pseudo NP-injective rings and give some characterizations and properties.

Throughout this paper, R will be an associative ring with identity and all modules are unitary right $R-$modules. For right $R-$modules M and N, $Hom_R(M, N)$ denotes the set of all R-homomorphisms from M to N and $S = End_R(M)$ denotes the endomorphism ring of M. If X is a subset of M the right (resp. left) annihilator of X in R (resp. S) is denoted by $r_R(X)$ (resp. $l_S(X)$). By notation, $N \subset^{\oplus} M$ $(N \subset^e M)$ we mean that N is a direct summand (an essential submodule) of M. We denote the singular submodule and the Jacobson radical of M by $Z(M)$ and $J(M)$, respectively.

2. Pseudo NP-injective Modules

A submodule N of a right $R-$module M is *essential* in M abbreviated $N \subset^e M$, in case for every nonzero submodule L of M, we have $N \cap L \neq 0$. An element $m \in M$ is called *singular* if $r_R(m) \subset^e R$. M is called *nonsingular* if it contains no nontrivial singular element.

Definition 2.1. Let R be a ring. A right R-module M is called *pseudo nonessential principally injective* (briefly, *pseudo* NP-*injective*) if, for each nonessential principal right ideal kR of R, any R-monomorphism from kR to M can be extended to an R-homomorphism from R to M.

Lemma 2.2.

(1) Any direct summand of a pseudo NP-injective module is again pseudo NP-injective.

(2) If $k \in R$ with $kR \not\subset^e R$ and kR is pseudo NP-injective, then $kR \subset^{\oplus} R$.

Proof. (1) Let M be pseudo NP-injective, $N \subset^{\oplus} M$, $k \in R$ with $kR \not\subset^e R$ and let $\varphi : kR \to N$ be an $R-$monomorphism. Since $\iota\varphi$ is monic where $\iota : N \to M$ is the injection map, there exists an $R-$homomorphism $\hat{\varphi} : R \to M$ which is an extension of φ. Then $\pi\hat{\varphi}$ extends φ where $\pi : M \to N$ is the projection map.

(2) Since kR is pseudo NP-injective, there exists an $R-$homomorphism $\varphi : R \to kR$ such that $\varphi\iota = 1_{kR}$ where $\iota : kR \to R$ is the inclusion map. Then by [1, Lemma 5.1], ι is a split monomorphism, therefore $kR \subset^{\oplus} R$. □

Let M be a right R-module. A right R-module N is called *pseudo* NPM-*injective* if, for each nonessential principal submodule mR of M, any R-

monomorphism from mR to N can be extended to an R-homomorphism from M to N. M is called *pseudo* NPQ-*injective* if it is pseudo NPM-injective.

Corollary 2.3.

(1) Any direct summand of a pseudo NPM-injective module is again pseudo NPM-injective.
(2) If $m \in M$ with $mR \not\subset^{e} M$ and mR is pseudo NPM-injective, then $mR \subset^{\oplus} M$.
(3) N is pseudo NPM-injective if and only if N is pseudo NPX-injective for every submodule X of M.

Proof. (1) and (2) Similar to Lemma 2.2.
(3) If $xR \not\subset^{e} X$, then $xR \not\subset^{e} M$ so (3) follows. □

Example 2.4. Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ where F is a field. Then R_R is pseudo NP-injective.

Proof. It is clear that only $X_1 = \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$, $X_2 = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ and $X_3 = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ are nonzero nonessential principal right ideals of R and $X_1 \simeq X_2$.
Let $\varphi : X_3 \to R$ be an R-monomorphism. Since $R = X_3 \oplus X_1$, $\hat{\varphi} : R \to R$ define by $\hat{\varphi}(a,b) = \varphi(a)$ for every $(a,b) \in X_3 \oplus X_1$ is an R-homomorphism and $\hat{\varphi}$ extends φ. By the similar proof of X_3, we can show for X_1 and it is clear for X_2. Then R_R is pseudo NP- injective. □

Since X_3 and X_1 are direct summand of R, X_3 and X_1 are pseudo NP- injective by Lemma 2.2, and so is X_2.

Theorem 2.5. Let R be a ring. If every nonessential principal right ideal of R is projective, then every factor module of a pseudo NP-injective module is pseudo NP- injective.

Proof. Let M be pseudo NP-injective, N a submodule of M, $k \in R$ with $kR \not\subset^{e} R$, and let $\varphi : kR \to M/N$ be an $R-$monomorphism. Then there exists an R homomorphism $\hat{\varphi} : kR \to M$ such that $\varphi = \eta\hat{\varphi}$ where $\eta : M \to M/N$ is the natural R-epimorphism. If $x \in \mathrm{Ker}(\hat{\varphi})$, then $\varphi(x) = \eta\hat{\varphi}(x) = N$ so $x = 0$ which shows that $\hat{\varphi}$ is monic. Since M is pseudo NP-injective, there exists an $R-$homomorphism $\alpha : R \to M$ which is an extension of $\hat{\varphi}$ to M. Then $\eta\alpha$ is an extension of φ to R. □

Theorem 2.6. Let kR and A be right ideals of R and let $kR \oplus A$ be pseudo NP-injective. If $\sigma : kR \to A$ is an R-monomorphism, then σ splits and kR is pseudo NPQ-injective.

Proof. Define $\varphi : \sigma(kR) \to kR \oplus A$ by $\varphi(\sigma(kr)) = (kr, 0)$ for every $r \in R$. It is clear that φ is an R-monomorphism. Since $kR \oplus A$ is pseudo NP-injective and $\sigma(kR) \not\subset^e R$, there exists an R-homomorphism $\hat{\varphi} : R \to kR \oplus A$ which is an extension of φ to R. Let $\iota_2 : A \to kR \oplus A$ and $\pi_1 : kR \oplus A \to kR$ be the injection map and the projection map, respectively, and set $\lambda = \pi_1 \hat{\varphi} \iota_2$. Then $\lambda\sigma = 1_{kR}$ so σ splits. Write $A = \sigma(kR) \oplus B$. Then $kR \oplus A = kR \oplus \sigma(kR) \oplus B$ and $kR \oplus \sigma(kR)$ is pseudo NP-injective.

We now show that kR is NPQ-injective. Let $tR \not\subset^e kR$ and $\alpha : tR \to kR$ be an R-monomorphism. Define $\beta : tR \to kR \oplus \sigma(kR)$ by $\beta(tr) = (tr, \sigma\alpha(tr))$ for every $r \in R$. It is clear that β is an R-monomorphism. Note that $tR \not\subset^e R$.

Then there exists an R-homomorphism $\hat{\beta} : R \to kR \oplus \sigma(kR)$ which is an extension of β to R. Let $\pi_2 : kR \oplus \sigma(kR) \to \sigma(kR)$ be the projection map. Set $\mu = \lambda \pi_2 \hat{\beta}$.
Then μ is an extension of α to R. □

Corollary 2.7. Let mR and N be submodules of M and let $mR \oplus N$ be pseudo NPM-injective. If $\sigma : mR \to N$ is an R-monomorphism, then σ splits and mR is pseudo NPQ-injective.

3. Pseudo NP-injective Rings

If R_R is pseudo NP-injective, then we call R a *right pseudo* NP-*injective ring*.

Lemma 3.1. Let R be a ring. Then R is right pseudo NP-injective if and only if for every $k, t \in R$ with $kR \not\subset^e R$, $r_R(k) = r_R(t)$, implies that $Rt \subset Rk$.

Proof. $(\Rightarrow)$ Let $r_R(k) = r_R(t)$, $k, t \in R$ with $kR \not\subset^e R$. Define $\varphi : kR \to R$ by $\varphi(kr) = tr$ for every $r \in R$. It is clear that φ is an R-monomorphism. Then there exists an R-homomorphism $\hat{\varphi} : R \to R$ which is an extension of φ to R. Hence $t = \hat{\varphi}(1)k \in Rk$ so $Rt \subset Rk$.
$(\Leftarrow)$ Let $k \in R$ with $kR \not\subset^e R$ and let $\varphi : kR \to R$ be an R-monomorphism.

Then $r_R(\varphi(k)) = r_R(k)$ so by assumption $R\varphi(k) \subset Rk$. Write $\varphi(k) = ak$ for some $a \in R$. Define $\hat{\varphi}: R \to R$ by $\hat{\varphi}(r) = ar$ for every $r \in R$. It is clear that $\hat{\varphi}$ is an R-homomorphism and extends φ. This shows that R is pseudo NP-injective. □

Corollary 3.2. Let M be a right R-module. Then M is pseudo NPQ-injective if and only if for every $m, n \in M$ with $mR \not\subset^e M$, $r_R(m) = r_R(n)$, implies that $Sn \subset Sm$.

Recall that a ring R is *semipotent*, also call I_0*-rings*, [7] if every ideal of R not contained in $J(R)$ contains a nonzero idempotent. We call a right R-module M is a *duo module* if every submodule of M is fully invariant. Thus R is a right duo ring if and only if R_R is a duo module.

Proposition 3.3. Let R be a right duo, right pseudo NP-injective ring. Then
(1) $J(R) \subset Z(R_R)$.
(2) If R is semipotent, then $J(R) = Z(R_R)$.
(3) If $k \in R$ and kR is a simple and nonessential right ideal of R, then Rk is a simple left ideal of R.

Proof. (1) Let $a \in J(R)$. If $r_R(a) \not\subset^e R$, then $r_R(a) \cap K = 0$, for some nonzero right ideal K of R. Take $k \in K$ such that $ak \neq 0$. Since $r_R(a) \cap kR = 0$ and R is right duo, $r_R(ak) = r_R(k)$ and $akR \not\subset^e R$. Then $Rk \subset Rak$ by Lemma 3.1. Write $k = sak$, $s \in R$. It follows that $(1-sa)k = 0$ and so $k = (1-sa)^{-1}0 = 0$, a contradiction.
(2) If $Z(R_R) \not\subset J(R)$, then $Z(R_R)$ contains a nonzero idempotent e because R is semipotent. But $r_R(e) \not\subset^e R$, a contradiction.
(3) If A is a nonzero submodule of Rk and $0 \neq ak \in A$, then $Rak \subset A$. Since akR is a nonzero homomorphic image of kR, akR is simple and $akR \not\subset^e R$. Define $\varphi: akR \to R$ by $\varphi(akr) = kr$ for every $r \in R$. Since $r_R(a) \cap kR = 0$, φ is well-defined. It is clear that φ is an R-monomorphism. Then there exists an $R-$homomorphism $\hat{\varphi}: R \to R$ such that $\varphi = \hat{\varphi}\iota$ where $\iota: akR \to R$ is the inclusion map. Thus $k = \hat{\varphi}(ak) \in Rak$ it follows that $A = Rk$. □

Let M be a right $R-$module with $S = End_R(M)$. Following [9], write

$$W(S) = \{s \in S: \ Ker(s) \subset^e M\}.$$

It is known that $W(S)$ is an ideal of S.

Corollary 3.4. Let M be a duo, pseudo NPQ-injective module. Then
(1) $J(S) \subset W(S)$.
(2) If S is semipotent, then $J(S) = W(S)$.
(3) If $m \in M$ and mR is a simple and nonessential right R-module, then Sm is a simple left S-module.

Theorem 3.5. Let R be a right duo, right pseudo NP-injective ring and $k, t \in R$ with $kR \not\subset^e R$.
(1) If kR embeds into tR, then Rk is an image of Rt.
(2) If $kR \simeq tR$, then $Rk \simeq Rt$.

Proof. (1) Let $\varphi : kR \to tR$ be an R-monomorphism. Since R is right pseudo NP-injective, there exists $\hat{\varphi} : R \to R$ such that $\iota_2\varphi = \hat{\varphi}\iota_1$ where $\iota_1 : kR \to R$ and $\iota_2 : tR \to R$ are the inclusion maps. Define $\sigma : Rt \to Rk$ by $\sigma(rt) = r\hat{\varphi}(k)$ for every $r \in R$. Since $\sigma(rt) = r\hat{\varphi}(k) = r\varphi(k) \in rtR$, σ is well-defined. It is clear that σ is an R-homomorphism. Then $\hat{\varphi}(kR) \subset kR$ because R is right duo so $\varphi(kR) \not\subset^e R$. Since $r_R(\varphi(k)) = r_R(k)$, $Rk \subset R\varphi(k)$ hence $k \in R\varphi(k) \subset \sigma(Rt)$.
(2) Clear □

Corollary 3.6. Let M be a duo, pseudo NPQ-injective module and $m, n \in M$ with $mR \not\subset^e M$.
(1) If mR embeds into nR, then Rm is an image of Rn.
(2) If $mR \simeq nR$, then $Rm \simeq Rn$.

Recall that a right R-module M is called C2 [6] if, every submodule of M that is isomorphic to a direct summand of M is itself a direct summand of M. M is called C3 if whenever N and K are direct summands of M with $N \cap K = 0$ then $N \oplus K$ also a direct summand of M.

Theorem 3.7. Let R be a right duo, right pseudo NP-injective ring.
(1) If $kR \simeq eR$, $1 \neq e = e^2$, then kR is a direct summand of R.
(2) If $eR \cap fR = 0$, $1 \neq e = e^2$, $1 \neq f = f^2$, then $eR \oplus fR$ is a direct summand of R.

Proof. (1) Let $\sigma : eR \to kR$ be an R-isomorphism.
Since R is right pseudo NP-injective and $eR \not\subset^e R$, there exists an R-homomorphism $\hat{\sigma} : R \to R$ such that $\iota_2\sigma = \hat{\sigma}\iota_1$ where $\iota_1 : eR \to R$ and $\iota_2 : kR \to R$ are the inclusion maps. It follows that $kR \not\subset^e R$. Since eR is pseudo NP- injective by Lemma 2.2, kR pseudo NP-injective hence kR is a direct summand of R.

(2) Let $eR \cap fR = 0$, $1 \neq e = e^2$, $1 \neq f = f^2$. Then $eR \oplus fR = eR \oplus (1-e)fR$. If $(1-e)f(R) = 0$, then $eR \oplus fR$ is a direct summand of R. If $(1-e)f(R) \neq 0$, then $(1-e)fR \simeq fR$ and hence $(1-e)fR = gR$ for some $g^2 = g \in R$ by (1). Let $h = e + g - ge$, then $h^2 = h$ and $eR \oplus fR = hR$. □

An element u of a right R-module M is called [9] *uniform* if uR is a uniform submodule of M.

Proposition 3.8. Let R be a right duo, right pseudo NP-injective ring. If $u \in R$ is uniform with $uR \not\subset^e R$, then $M_u = \{a \in R : uR \cap r_R(a) \neq 0\}$ is the unique maximal left ideal of R containing $l_R(u)$.

Proof. Since uR is uniform, M_u is a left ideal of R. Clearly, $l_R(u) \subset M_u \neq R$. Let A be a left ideal of R containing $l_R(u)$ and $A \neq R$. If $a \in A \setminus M_u$, then $r_R(a) \cap uR = 0$ so $r_R(u) = r_R(au)$. Since R is duo, $auR \not\subset^e R$. Then by Lemma 3.1, $Ru \subset Rau$, write $u = sau$ where $s \in R$, so $1 - sa \in l_R(u) \subset A$, a contradiction. Hence $A \subset M_u$. □

Theorem 3.9. If R is right pseudo NP-injective, so is eRe for all $e^2 = e \in R$ satisfying $ReR = R$.

Proof. Write $S = eRe$ and let $r_S(k) = r_S(t)$ where $k, t \in S$ and $kS \not\subset^e S$. Then there exists a nonzero right ideal A of S such that $kS \cap A = 0$. Let $0 \neq a \in A$. Then $aR \neq 0$ is a right ideal of R. If $kR \cap aR \neq 0$, then there exists $r, s \in R$ such that $kr = as \neq 0$. Hence $krReR \neq 0$, so there exists $0 \neq b \in R$ such that $0 \neq krbe \in kRe = kS$. Then $krbe = asbe = aesbe \in aS \subset A$, a contradiction so $kR \cap aR = 0$. We now show that $r_R(k) = r_R(t)$. Let $kr = 0, r \in R$ and $1 = \sum_{i=1}^{n} a_i e b_i$ where $a_i, b_i \in R$. Then $k(era_i e) = (kr)a_i e = 0$ for each i, so $t(era_i e) = 0$ because $r_S(k) = r_S(t)$. Thus $tr = \sum_{i=1}^{n} t(era_i e)b_i = 0$ and hence $r_R(k) \subset r_R(t)$. Similarly, $r_R(t) \subset r_R(k)$. Since $kR \not\subset^e R$ and R is right pseudo NP-injective, $Rt \subset Rk$ by Lemma 3.1, it follows that $St \subset Sk$. Therefore S is right pseudo NP-injective. □

Theorem 3.10. Let R be pseudo NP-injective and $k \in R$ with $kR \not\subset^e R$. Then the following conditions are equivalent.

(1) kR is projective.
(2) kR is a direct summand of R.
(3) kR is pseudo NP-injective.

Proof. $(2) \Rightarrow (1)$ Clear.
$(1) \Rightarrow (3)$ Since the sequence $0 \to kR \to R \to kR \to 0$ splits, kR is isomorphic to a direct summand of R. Then by Lemma 2.2, kR is pseudo NP-injective.
$(3) \Rightarrow (2)$ It follows from Lemma 2.2. □

A right R-module M is said [11] to has the *summand intersection property* [SIP] if the intersection of two direct summands of M is again summand of M. The module M is said [4] to has the *summand sum property* [SSP] if the sum of two summands of M is again summand of M. We prove a similar result for a pseudo NP-injective ring**.**

Theorem 3.11. Let R be a right duo, right pseudo NP-injective ring and let $e, g \in R$ with $1 \neq e = e^2$, $1 \neq g = g^2$. Then $eR \cap gR$ and $eR + gR$ are summands of R.

Proof. Write $R = eR \oplus K$ and $R = gR \oplus L$. Since R is duo, $eR = e(gR \oplus L) = egR + eL \subset (eR \cap gR) \oplus (eR \cap L) \subset eR$, $eR \cap gR \subset^{\oplus} R$. We now write $R = eR \cap gR \oplus A$.
Then $gR = gR \cap ((eR \cap gR) \oplus A) = eR \cap gR \oplus gR \cap A$, by the Modular law. Hence $eR + gR = eR + (eR \cap gR \oplus gR \cap A) = eR \oplus gR \cap A$. Since eR and $gR \cap A$ are direct summands of R and $gR \cap A \subset^{\oplus} gR$, it follows that $gR \cap A = hR$ where $1 \neq h = h^2$. Then $eR + gR$ is a direct summand of R by Theorem 3.7. □

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