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Tribonacci and Tribonacci-Lucas Hybrid Numbers

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Abstract

In this paper, we define the Tribonacci and Tribonacci-Lucas hybrid numbers and obtain Binet's formulas and generating functions for these numbers. Then we present some Hessenberg matrices with applications to the Tribonacci and Tribonacci-Lucas hybrid numbers and show that the determinants and permanents of these Hessenberg matrices are equal to the nth Tribonacci hybrid number and nth Tribonacci-Lucas hybrid number.

Keywords: Tribonacci number, Tribonacci-Lucas number, hybrid number, Hessenberg matrix

1 Introduction

Tribonacci and Tribonacci Lucas sequences which are sequences of integer number defined by recurrence relations are well-known third order recurrence sequences in all of mathematics and generalizations of the Fibonacci sequence. Most of the authors introduced Fibonacci pattern based sequences in many ways which are known as generalization of Fibonacci sequence (see, for example, [3], [5], [6], [12]).

In [4], the Tribonacci sequence originally was studied in 1963 by M. Feinberg. The Tribonacci sequence $\{T_n\}_{n\geq 0}$ and Tribonacci-Lucas sequence $\{K_n\}_{n\geq 0}$ are defined by the recurrence relations as follows

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}, \qquad T_0 = 0, T_1 = 1, T_2 = 1 \tag{1}$$

and

$$K_n = K_{n-1} + K_{n-2} + K_{n-3}, \qquad K_0 = 3, K_1 = 1, K_2 = 3 \tag{2}$$

respectively. The first terms of the Tribonacci sequence are

$$0,1,1,2,4,7,13,24,44,81,149,274,504,927,1705, \ldots$$

and the first terms of the Tribonacci-Lucas sequence are

$$3,1,3,7,11,21,39,71,131,241,443,815,1499,2757,5071,\ldots$$

The Tribonacci sequence and Tribonacci-Lucas sequence for negative subscripts are defined by the recurrence relations as follows

$$T_{-n} = -T_{-(n-1)} - T_{-(n-2)} + T_{-(n-3)}$$

and

$$K_{-n} = -K_{-(n-1)} - K_{-(n-2)} + K_{-(n-3)}$$

for $n \geq 1$, respectively. The Binet's formulas for the nth Tribonacci number and nth Tribonacci-Lucas number with positive and negative subscripts are given by

$$T_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)} \qquad (3)$$

$$T_{-n} = \frac{\alpha^{-n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{-n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{-n+1}}{(\gamma-\alpha)(\gamma-\beta)} \qquad (4)$$

and

$$K_n = \alpha^n + \beta^n + \gamma^n \qquad (5)$$

$$K_{-n} = \alpha^{-n} + \beta^{-n} + \gamma^{-n} \qquad (6)$$

respectively, where α, β and γ are the roots of the cubic equation $x^3 - x^2 - x - 1 = 0$ and

$$\alpha = \frac{1+\sqrt[3]{19+3\sqrt{33}}+\sqrt[3]{19-3\sqrt{33}}}{3},$$

$$\beta = \frac{1+\omega\sqrt[3]{19+3\sqrt{33}}+\omega^2\sqrt[3]{19-3\sqrt{33}}}{3},$$

$$\gamma = \frac{1+\omega^2\sqrt[3]{19+3\sqrt{33}}+\omega\sqrt[3]{19-3\sqrt{33}}}{3}$$

where $\omega = \frac{-1+i\sqrt{3}}{2} = \exp(\frac{2\pi i}{3})$ is a primitive cube rot of unity. Note that we have the following identities

$$\alpha + \beta + \gamma = 1$$
$$\alpha\beta + \alpha\gamma + \beta\gamma = -1$$
$$\alpha\beta\gamma = 1.$$

(see, for example, [7]).

Many authors studied the sequences of integer number defined by recurrence relations and their generalizations. For instance, several authors have defined new classes of hybrid numbers associated with these sequences of integer number. Now we give information about some special classes of hybrid numbers were introduced and studied in the literature.

A hybrid number is a generalization of complex numbers, dual numbers and hyperbolic numbers.

Complex, dual and hyperbolic numbers are well known two-dimensional number systems. The sets of complex numbers, dual numbers and hyperbolic numbers are

$$\mathbb{C} = \{a + \boldsymbol{i}b : a, b \in \mathbb{R}, \boldsymbol{i}^2 = -1\},$$
$$\mathbb{D} = \{a + \boldsymbol{\varepsilon}b : a, b \in \mathbb{R}, \boldsymbol{\varepsilon}^2 = 0\},$$
$$\mathbb{P} = \{a + \boldsymbol{h}b : a, b \in \mathbb{R}, \boldsymbol{h}^2 = 1\},$$

respectively (see, for example, [13]). Özdemir [11] defined a new generalized of complex, dual and hyperbolic numbers different from known generalizations. In this generalization, the author gave a system of such numbers that consists of all three number systems together. This set was called hybrid numbers, denoted by $\mathbb{K}$, is defined as

$$\mathbb{K} = \{a + b\boldsymbol{i} + c\boldsymbol{\varepsilon} + d\boldsymbol{h}: a, b, c, d \in \mathbb{R}, \boldsymbol{i}^2 = -1, \boldsymbol{\varepsilon}^2 = 0, \boldsymbol{h}^2 = 1, \boldsymbol{ih} = -\boldsymbol{hi} = \boldsymbol{\varepsilon} + \boldsymbol{i}\}.$$

Let $\boldsymbol{Z} = a + b\boldsymbol{i} + c\boldsymbol{\varepsilon} + d\boldsymbol{h}$ be a hybrid number. The real number a is called the scalar part and is denoted by $S(\boldsymbol{Z})$. The part $bi + c\varepsilon + dh$ is also called the vector part and is denoted by $V(\boldsymbol{Z})$. Let $\boldsymbol{Z_1} = a_1 + b_1\boldsymbol{i} + c_1\boldsymbol{\varepsilon} + d_1\boldsymbol{h}$ and $\boldsymbol{Z_2} = a_2 + b_2\boldsymbol{i} + c_2\boldsymbol{\varepsilon} + d_2\boldsymbol{h}$ be any two hybrid numbers. The equality, addition, substraction and multiplication by scalar are defined as follows

Equality: $\boldsymbol{Z_1} = \boldsymbol{Z_2}$ only if $a_1 = a_2, b_1 = b_2, c_1 = c_2, d_1 = d_2$
Addition: $\boldsymbol{Z_1} + \boldsymbol{Z_2} = (a_1 + a_2) + (b_1 + b_2)\boldsymbol{i} + (c_1 + c_2)\boldsymbol{\varepsilon} + (d_1 + d_2)\boldsymbol{h}$
Substraction: $\boldsymbol{Z_1} - \boldsymbol{Z_2} = (a_1 - a_2) + (b_1 - b_2)\boldsymbol{i} + (c_1 - c_2)\boldsymbol{\varepsilon} + (d_1 - d_2)\boldsymbol{h}$
Multiplication by scalar $\lambda \in \mathbb{R}$: $\lambda\boldsymbol{Z_1} = \lambda a_1 + \lambda b_1\boldsymbol{i} + \lambda c_1\boldsymbol{\varepsilon} + \lambda d_1\boldsymbol{h}$.

Table 1. The multiplication table for the basis of $\mathbb{K}$.

$\times$	1	i	ε	h
1	1	i	ε	h
i	i	-1	$1-h$	$\varepsilon+i$
ε	ε	$1+h$	0	$-\varepsilon$
h	h	$-\varepsilon-i$	ε	1

The Table 1 shows us that the multiplication operation in the hybrid numbers is not commutative. But it has the property of associativity. Addition operation in the hybrid numbers is both commutative and associative. Zero is the null element. With respect to the addition operation, the inverse element of $\boldsymbol{Z}$ is $-\boldsymbol{Z} = -a - bi - c\varepsilon - dh$. This implies that, $(\mathbb{K}, +)$ is an Abelian group.
Let $n \geq 0$ be an integer. The nth Horadam hybrid number H_n is defined as follows

$$H_n = W_n + iW_{n+1} + \varepsilon W_{n+2} + hW_{n+3}$$

where W_n is the nth Horadam number (see, for example, [8]). Special cases of Horadam hybrid numbers are definitions of the nth Fibonacci hybrid number and nth Pell hybrid number as follows

$$FH_n = F_n + iF_{n+1} + \varepsilon F_{n+2} + hF_{n+3}$$
$$PH_n = P_n + iP_{n+1} + \varepsilon P_{n+2} + hP_{n+3}$$

where F_n is the nth Fibonacci number and P_n is the nth Pell number, respectively. The nth Jacobsthal hybrid number and nth Jacobsthal-Lucas hybrid number are defined by

$$JH_n = J_n + J_{n+1}i + J_{n+2}\varepsilon + J_{n+3}h$$
$$jH_n = j_n + j_{n+1}i + j_{n+2}\varepsilon + j_{n+3}h$$

where J_n is the nth Jacobsthal number and j_n is the nth Jacobsthal-Lucas number, respectively (see, for example, [9]). Catarino [2] introduced a new sequence of numbers called k-Pell hybrid numbers as follows

$$HP_{k,n} = P_{k,n} + P_{k,n+1}i + P_{k,n+2}\varepsilon + P_{k,n+3}h$$

where $P_{k,n}$ is the nth k-Pell number.

2 Tribonacci and Tribonacci-Lucas Hybrid Numbers

In this section, we define the Tribonacci and Tribonacci-Lucas hybrid numbers and give Binet's formulas and generating functions for these numbers.

Definition 1. The nth Tribonacci hybrid number $\mathbb{H}T_n$ and nth Tribonacci-Lucas hybrid number $\mathbb{H}K_n$ are defined by with the basis $\{1, i, \varepsilon, h\}$ where i, ε, h satisfy the conditions $i^2 = -1, \varepsilon^2 = 0, h^2 = 1, ih = -hi = \varepsilon + i$ as follows

$$\mathbb{H}T_n = T_n + T_{n+1}i + T_{n+2}\varepsilon + T_{n+3}h \tag{7}$$

and

$$\mathbb{H}K_n = K_n + K_{n+1}i + K_{n+2}\varepsilon + K_{n+3}h \tag{8}$$

where T_n is the nth Tribonacci number and K_n is the nth Tribonacci-Lucas number, respectively.

Using the equations (7) and (8) we can write the first Tribonacci and Tribonacci-Lucas hybrid numbers as follows

$$i + \varepsilon + 2h, 1 + i + 2\varepsilon + 4h, 1 + 2i + 4\varepsilon + 7h, 2 + 4i + 7\varepsilon + 13h, \ldots$$

and

$$3 + i + 3\varepsilon + 7h, 1 + 3i + 7\varepsilon + 11h, 3 + 7i + 11\varepsilon + 21h, \ldots$$

respectively.

We now present the following theorem for the recurrence relations of the Tribonacci and Tribonacci-Lucas hybrid numbers.

Theorem 1. Let $\mathbb{H}T_n$ be the nth Tribonacci hybrid number and $\mathbb{H}K_n$ be nth Tribonacci-Lucas hybrid number. Then we give the following recurrence relations

$$\mathbb{H}T_n = \mathbb{H}T_{n-1} + \mathbb{H}T_{n-2} + \mathbb{H}T_{n-3} \tag{9}$$

and

$$\mathbb{H}K_n = \mathbb{H}K_{n-1} + \mathbb{H}K_{n-2} + \mathbb{H}K_{n-3}. \tag{10}$$

Proof. Using the equations (1) and (7) we have

$$\begin{aligned}\mathbb{H}T_{n-1} + \mathbb{H}T_{n-2} + \mathbb{H}T_{n-3} &= (T_{n-1} + T_{n-2} + T_{n-3}) + (T_n + T_{n-1} + T_{n-2})i \\ &\quad + (T_{n+1} + T_n + T_{n-1})\varepsilon + (T_{n+2} + T_{n+1} + T_n)h \\ &= T_n + T_{n+1}i + T_{n+2}\varepsilon + T_{n+3}h \\ &= \mathbb{H}T_n.\end{aligned}$$

Similarly, we can obtain equation (10) by using equations (2) and (8). ■

Using the equations (9) and (10) we obtain the recurrence relations of the Tribonacci and Tribonacci-Lucas hybrid numbers for negative subscripts as follows

$$\mathbb{H}T_{-n} = -\mathbb{H}T_{-(n-1)} - \mathbb{H}T_{-(n-2)} + \mathbb{H}T_{-(n-3)}$$

and

$$\mathbb{H}K_{-n} = -\mathbb{H}K_{-(n-1)} - \mathbb{H}K_{-(n-2)} + \mathbb{H}K_{-(n-3)}$$

respectively. Thus the equations (9) and (10) holds for all integer n.

The following theorem gives the Binet's formulas for the nth Tribonacci hybrid number and nth Tribonacci-Lucas hybrid number.

Theorem 2. Let $\mathbb{H}T_n$ be the nth Tribonacci hybrid number and $\mathbb{H}K_n$ be nth Tribonacci-Lucas hybrid number. For $n \geq 0$, the nth Tribonacci hybrid number and nth Tribonacci-Lucas hybrid number are given by

$$\mathbb{H}T_n = \frac{\hat{\alpha}\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\hat{\beta}\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\hat{\gamma}\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)} \tag{11}$$

and

$$\mathbb{H}K_n = \hat{\alpha}\alpha^n + \hat{\beta}\beta^n + \hat{\gamma}\gamma^n \tag{12}$$

respectively, where $\hat{\alpha} = 1 + \alpha i + \alpha^2\varepsilon + \alpha^3 h$, $\hat{\beta} = 1 + \beta i + \beta^2\varepsilon + \beta^3 h$, $\hat{\gamma} = 1 + \gamma i + \gamma^2\varepsilon + \gamma^3 h$. Moreover, α, β and γ are the roots of the cubic equation $x^3 - x^2 - x - 1 = 0$ and

$$\alpha = \frac{1+\sqrt[3]{19+3\sqrt{33}}+\sqrt[3]{19-3\sqrt{33}}}{3},$$

$$\beta = \frac{1+\omega\sqrt[3]{19+3\sqrt{33}}+\omega^2\sqrt[3]{19-3\sqrt{33}}}{3},$$

$$\gamma = \frac{1+\omega^2\sqrt[3]{19+3\sqrt{33}}+\omega\sqrt[3]{19-3\sqrt{33}}}{3}.$$

where $\omega = \frac{-1+i\sqrt{3}}{2} = \exp\left(\frac{2\pi i}{3}\right)$ is a primitive cube rot of unity.

Proof. Using equation (3) in equation (7) we have

$$\begin{aligned}
\mathbb{H}T_n &= T_n + T_{n+1}i + T_{n+2}\varepsilon + T_{n+3}h \\
&= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)} \\
&\quad + \left(\frac{\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)}\right) i \\
&\quad + \left(\frac{\alpha^{n+3}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+3}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+3}}{(\gamma-\alpha)(\gamma-\beta)}\right) \varepsilon \\
&\quad + \left(\frac{\alpha^{n+4}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+4}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+4}}{(\gamma-\alpha)(\gamma-\beta)}\right) h \\
&= \frac{(1+\alpha i+\alpha^2\varepsilon+\alpha^3 h)\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{(1+\beta i+\beta^2\varepsilon+\beta^3 h)\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{(1+\gamma i+\gamma^2\varepsilon+\gamma^3 h)\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)} \\
&= \frac{\hat{\alpha}\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\hat{\beta}\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\hat{\gamma}\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}.
\end{aligned}$$

where $\hat{\alpha} = 1 + \alpha i + \alpha^2\varepsilon + \alpha^3 h$, $\hat{\beta} = 1 + \beta i + \beta^2\varepsilon + \beta^3 h$, $\hat{\gamma} = 1 + \gamma i + \gamma^2\varepsilon + \gamma^3 h$.

Similarly, we can obtain equation (12) by using equation (5) in equation (8). ■

We can introduce Binet's formulas for the nth Tribonacci hybrid number and nth Tribonacci-Lucas hybrid number with negative subscripts as follows

$$\mathbb{H}T_{-n} = \frac{\hat{\alpha}\alpha^{-n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\hat{\beta}\beta^{-n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\hat{\gamma}\gamma^{-n+1}}{(\gamma-\alpha)(\gamma-\beta)}$$

and

$$\mathbb{H}K_{-n} = \hat{\alpha}\alpha^{-n} + \hat{\beta}\beta^{-n} + \hat{\gamma}\gamma^{-n}$$

respectively.

Next, we give generating functions for the Tribonacci and Tribonacci-Lucas hybrid numbers.

Theorem 3. Let $\mathbb{H}T_n$ be the nth Tribonacci hybrid number and $\mathbb{H}K_n$ be nth Tribonacci-Lucas hybrid number. The generating functions for the Tribonacci and Tribonacci-Lucas hybrid numbers are

$$g(t) = \sum_{n=0}^{\infty} \mathbb{H}T_n t^n = \frac{\mathbb{H}T_0 + (\mathbb{H}T_1 - \mathbb{H}T_0)t + \mathbb{H}T_{-1}t^2}{1-t-t^2-t^3} \tag{13}$$

and

$$r(t) = \sum_{n=0}^{\infty} \mathbb{H}K_n t^n = \frac{\mathbb{H}K_0 + (\mathbb{H}K_1 - \mathbb{H}K_0)t + \mathbb{H}K_{-1}t^2}{1-t-t^2-t^3} \tag{14}$$

respectively.

Proof. Let $g(t) = \sum_{n=0}^{\infty} \mathbb{H}T_n t^n$ be generating function for the Tribonacci hybrid numbers. On the other hand, since

$$g(t) = \mathbb{H}T_0 + \mathbb{H}T_1 t + \mathbb{H}T_2 t^2 + \cdots + \mathbb{H}T_n t^n + \cdots$$
$$g(t)t = \mathbb{H}T_0 t + \mathbb{H}T_1 t^2 + \mathbb{H}T_2 t^3 + \cdots + \mathbb{H}T_n t^{n+1} + \cdots$$
$$g(t)t^2 = \mathbb{H}T_0 t^2 + \mathbb{H}T_1 t^3 + \mathbb{H}T_2 t^4 + \cdots + \mathbb{H}T_n t^{n+2} + \cdots$$
$$g(t)t^3 = \mathbb{H}T_0 t^3 + \mathbb{H}T_1 t^4 + \mathbb{H}T_2 t^5 + \cdots + \mathbb{H}T_n t^{n+3} + \cdots$$

we obtain that

$$(1 - t - t^2 - t^3)g(t) = \mathbb{H}\mathrm{T}_0 + \mathbb{H}T_1 t + \mathbb{H}T_2 t^2 - \mathbb{H}\mathrm{T}_0 t - \mathbb{H}T_1 t^2 - \mathbb{H}\mathrm{T}_0 t^2$$

where $\mathbb{H}T_n = \mathbb{H}T_{n-1} + \mathbb{H}T_{n-2} + \mathbb{H}T_{n-3}$ from equation (9). Here the coefficients of t^n for $n \geq 3$ are equal to zero. Then generating function for the Tribonacci hybrid numbers is

$$\sum_{n=0}^{\infty} \mathbb{H}T_n t^n = \frac{\mathbb{H}T_0 + (\mathbb{H}T_1 - \mathbb{H}T_0)t + \mathbb{H}T_{-1}t^2}{1-t-t^2-t^3}.$$

Similarly, we can obtain equation (14). ■

3 Tribonacci and Tribonacci-Lucas Hybrid Numbers by Hessenberg Matrices

In this section, we define four type lower Hessenberg matrices and investigate the relationships between these matrices and the Tribonacci and Tribonacci-Lucas hybrid numbers. We then show that the determinants and permanents of these Hessenberg matrices are the nth Tribonacci hybrid number and nth Tribonacci-Lucas hybrid number.

A nxn matrix $M_n = (m_{ij})$ is called lower Hessenberg matrix if $m_{ij} = 0$ when $j - i > 1$, i.e.,

$$M_n = \begin{pmatrix} m_{1,1} & m_{1,2} & 0 & 0 & \cdots & 0 \\ m_{2,1} & m_{2,2} & m_{2,3} & 0 & \cdots & 0 \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ m_{n-1,1} & m_{n-1,2} & m_{n-1,3} & m_{n-1,4} & \cdots & m_{n-1,n} \\ m_{n,1} & m_{n,2} & m_{n,3} & m_{n,4} & \cdots & m_{n,n} \end{pmatrix}.$$

In [1], for $n \geq 2$, it was given the following formula

$$det(M_n) = m_{n,n}\, det(M_{\mathrm{n}-1}) + \sum_{i=1}^{n-1}\left[(-1)^{n-i} m_{n,i} \prod_{j=i}^{n-1} m_{j,j+1} det(M_{\mathrm{i}-1})\right] \quad (15)$$

with $det(M_0) = 1$ and $det(M_1) = m_{11}$. Also in [10], for $n \geq 2$, it was given the following formula

$$per(M_n) = m_{n,n}\, per(M_{\mathrm{n}-1}) + \sum_{i=1}^{n-1}\left[m_{n,i} \prod_{j=i}^{n-1} m_{j,j+1} per(M_{i-1})\right] \quad (16)$$

with $per(M_0) = 1$ and $per(M_1) = m_{11}$.

Theorem 4. Let $n \times n$ Hessenberg matrices $P_n = (p_{ij})$ and $Q_n = (q_{ij})$ be as follows

$$P_n = \begin{pmatrix} \mathbb{H}T_0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \mathbb{H}T_1/\mathbb{H}T_0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -\mathbb{H}T_{-1} & 1 & 1 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & \cdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 \end{pmatrix}$$

and

$$Q_n = \begin{pmatrix} \mathbb{H}K_0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \mathbb{H}K_1/\mathbb{H}K_0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -\mathbb{H}K_{-1} & 1 & 1 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & \cdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 \end{pmatrix}.$$

Then for $n \geq 1$,

$$detP_n = \mathbb{H}T_{n-1} \quad (17)$$

and

$$detQ_n = \mathbb{H}K_{n-1} \quad (18)$$

where $\mathbb{H}T_n$ is the nth Tribonacci hybrid number and $\mathbb{H}K_n$ is the nth Tribonacci-Lucas hybrid number.

Proof. We can use the mathematical induction method on n to prove $detP_n = \mathbb{H}T_{n-1}$. Then
$n = 1$, $detP_1 = \mathbb{H}T_0$

$n = 2, detP_2 = \mathbb{H}T_0(\mathbb{H}T_1/\mathbb{H}T_0) - 0 = \mathbb{H}T_1$
$n = 3, detP_3 = \mathbb{H}T_0(\mathbb{H}T_1/\mathbb{H}T_0 + 1) - 1(-\mathbb{H}T_{-1}) = \mathbb{H}T_1 + \mathbb{H}T_0 + \mathbb{H}T_{-1} = \mathbb{H}T_2$
$\vdots$
We assume that it is true for $n \in \mathbb{Z}^+$, namely

$$detP_n = \mathbb{H}T_{n-1}, detP_{n-1} = \mathbb{H}T_{n-2}, detP_{n-2} = \mathbb{H}T_{n-3}, \ldots$$

and we shall show that it is true for $n + 1$. Using the equation (15) we have

$$\begin{aligned} detP_{n+1} &= p_{n+1,n+1}detP_n + \sum_{i=1}^{n}\left[(-1)^{n+1-i}p_{n+1,i}\prod_{j=i}^{n}p_{j,j+1}detP_{i-1}\right] \\ &= (1)detP_n + \sum_{i=1}^{n-2}\left[(-1)^{n+1-i}p_{n+1,i}\prod_{j=i}^{n}p_{j,j+1}detP_{i-1}\right] \\ &\quad +(-1)p_{n+1,n}p_{n,n+1}detP_{n-1} + (-1)^2p_{n+1,n-1}p_{n-1,n}p_{n,n+1}detP_{n-2} \\ &= detP_n + 0 + (-1)(1)(-1)detP_{n-1} + (-1)^2(1)(-1)(-1)detP_{n-2} \\ &= detP_n + detP_{n-1} + detP_{n-2} \\ &= \mathbb{H}T_{n-1} + \mathbb{H}T_{n-2} + \mathbb{H}T_{n-3} \\ &= \mathbb{H}T_n. \end{aligned}$$

Similarly, we can obtain equation (18).

■

Theorem 5. Let $n \times n$ Hessenberg matrices $R_n = (r_{ij})$ and $S_n = (s_{ij})$ be as follows

$$R_n = \begin{pmatrix} \mathbb{H}T_0 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \mathbb{H}T_1/\mathbb{H}T_0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -\mathbb{H}T_{-1} & 1 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & \cdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 \end{pmatrix}$$

and

$$S_n = \begin{pmatrix} \mathbb{H}K_0 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \mathbb{H}K_1/\mathbb{H}K_0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -\mathbb{H}K_{-1} & 1 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & \cdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 \end{pmatrix}.$$

Then for $n \geq 1$,

$$perR_n = \mathbb{H}T_{n-1} \tag{19}$$

and

$$perS_n = \mathbb{H}K_{n-1} \tag{20}$$

where $\mathbb{H}T_n$ is the nth Tribonacci hybrid number and $\mathbb{H}K_n$ is the nth Tribonacci-Lucas hybrid number

Proof. We can use the mathematical induction method on n to prove $perR_n = \mathbb{H}T_{n-1}$. Then
$n = 1, perR_1 = \mathbb{H}T_0$
$n = 2, perR_2 = \mathbb{H}T_0(\mathbb{H}T_1/\mathbb{H}T_0) + 0 = \mathbb{H}T_1$

$n = 3, perR_3 = \mathbb{H}T_0(\mathbb{H}T_1/\mathbb{H}T_0 + 1) - 1(-\mathbb{H}T_{-1}) = \mathbb{H}T_1 + \mathbb{H}T_0 + \mathbb{H}T_{-1} = \mathbb{H}T_2$
$\vdots$
We assume that it is true for $n \in \mathbb{Z}^+$, namely

$$perR_n = \mathbb{H}T_{n-1}, perR_{n-1} = \mathbb{H}T_{n-2}, perR_{n-2} = \mathbb{H}T_{n-3}, \dots$$

and we shall show that it is true for $n + 1$. Using the equation (16) we have

$$\begin{aligned} per(R_{n+1}) &= r_{n+1,n+1}\, perR_{\mathrm{n}} + \textstyle\sum_{i=1}^{n}\left[r_{n+1,i} \prod_{j=i}^{n} r_{j,j+1} per(R_{i-1})\right] \\ &= (1)perR_n + \textstyle\sum_{i=1}^{n-2}\left[r_{n+1,i} \prod_{j=i}^{n} r_{j,j+1} perR_{i-1}\right] \\ &\quad + r_{n+1,n} r_{n,n+1} perR_{n-1} + r_{n+1,n-1} r_{n-1,n} r_{n,n+1} perR_{n-2} \\ &= perR_n + 0 + (1)(1)perR_{n-1} + (1)(1)(1)perR_{n-2} \\ &= perR_n + perR_{n-1} + perR_{n-2} \\ &= \mathbb{H}T_{n-1} + \mathbb{H}T_{n-2} + \mathbb{H}T_{n-3} \\ &= \mathbb{H}T_n. \end{aligned}$$

Similarly, we can obtain equation (20). ∎

4 Discussion and Conclusions

In this paper, we define the Tribonacci and Tribonacci-Lucas hybrid numbers by using a new generalized of complex, dual and hyperbolic numbers and obtain their recurrence relations. We also states the expression for negative subscripts. Then we find Binet's formulas for nth number and generating functions that plays an important role in the literature. Moreover, some Hessenberg matrices whose entries are the Tribonacci and Tribonacci-Lucas hybrid numbers was presented as a different way to obtain the nth Tribonacci hybrid number and nth Tribonacci-Lucas hybrid number and show that the determinants and permanents of these Hessenberg matrices are equal to the nth Tribonacci hybrid number and nth Tribonacci-Lucas hybrid number.

In the future, we consider all subsequences of the Tribonacci and Tribonacci-Lucas hybrid numbers of forms $(\mathbb{H}T_{mn+s})$ and $(\mathbb{H}K_{mn+s})$ for arbitrary integers n, s and m with $0 \leq s < m$. We intend to discuss Binet's formulas, generating functions and the summation formulas for subsequences of the Tribonacci and Tribonacci-Lucas hybrid numbers of forms $(\mathbb{H}T_{mr+s})$ and $(\mathbb{H}K_{mr+s})$ with positive and negative subscripts by using the new definitions presented in this paper.

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